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Face-regular 3-valent two-faced spheres and tori

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Abstract

Call *two-faced map* and, specifically, (p, q)*-map* a 3-valent map (on sphere or torus) with only p- and q-gonal faces (at least one each), for given integers $3 \le p < q$; so, $3 \le p \le 5$. Two-faced maps (especially, $(5, 6)$-polyhedra, called *fullerenes*) are prominent molecular models in Chemistry.

We say that a (p, q)-map is pR_i if any p-gon has the same number i of p-gonal neighbors; it is qR_j if each q-gon has the same number j of q-gonal neighbors. Call a (p, q)-map *strictly face-regular* if it is both, pR_i and qR_j, for some i,j; call it *weakly face-regular*, if it is only pR_i *or* qR_j.

All strictly face-regular (p, q)-polyhedra are ([BrDe99], [De02]) $Prism_m$, $Barrel_m$ ($m \ge 3$) and 55 sporadic polyhedra. By $Barrel_m$ we denote $4m$-vertex $(5, m)$-polyhedron, consisting of two m-gons separated by two m-rings of 5-gons. All 23 parameter-sets $(p, q; i, j)$ for strictly face-regular (p, q)-tori are found ([De02]); the number of minimal tori is one for 7 of them and an infinity for 16 others.

Here we address the characterization of all *weakly* face-regular (p, q)-maps on sphere or torus. Examples of obtained results are:

1. Any $(3, q)$-map, which is $3R_0$, has $4 \le q \le 12$. It is strongly face-regular for $q = 4, 5$ ($Prism_3$ and $Barrel_3$ only) and $q = 11, 12$ (only tori, unique for $q = 12$). All such weakly face-regular maps are infinities of polyhedra and tori for each $7 \le q \le 10$ and (characterized) infinity of $(3, 6)$-polyhedra.
2. Weakly face-regular $(5, q)$-polyhedra $5R_j$ exist for $j = 3, 6 \le q \le 10$, and $j = 2, q \ge 8$.
3. The following general conjecture: the number of (p, q)-polyhedra, which are qR_j, is infinite if and only the corresponding torus exist.
4. If a (p, q)-polyhedron is qR_j, then $j \le 5$. The number of $(5, q)$-polyhedra is infinite if and only if $q \ge 12, 10, 8, 7, 7, 7$ for $j = 0, 1, 2, 3, 4, 5$, respectively (except some undecided cases for $j = 0, 3, 5$).
5. The number of $(4, q)$-polyhedra qR_j is finite (all are classified) for $j \le 3$. For $j = 4$, they are classified (and their number is infinite) if $q = 8$; we conjecture infiniteness if and only if $q \ge 8$.
6. For many classes qR_j, the number of possibilities is very large. However, for the critical value of q, starting at which an infinity of graphs occurs, we are often able to describe the structure and obtain a complete classification.

We used large computations, variations of Euler formula, analysis of coronae of faces, the decomposition of (p, q)-maps into *elementary* $(p, 3)$*-polycycles* and many ad hoc arguments.

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1 Introduction

1.1 Main notions

Call a *two-faced map* and, specifically, (p,q)*-map* any 3-valent map (on sphere or torus) with only p- and q-gonal faces for given integers $2 < p < q$; we will also use terms (p,q)*-sphere* (moreover, (p,q)*-polyhedron* if it is 3-connected) or (p,q)*-torus*, respectively. In fact, the term *map* will be used only for *sphere or torus.* $(5,6)$-polyhedra are called *fullerenes* in Organic Chemistry; one of the purposes of this paper is possible application in Chemistry, where fullerenes and other two-faced maps are prominent molecular models.

We say that a (p,q)-map is pR_i, if any p-gon has the same number i of p-gonal neighbors; we say that it is qR_j, if each q-gon has the same number j of q-gonal neighbors.

Call a (p,q)-map *strictly face-regular* if it is both, pR_i and qR_j, for some i,j; call it *weakly face-regular*, if it is only pR_i *or* qR_j. All 3-connected strictly face-regular (p,q)-maps are known. On the sphere they are ([BrDe99], [De02]): $Prism_m$ (for all $m \geq 3$, except 4), $Barrel_m$ (for all $m \geq 3$, except 5) and 55 sporadic simple polyhedra; on torus ([De02], in equivalent terms of plane partitions): 23 parameter-sets $(p,q;i,j)$. By $Barrel_m$ above we denote $4m$-vertex $(5,m)$-polyhedron, consisting of two m-gons separated by two m-rings of 5-gons.

In this paper we address the following problem: characterize all *weakly* face-regular (p,q)-maps on sphere or torus. This problem for $j=2$ on the sphere has been considered in the papers [DGr02] and [DDS04].

Let us call *gonality* of a face the number of its vertices. Denote by v, e and f the number of vertices, edges and faces, respectively, of a given map. Denote by f_i the number of its i-gonal faces and by e_{a-b} (for fixed $3 \leq a \leq b$) the number of $(a-b)$-edges, i.e., edges separating a- and b-gonal faces. Call *corona* (of a face) the sequence of gonalities of all its consecutive neighbors. Denote by *Aut* the automorphism group of a given (p,q)-map.

Call (p,q)*-plane* any 3-valent partition of Euclidean plane by p- and q-gons only, for given $3 \leq p < 6 < q$. We have (Euler's formula for (p,q)-torus) $f_p(6-p) = f_q(q-6)$, i.e., proportion of p-and q-gons in (p,q)-plane is $6-p : q-6$. Call a (p,q)-plane *decorated* $\{6^3\}$ (or just *decorated graphite*), if it became the 3-valent partition of the plane by hexagons after deletion of all $(p-p)$-edges and removal of vertices of degree 2. Call *matching* M of a graph G a set of edges of G such that every vertex belongs to at most one edge from M; M is called *perfect* if every vertex belongs to exactly one edge from M.

The *Goldberg-Coxeter construction* takes a 3- or 4-valent orientable map G, two integers $(i,j) \neq (0,0)$ and returns a 3- or 4-valent map, denoted by $GC_{i,j}(G)$, see [Gold37], [Cox71] and [DuDe03] for detailed description.

We used computer methods (by the consideration of all possibilities), when this approach worked, and theoretical, otherwise. The numerical approach cannot work in the torus case, since, given a (p,q)-torus, which is qR_j or pR_i, one can obtain a (p,q)-torus with the same property and arbitrary large number v of vertices.

However, for many subcases, the torus case is simpler, since the Euler formula takes the form $v-e+f=0$, instead of more complicated $v-e+f=2$. To illustrate this

q j	5	6	7	8	9	10
0	**0**	**0**	**0**	**0**	**0**	**0**
strictly	**0**	**0**	**0**	**0**	**0**	**0**
1	**0**	**0**	**0**	**0**	**0**	**0**
strictly	**0**	**0**	**0**	**0**	**0**	**0**
2	**0**	**1**	**1**	**2**	**1**	**1**
strictly	**Nr. 16**	**Nr. 19**	**0**	**0**	**0**	**0**
3	**0**	**0**	**3**	**5**	**5**	**6**
Figures			Fig. 45	Fig. 46	Fig. 47	Fig. 48
strictly	**Nr. 17**	**Nrs. 21,22**	**0**	**0**	**Nr. 35**	**0**
4	**0**	**0**	**3**	∞	4(58)	8(56)
Figures			Fig. 74	Fig. 55–66	Fig. 77	Fig. 79
strictly	**Nr. 18**	**Nrs. 20, 23-25**	**Nrs. 27,28, 30,31**	**Nrs. 32-34**	**Nr. 38**	**0**
5	**0**	**0**	$0(62)+\geq 3$	2(56)	3(58)	2(56)
Figures			Fig. 75	Fig. 76	Fig. 78	Fig. 80
strictly	**0**	**Nr. 26**	**Nr. 29**	**0**	**Nrs. 36,37**	**0**

Table 1: The number of $(4, q)$-spheres qR_j, different from Cube and $Prism_q$, with $q \leq 10$ and $j \leq 5$

point, a $(5, 7)$-sphere, which is $7R_2$, satisfies to $x_0 + x_3 + f_7 = 20$, which allows to have an upper bound on v and enumerate such spheres, while a $(5, 7)$-torus, which is $7R_2$, satisfies to $x_0 + x_3 + f_7 = 0$ and hence, does not exist at all.

The shape of the results is also interesting. If the qR_j (p, q)-tori admit classification, then, usually, there is more freedom for the qR_j (p, q)-spheres (compare Theorems 6.2 and 6.3). However, if the pR_i (p, q)-tori admit classification, then, usually, the possibilities for pR_i (p, q)-sphere are more restricted (see Theorem 5.19).

Tables 1 and 2 illustrate the present knowledge about the finiteness of the number of face-regular (p, q)-spheres qR_j with $q \leq 10$ and $j \leq 5$; in parenthesis are given the number of vertices, until which the computations were done. The definite numbers there are given boldface. The strictly face-regular (p, q)-polyhedra are indicated by their numbers in Table 3. We took $j \leq 5$, because if a (p, q)-polyhedron is qR_j, then $j \leq 5$ (see Corollary 4.3 below).

Remarks on Table 1:

(i) All $(4, q)$-spheres qR_j with $j \leq 3$ are given in Theorems 4.10 and 9.1.

(ii) All strictly face-regular $(4, q)$-spheres (different from Cube and $Prism_q$), which are not given in Table 1, are Nrs. 39-43 (they have (q, j)=$(11, 5)$, $(11, 5)$, $(12, 4)$, $(13, 5)$, $(15, 5)$).

(iii) For strictly face-regular $(4, q)$-spheres Nrs. 35-43, Table 1 in [De02] contained a misprint: the properties pR_i, qR_j are denoted by $t_p = i$ and $t_q = j$; the values, given there

q j	6	7	8	9	10
0	**1**	**0**	**0**	**2**	**3**
Figures				Fig. 29	Fig. 30
strictly	**Nr. 45**	**0**	**Nr. 58**	**0**	**Nr. 60**
1	**2**	**3**	4(74)	5(68)	10(80)
Figures		Fig. 37	Fig. 38	Fig. 39	Fig. 40
strictly	**0**	**Nr. 56**	**0**	**0**	**0**
2	**6**	**25**[DDS]	∞	$11+\infty$	∞
Figures			Fig. 82	Fig. 85	Fig. 88
Fig.[DDS]		Fig. 10,11	Fig. 12	Fig. 13	Fig. 14
strictly	**Nrs. 46,47**	**Nr. 57**	**Nr. 59**	**0**	**0**
3	**2**	4(76)+≥ 2	4(74)	2(68)+∞	3(80)+∞
Figures		Fig. 81	Fig. 83	Fig. 86	Fig. 89
strictly	**Nrs. 48,49,51**	**0**	**0**	**0**	**0**
4	**1**	$2(76)+\infty$	$3(74)+\infty$	0(68)	3(80)+∞
Figures		Fig. 67	Fig. 84		Fig. 90
strictly	**Nrs. 50,52-54**	**0**	**0**	**0**	**0**
5	**0**	1(278)	1(92)+∞	1(68)+∞	0(80)+∞
Figures		Fig. 73	Fig. 72	Fig. 87	
strictly	**Nr. 55**	**0**	**0**	**0**	**0**

Table 2: The number of $(5, q)$-spheres qR_j, different from Dodecahedron and $Barrel_q$, with $q \leq 10$ and $j \leq 5$

as t_q, are, actually, $q - t_q$ and so, remark (v) there should be deleted.

Remarks on Table 2:

(i) All 16 strictly face-regular $(5, q)$-spheres, different from Dodecahedron and $Barrel_q$, are in Table 2.

(ii) The list of $(5, q)$-spheres qR_0 is known for $q \leq 12$ (see Figures 29, 30, 31, 32 for $q = 9, 10, 11, 12$); it is, conjecturally, infinite if and only if $q \geq 12$ (see known spheres for $13 \leq q \leq 16$ on Figures 33, 34, 35, 36).

Here is a summary of the results and conjectures on finiteness of the number of $(4, q)$-spheres:

- The number of $(4, q)$-spheres qR_j is 0 for $j \geq q - 2 \geq 6$ (Theorem 12.2); it is finite for $j \leq 3$ (all such spheres are given in Theorem 4.10 for $j \leq 2$ and in Theorem 9.1(ii) for $j = 3$); it is infinite for $j = 4$ (all such spheres with $q = 8$ are described in Theorem 10.3 and we conjecture it for $q \geq 9$);

- The number of $(4, q)$-spheres $4R_0$ is infinite for $q = 6$ and 7 only (see Theorems 5.1(i), 5.2 and 5.1)

- The number of $(4, q)$-spheres $4R_1$ is infinite for $q = 6$, 7, 8 and 9 (see Theorems 5.3, 5.4, 5.6 and 5.1).

- The number of $(4, q)$-spheres $4R_2$ (different from $Prism_q$) is infinite for $6 \leq q \leq 13$ and $q = 15$ and such spheres do not exist for other values of q (see Theorems 5.7(ii), 5.8, 5.9, 5.10, 5.11, 5.13, 5.15, 5.12 and 5.14).

- Remarks 13.1–13.4 treat $(4, q)$-spheres with $7 \leq q \leq 10$ in detail, completing Table 1.

Here is a summary of results and conjectures on existence of $(4, q)$-tori:

- A $(4, q)$-torus qR_j does not exist if $j \leq 3$ (Theorems 4.10(ii) and 9.1(i)).

  For $j = 4$, it exists if and only if $q \geq 8$; moreover, it is $4R_0$ for $q = 8$ (Theorem 10.2 (i), (ii)) and characterized (Theorem 10.2 (iii), (iv)) for $q = 9$.

  For $j = 5$, it exists if and only if $q \geq 7$; moreover, it is $4R_0$ for $q = 7$ (Theorem 11.1).

  For $j = 6$ and if the $(4, q)$-torus is 3-connected, it is also $4R_2$ (Theorem 12.1).

- A $(4, q)$-torus $4R_i$ can exist only for $(i, q) = (0, 7)$, $(1, 7)$, $(1, 8)$, $(1, 9)$, $(2, 7 \leq q \leq 16)$, $(2, 18)$ (Theorems 5.1(ii) and 5.7(i)). See Figure 18 for an example with $(i, q) = (1, 9)$.

Here is the summary of results and conjectures on finiteness of the number of $(5, q)$-spheres:

- The number of $(5, q)$-spheres qR_0 is finite if and only if $6 \leq q \leq 11$ (Theorem 6.3 lists them for $q \leq 12$ and proves it for $q = 12$; we conjecture it for $q > 12$).

- The number of $(5, q)$-spheres qR_1 is finite if and only if $6 \leq q \leq 9$ (Figure 37 lists them for $q = 7$; Theorems 4.4(ii) and 7.2(i) prove it for $q = 8$ and $q = 9$, respectively; we conjecture it for $q > 9$).

- The number of $(5, q)$-spheres qR_2 is finite if and only if $6 \leq q \leq 7$ (Theorem 8.3 proves it for $q \geq 8$; for $q = 6$ and $q = 7$ all were listed in [DeGr01] and [DDS04], respectively).

- The number of $(5, q)$-spheres qR_3 is finite if and only if $q = 6$ (it is Conjecture 9.4; Theorem 9.5 proves it for $q = 9, 10, 12$).

- The number of $(5, q)$-spheres qR_4 is finite if and only if $q = 6$ (Theorems 10.4, 10.5, 10.7 prove it for $q = 7, 8, 10, 13, 16$).

- The number of $(5, q)$-spheres qR_5 is finite if and only if $q = 6$ (Theorems 11.6, 11.7 prove it for $q \leq 21$, except of undecided cases $q = 7, 10, 13, 16, 19$; we conjecture it for all q and Figures 73, 72, 87 give examples for $q = 7$, 8, 9, respectively).

- By Theorem 5.1, a $(5,q)$-sphere $5R_i$ can exist only for $i=2$ (see [DGr02]) or $i=3$. A $(5,q)$-sphere $5R_3$ exists if and only if $6 \le q \le 10$ (Theorem 5.19 lists them for $q=7$ and proves it for $q \le 10$, Theorem 5.24 gives an infinite series and seven examples for $q=8$, Theorem 5.26 gives unique case for $q=9$, Theorem 5.27 gives three examples for $q=10$).

- Remarks 14.1–14.4 treat $(5,q)$ with $7 \le q \le 10$ in detail, complementing Table 2.

Here is a summary of the results and conjectures on existence of $(5,q)$-tori:

- $(5,q)$-torus qR_0 exists if and only if $q \ge 12$; for $q=12$ it is qR_0 if and only if it is $5R_3$ (Theorems 6.2 and 5.19(ii)).

- $(5,q)$-torus qR_1 exists if and only if $q \ge 10$; for $q=10$ it corresponds to some perfect matching of a 6-valent tessellation of the torus by triangles (Theorem 7.2(ii), 7.5 and 7.3).

- $(5,q)$-torus qR_2 exists if and only if $q \ge 8$; for $q=8$ it is qR_2 if and only if it is $5R_2$ (Theorems 8.1 and 8.2).

- $(5,q)$-torus qR_3 exists if and only if $q \ge 7$; for $q=7$ it is qR_3 if and only if it is $5R_1$ (Theorems 9.3 and 9.2).

- $(5,q)$-torus qR_4 exists if and only if $q \ge 7$ (Theorem 10.6).

- By Theorem 5.1, a $(5,q)$-torus $5R_i$ can exist only for $i=2$ and $i=3$. A $(5,q)$-torus $5R_2$ exists if and only if $q=7$ or 8 (see Theorem 5.17). A $(5,q)$-torus $5R_3$ exists if and only if $q=8,10,11,12$; moreover, it is qR_{12-q} for $q=10,11,12$ (Theorems 5.19, 5.26(ii), Lemma 5.22(ii) and Conjecture 5.28 for $q=10$).

In view of above summaries for spheres and tori, we conjecture:

Conjecture 1.1 *The number of (p,q)-spheres qR_j is infinite if and only if a (p,q)-torus qR_j exists.*

But the similar conjecture for pR_i does not hold, for example, for $(p,q;i)=(5,11;3)$ (Lemma 5.21) or $(5,12;3)$ (Lemma 5.20).

Given a torus, its universal cover is a periodic infinite map on the plane. By choosing a finite index subgroup H of the group G of covering transformations and taking the quotient, one can obtain bigger tori; those tori have a translation subgroup, which is isomorphic to the quotient G/H. On the other hand, given a torus with non-trivial translation group, there exist a unique *minimal torus* with the same universal cover and trivial translation subgroup. Those minimal tori correspond, in a one-to-one way, to periodic tilings of the plane.

It is shown in [Mo97] that a combinatorial map of arbitrary genus admits an unique *primal-dual circle representation* on a Riemann surface of the same genus (i.e., sphere, torus or surface with handle). More precisely, this means that the combinatorics of the map determines an unique complex analytic structure. This implies that every combinatorial symmetry of the map correspond to an isometry of the Riemann surface (in the case of sphere, this is Mani's theorem [Ma71]). The finite groups of isometries of the sphere have been determined a long time ago and are called *point groups*. They are described, for example, in [D1] using the Schoenflies notation, which is used in this work (for the Hermann-Maugin notation, see Chapter 3 of [KeHy96]).

The enumeration of (p, q)-spheres (respectively, tori) in this paper was done using the programs CPF (respectively, CGF) by T.Harmuth (see [BFDH97] and [Ha]). The enumeration of face-regular maps was done using the GAP computer algebra program (see [GAP02]), the PlanGraph package ([Du02]). The drawings were done using CaGe (see [BFDH97]) and TorusDraw (see [Du04], which uses the theory explained in [Mo97]).

For spheres, we presented all found plane graphs up to the capacity of CPF. Due to the sheer number of existing tori and the uninterestness of most, we selected the ones, which are presented. All (p, q)-tori are represented on the plane by their universal cover, called (p, q)-*planes*. For each of them we give parameters (v, f_p, f_q) of their minimal tori. The symmetry groups belong to one of 17 possibilities, which are explained, for example, in Chapter 1 of [KeHy96].

Face-regular maps are of interest for Chemistry and Physics, because many of them appear already there. For example, many of known polyhedral (energy) minimizers in Thomson problem (for given number of particles on sphere) or Skyrme problem (for given integer barionic number) are face-regular $(5, 6)$-polyhedra. Face-regular $(5, 7)$-planes are related to a putative "metallic carbon" deformation of the graphite lattice. Also, for example, all known polyhedra P, such that their skeleton is an isometric subgraph of a hypercube or a half-hypercube, have either P, or its dual P^* face-regular.

1.2 $(p, 3)$-polycycles

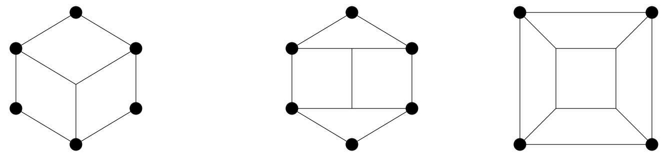
A $(p, 3)$-*polycycle* is a plane graph, such that all interior faces are p-gons, all interior vertices are 3-valent and any vertex of the *boundary* (i.e., the exterior face) has valency within $\{2, 3\}$.

This simple, yet powerful, notion allow us to prove some of the most interesting results of this paper. However, in some cases, we must generalize this notion. This can be done in two direction: by allowing several gonalities for interior faces and by allowing several boundaries. If we allow different gonalities for interior faces, we will use the term 3-*patch*. If we allow several boundaries, then we speak of *generalized* $(p, 3)$-*polycycle*. If we allow both different gonalities of interior faces and several boundaries, then we just call it loosely *group of faces*.

The *boundary sequence* of a $(p, 3)$-polycycle P is the sequence $b(P)$ enumerating, up to a cyclic shift or reversal, the consecutive valencies of all vertices incident to the exterior face. Call a sequence $(p, 3)$-*fillable* if there exists a $(p, 3)$-polycycle having it as the boundary sequence.

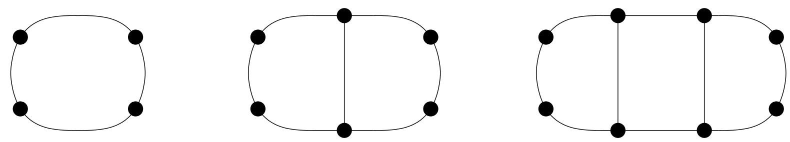
Now we present the information on $(p, 3)$-polycycles, which will be extensively used in many proofs below.

Any $(4,3)$-polycycle is either one of the following graphs S_1, S_2, S_3:



$b(S_1) = 232323$ $b(S_2) = 233233$ $b(S_3) = 3333$

or belongs to the infinite series $(P_{2k})_{k \geq 2}$, shown below for first values $k = 2, 3, 4$:



$b(P_4) = 2222$ $b(P_6) = 223223$ $b(P_8) = 22332233$

Theorem 1.2 *Given a $(4,q)$-map, which is not $Prism_q$, we can decompose its set of 4-gons is decomposed into $(4,3)$-polycycles.*

Proof. We will get the result if we prove that the only generalized $(4,3)$-polycycles are $(4,3)$-polycycles or $Prism_q$.

Take an initial 4-gon and add 4-gons in any possible way. At first one obtains P_6 and $Prism_2$, then P_8, S_1 and $Prism_3$. The next steps are S_2 and *Cube*. After that the only possibilities are P_{2k} and $Prism_k$. $\square$

A $(p,3)$-polycycle is called *elementary* if it cannot be separated into two other $(p,3)$-polycycles by splitting an edge. Call an edge *open* if both its ends have degree 2.

Theorem 1.3 *([DSt01], [DSt02] and [DDS1]) Every finite generalized $(5,3)$-polycycle is either $Barrel_n$ for some $n \geq 3$, or is formed by the agglomeration of elementary ones, listed on Figure 1, along open edges.*

Remark that for $p > 5$, the set of elementary $(p,3)$-polycycles becomes uncountable.

The following proposition, which is a slight generalization of the one in [DDS04], is very helpful in deriving classification results.

Proposition 1.4 *Let P be a generalized $(p,3)$-polycycle with t boundaries. Denote by v_2 and v_3 the number of vertices of degree 2, 3 on the boundary. Let x and f_p be the number of interior vertices and p-gonal interior faces. Then, one has:*

$$\left\{ \begin{array}{rcl} pf_p - 3x & = & v_2 + 2v_3 \\ f_p - \frac{x}{2} & = & (2-t) + \frac{v_3}{2} \end{array} \right. .$$

If $p \neq 6$, then the system of equations has the solution:

$$\left\{ \begin{array}{rcl} f_p & = & \frac{1}{p-6}\{v_2 - v_3 - 6(2-t)\} \\ x & = & \frac{1}{p-6}\{2v_2 - (p-4)v_3 - 2p(2-t)\} \end{array} \right. .$$

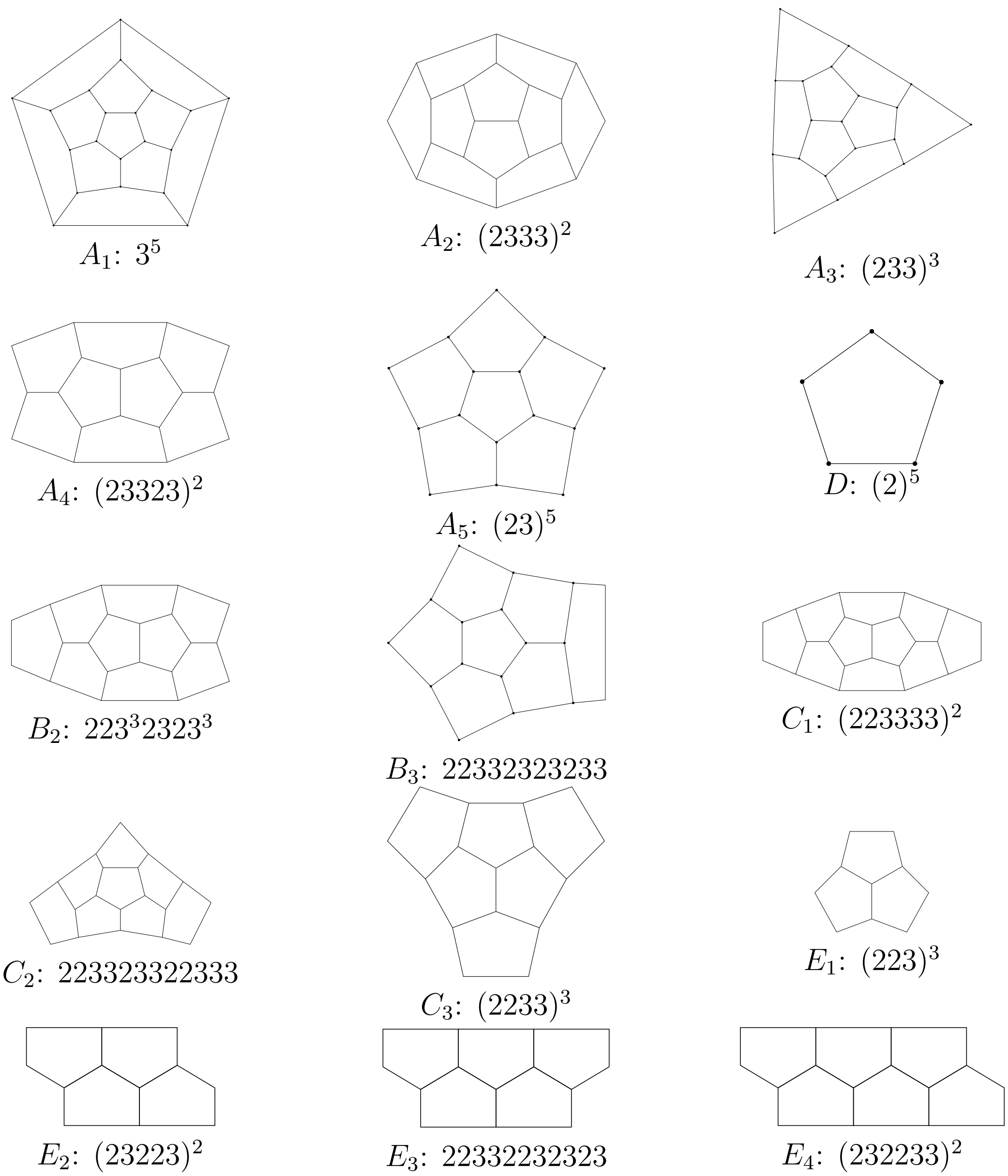


Figure 1: All finite elementary $(5,3)$-polycycles and their boundary sequences (infinite series E_n is illustrated by its first members with $1 \leq n \leq 4$)

Proof. We consider this set of 5-gonal faces as a plane graph with 5-gons and t other faces.

By counting in two different ways the number e of edges, one obtains $2e = pf_p + v_2 + v_3 = 2v_2 + 3v_3 + 3x$, which implies $pf_p - 3x = v_2 + 2v_3$.

On the other hand, Euler formula $v - e + f = 2$ implies, by writing $v = v_2 + v_3 + x$ and $f = t + f_p$, the relation $f_p - \frac{x}{2} = (2-t) + \frac{v_3}{2}$. The solution come by solving the linear system. $\square$

Theorem 1.5 *If the set of 5-gonal faces of a $(5,q)$-sphere qR_j contains at least two $(5,3)$-polycycles A_2 or at least two $(5,3)$-polycycles A_3, then there exists an infinity of $(5,q)$-spheres, which are qR_j.*

Proof. The polycycle A_2 (see Figure 1) has a central edge; by removing it, one obtains a generalized $(5,3)$-polycycle, which is the union of two $(5,3)$-polycycles E_2. This generalized $(5,3)$-polycycle has two boundaries, which have the same boundary sequence $(23^3)^2$. Hence, both sides can be filled by the same structure, which again has at least two $(5,3)$-polycycles A_2 and is again qR_j. This construction can, obviously, be repeated and one obtains an infinite series.

Take an elementary $(5,3)$-polycycle A_3 (see Figure 1) and remove its central vertex. The result is a generalized $(5,3)$-polycycle with two boundary sequences. It turns out, that those boundary sequences are identical, namely, $(3^2 2)^3$. Hence, we can fill both those boundaries by the same structure, which is again qR_j. So, one obtains a larger $(5,8)$-sphere, which is $8R_5$. This operation can, obviously, be repeated and one obtains larger $(5,q)$-spheres, which are qR_j. By creating a chain of such spheres, we get an infinity of them. $\square$

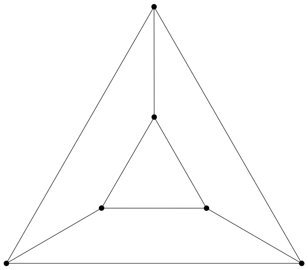
2 The list of 3-valent strictly face-regular (p,q)-maps

2.1 (p,q)-polyhedra

See in Table 3 all 3-valent strictly face-regular (p,q)-polyhedra; they were enumerated in [BrDe99] and [De02].

Nr.	p,q	v	i,j	Aut	Polyhedron
1	3,4	6	0,2	D_{3h}	$Prism_3$
2	3,5	12	0,4	D_{3d}	$Barrel_3$
3	3,6	12	0,3	T_d	Truncated Tetrahedron=$GC_{1,1}(Tetrahedron)$
4	3,6	16	0,4	T_d	4-truncated Cube=$GC_{2,0}(Tetrahedron)$
5	3,6	16	0,4	D_{2h}	twisted Nr. 4
6	3,6	28	0,5	T	4-truncated Dodec.=$G_{2,1}(Tetrahedron)$
7	3,7	20	0,4	D_{3d}	6-truncated Cube
8	3,7	36	0,5	T_h	8-truncated Dodecahedron
9	3,7	36	0,5	D_3	twisted Nr. 8
10	3,8	24	0,4	O_h	Truncated Cube
11	3,8	44	0,5	T_h	12-truncated Dodecahedron
12	3,8	44	0,5	D_3	twisted Nr. 11
13	3,9	52	0,5	T	16-truncated Dodecahedron
14	3,10	60	0,5	I_h	Truncated Dodecahedron
15	4,$\mathbf{q}$	2$\mathbf{q}$	2,0	$D_{\mathbf{q}d}$	serie $Prism_{\mathbf{q}}$, $\mathbf{q} \geq 5$
16	4,5	12	1,2	D_{2d}	decorated Cube
17	4,5	14	0,3	D_{3h}	(q-cap $Prism_3$)*
18	4,5	16	0,4	D_{4d}	(q-cap $APrism_4$)*=$Barrel_4$
19	4,6	14	2,2	D_{3h}	4-triakon Nr. 1
20	4,6	20	2,4	D_{3d}	4-triakon Nr. 2
21	4,6	20	1,3	D_3	4-halved Nr. 17
22	4,6	24	0,3	O_h	Truncated Octahedron=$GC_{1,1}(Cube)$
23	4,6	26	1,4	D_{3h}	decorated Nr. 17
24	4,6	32	0,4	O_h	$GC_{2,0}(Cube)$
25	4,6	32	0,4	D_{3h}	twisted $GC_{2,0}(Cube)$
26	4,6	56	0,5	O	$GC_{2,1}(Cube)$
27	4,7	44	1,4	T_h	4-halved Nr. 24
28	4,7	44	1,4	D_3	4-halved Nr. 25
29	4,7	44	2,5	T	4-triakon Nr. 6
30	4,7	80	0,4	O_h	(q-cap Rhombicuboctahedron)*
31	4,7	80	0,4	D_{4d}	(q-cap tw.Rhombicuboctahedron)*
32	4,8	32	2,4	T_d	4-triakon Nr. 4
33	4,8	32	2,4	D_{2h}	4-triakon Nr. 5
34	4,8	80	1,4	D_3	decorated Nr. 20
35	4,9	28	2,3	T_d	4-triakon Nr. 3
36	4,9	68	2,5	T_h	4-triakon Nr. 8
37	4,9	68	2,5	D_3	4-triakon Nr. 9
38	4,10	44	2,4	D_{3d}	4-triakon Nr. 7
39	4,11	92	2,5	T_h	4-triakon Nr. 11
40	4,11	92	2,5	D_3	4-triakon Nr. 12
41	4,12	56	2,4	O_h	4-triakon Nr. 10
42	4,13	116	2,5	T	4-triakon Nr. 13
43	4,15	140	2,5	I_h	4-triakon Nr. 14
44	5,$\mathbf{q}$	4$\mathbf{q}$	4,0	$D_{\mathbf{q}d}$	series $Barrel_{\mathbf{q}}$, $\mathbf{q} \geq 6$
45	5,6	28	3,0	T_d	(q-cap truncated Tetrahedron)*
46	5,6	32	3,2	D_{3h}	decorated Nr. 23
47	5,6	38	2,2	C_{3v}	decorated $Barrel_6$
48	5,6	44	2,3	T	5-triakon Nr. 45
49	5,6	52	1,3	T	decorated Nr. 48
50	5,6	56	2,4	T_d	5-triakon Nr. 22
51	5,6	60	0,3	I_h	Truncated Icosahedron=$GC_{1,1}(Dodecahedron)$
52	5,6	68	1,4	T_d	decorated Nr. 50
53	5,6	80	0,4	I_h	(q-cap Icosidodecahedron)*
54	5,6	80	0,4	D_{5h}	$GC_{2,0}(Dodecahedron)$
55	5,6	140	0,5	I	$GC_{2,1}(Dodecahedron)$
56	5,7	44	3,1	D_{3h}	6-halved Nr. 46
57	5,7	92	2,2	C_{3v}	decorated Nr. 47
58	5,8	56	3,0	O_h	(q-cap Truncated Cube)*
59	5,8	92	3,2	T_d	decorated Truncated Octahedron
60	5,10	140	3,0	I_h	(q-cap Truncated Dodecahedron)*

Table 3: All strictly face-regular (p,q)-polyhedra pR_i and qR_j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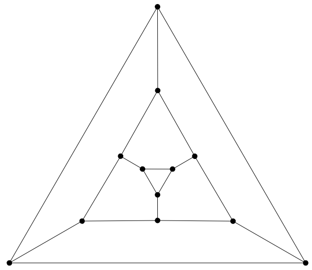
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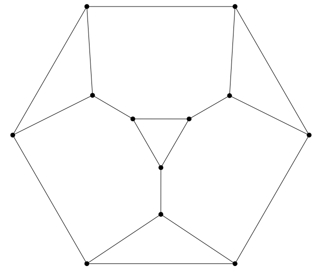
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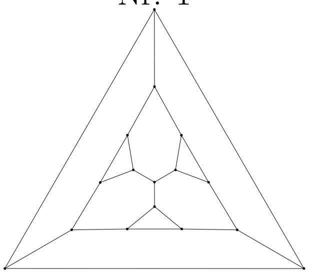
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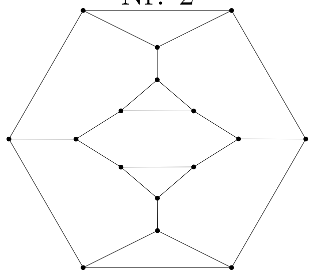
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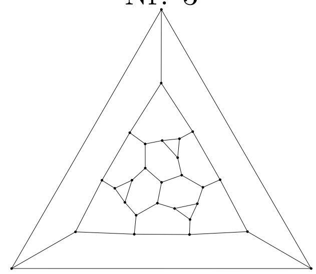
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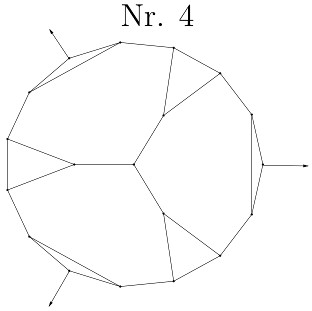
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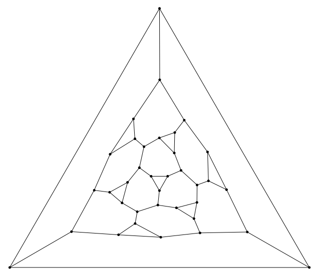
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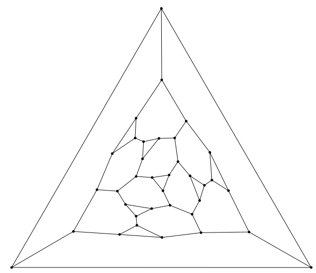
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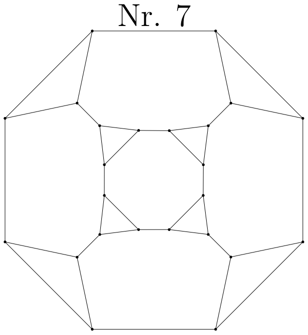
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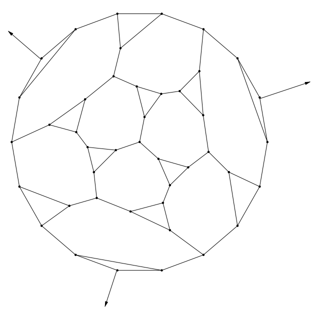
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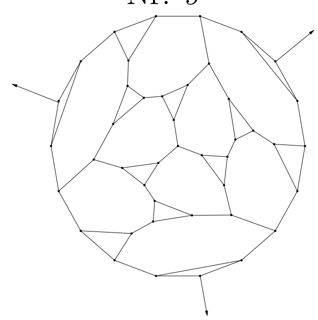
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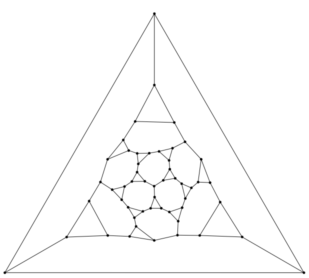
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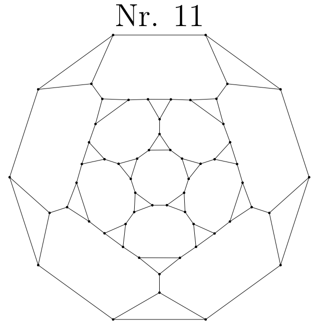
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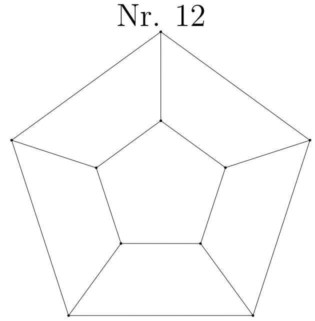
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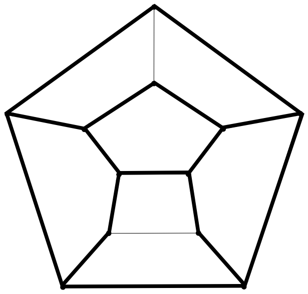

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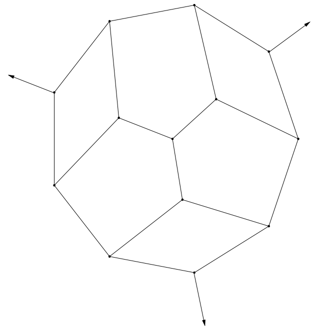

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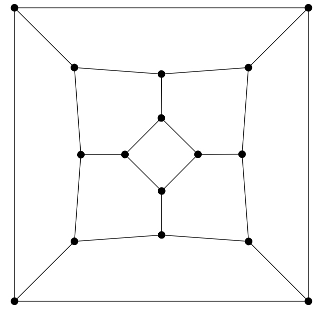

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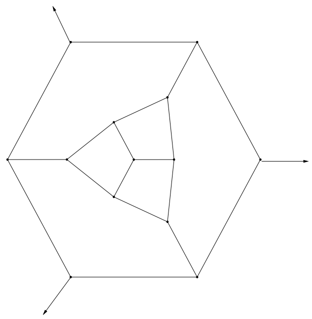

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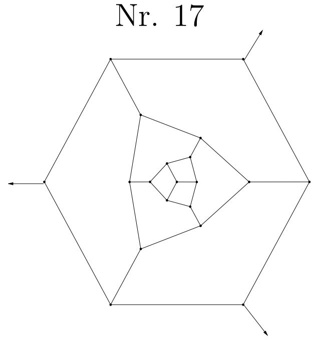

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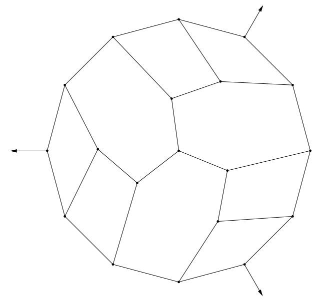

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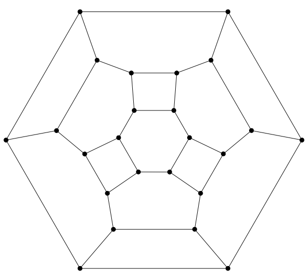

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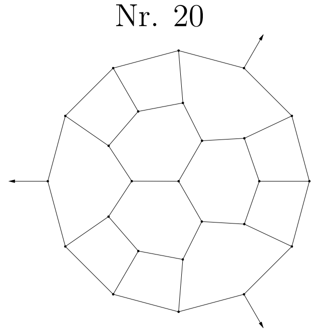

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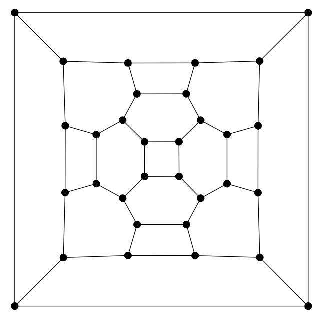

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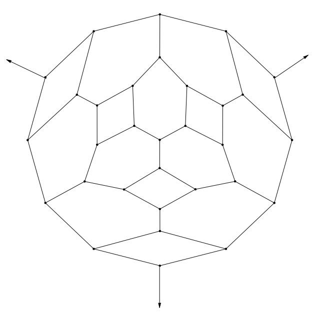

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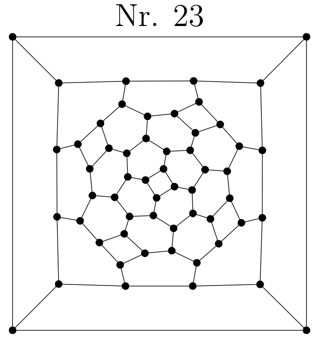

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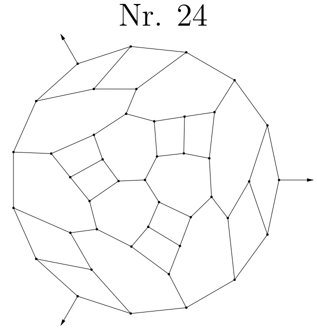

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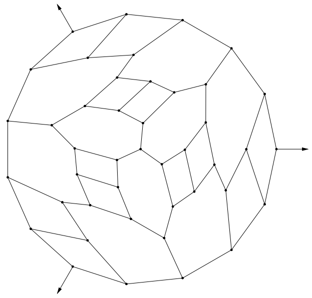

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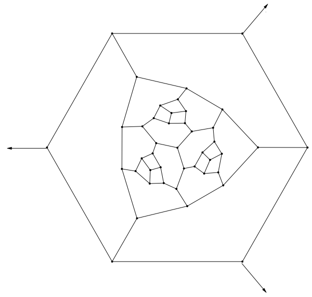

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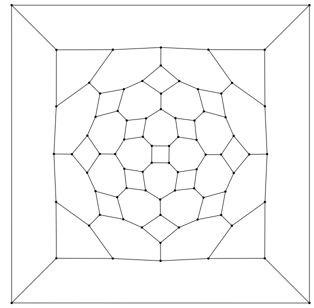

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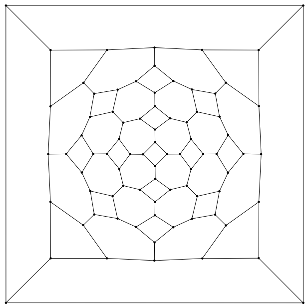

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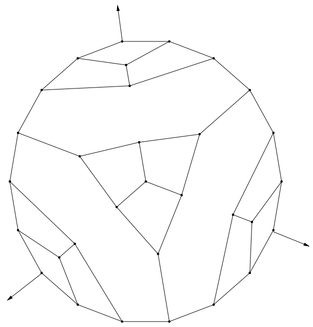

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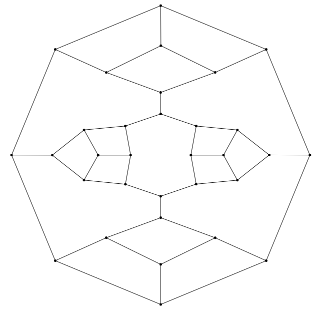

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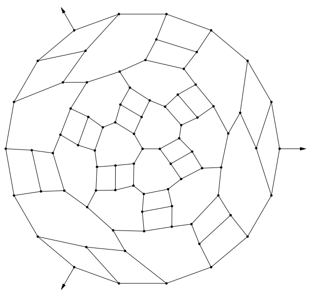

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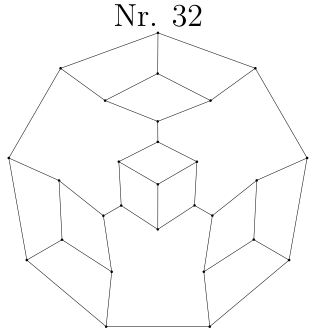

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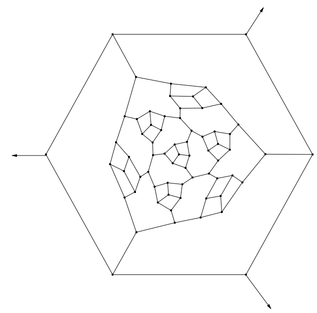

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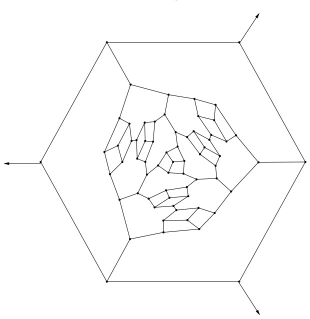

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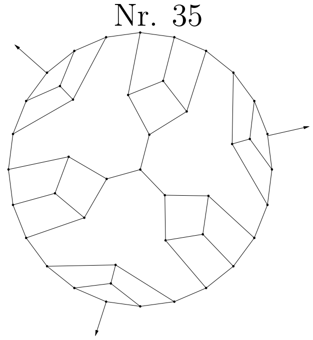

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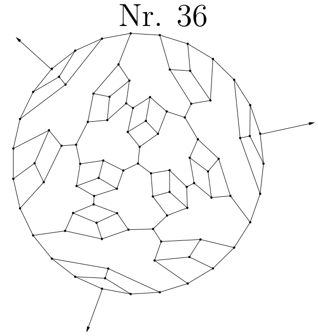

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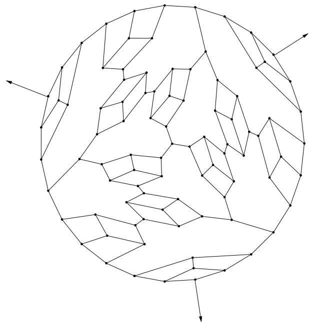

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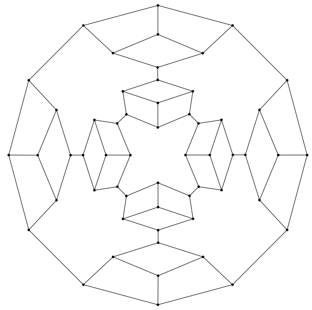

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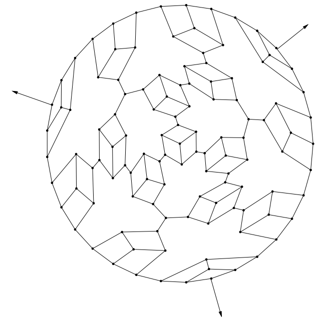

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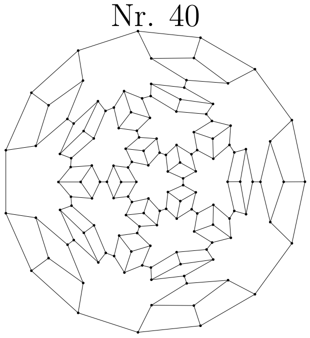

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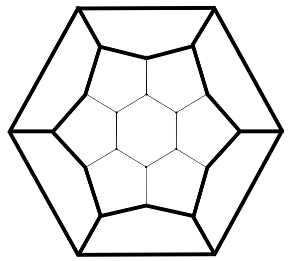

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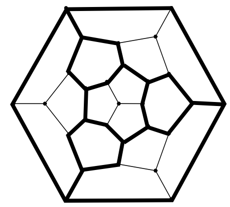

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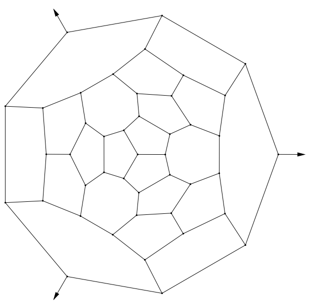

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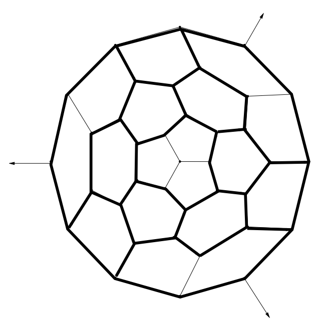

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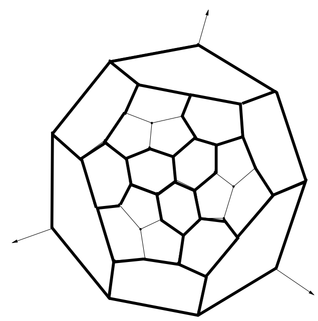

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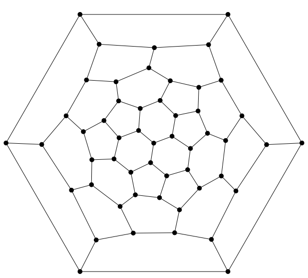

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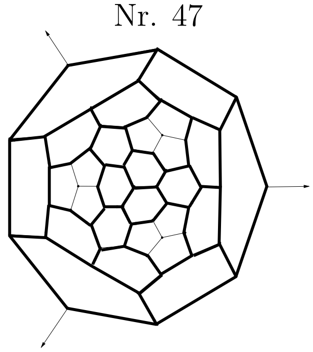

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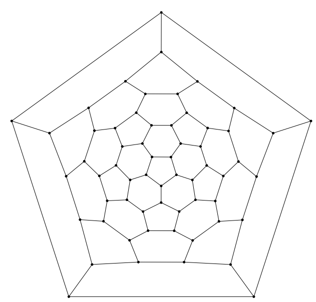

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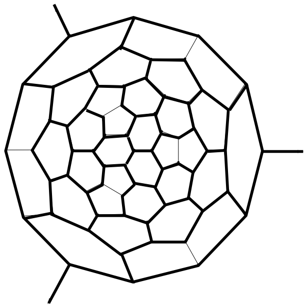

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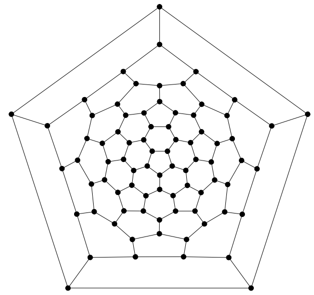

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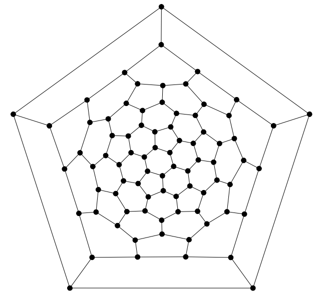
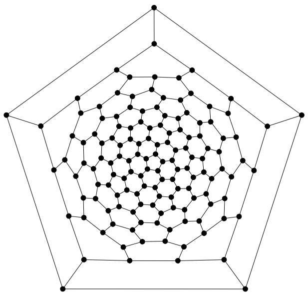
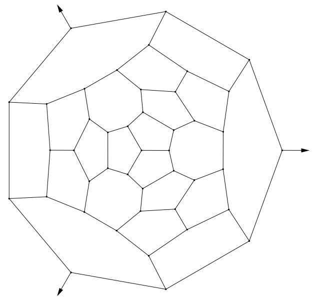


Nr. 54



Nr. 55



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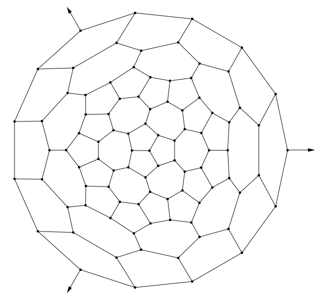

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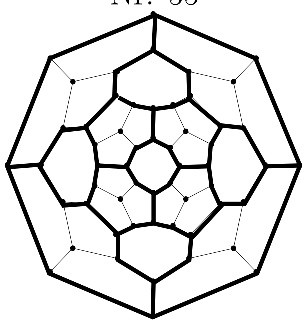

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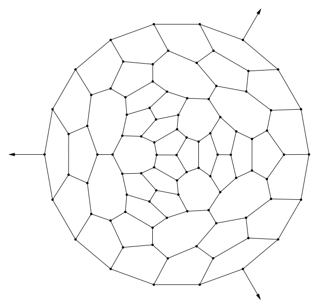
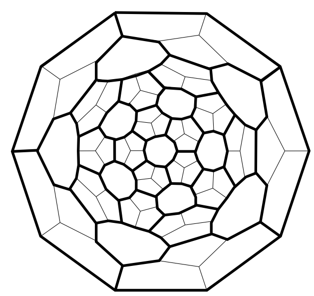


Nr. 59



Nr. 60

The list of *only* 2*-connected*, i.e., 2- but not 3-connected ones, is described below.

One can prove that a strictly face-regular (p, q)-map, which is 2-connected but not 3-connected has $p = 2$ or 3. All strictly face-regular $(3, q)$-maps, which are not $3R_0$ and different from Tetrahedron, are (only) 2-connected $(3, q)$-maps $3R_1$ and qR_j. Such maps correspond (via removal of the edge of adjacency in each pair of adjacent triangles and both vertices of this edge) to the $(2, q_1 = \frac{q+j}{2})$-map $2R_0$ and q_1R_j on the same surface (sphere or torus). Furthermore, by removal of each 2-gon (and both its vertices), one obtains the regular 3-valent map $\{p^3\}$ with $p = q_1 - (q - j) = \frac{3j-q}{2}$; now, $2 \leq p \leq 5$ for sphere and $p = 6$ for torus. So, all strictly face-regular $(3, q)$-spheres $3R_1$ and qR_j have $q = 3j - 2p$ for all $2 \leq p \leq 5$ and all $j > 3$ with $p + 1 \leq j \leq 2p$. All strictly face-regular $(3, q)$-tori $3R_1$ and qR_j have $q = 3(j - 4)$ for all $7 \leq j \leq 12$.

The problem can then be expressed in the following terms: To every edge e of $\{p^3\}$, one associate an integer parameter x_e, which is the number of 2-gons (or pair of adjacent triangles) on this edge. Then the equation are:

$$\sum_{e\in F} x_e = C = \frac{q-j}{2}.$$

for every face F of $\{p^3\}$. This is an integer problem and some systematic methods for finding all solutions exist, namely:

- It is known that the number of solutions is asymptotically equivalent to αC^w with w being the dimension of the following polytope

  $$\sum_{e\in F} x_e = 1 \text{ and } x_e \geq 0$$

  and α being proportional to its volume.

- The exact number of solutions is a polynomial, which can be, theoretically, found by computing the Hilbert basis of a polyhedral cone.

- If one wants to find the number of solutions up to isomorphy, this is still possible. One computes first the symmetry group G of the map $\{p^3\}$, which ought to be well known. Then, one compute all subgroups H of G. Then use previous method to compute all solution of the problem with respect to the symmetry group H. The number of solutions up to isomorphy is then found by using the subgroup lattice of G and solving the corresponding linear system.

2.2 (p, q)-tori

The list of strictly face regular tilings of torus was obtained in [De02] (in terms of (p, q)-planes) by the same technique as for polyhedra.

This list consists of 23 different cases. In some cases the number of possibilities is infinite, while in others there is only one possibility.

While the list of parameter sets in [De02] is correct, some tilings were missed in following cases:

(i) The number of all tilings should be infinite in the cases 1-5 and so, in cases 7-11.

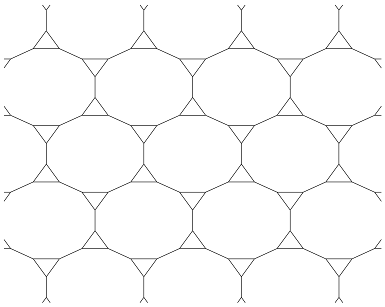
(ii) The description in cases 13, 15 being incomplete, see full description in Theorems 2.2 and 2.1.

Tilings 6, 12, 14, 20–23 are unique; they are represented on Figure 2.

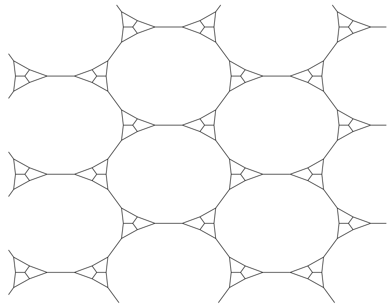
Cases 1–6, i.e., $(3, q)$-tori $3R_0$, qR_6 are obtained by taking a 3-valent tessellation of the torus by 6-gons and a set Y_q of vertices, such that every face is incident to exactly $q-6$ vertices in Y_q. By doing the truncation over such a set Y_q, one obtains a $(3, q)$-torus, which is $3R_0$ and qR_0. Of course, by complementing a set Y_q, one obtains a set Y_{18-q}, thus establishing a one-to-one mapping between the classes 1 and 5, as well as between the classes 2 and 4. Of course, since Y_6 is unique, the $(3, 12)$-plane $3R_0$, $12R_0$ is unique, but

Case	p,q	i,j	Quantity	Tilings
1	3,7	0,6	∞	$\frac{1}{6}$-truncated $\{6^3\}$
2	3,8	0,6	∞	$\frac{1}{3}$-truncated $\{6^3\}$
3	3,9	0,6	∞	$\frac{1}{2}$-truncated $\{6^3\}$
4	3,10	0,6	∞	$\frac{2}{3}$-truncated $\{6^3\}$
5	3,11	0,6	∞	$\frac{5}{6}$-truncated $\{6^3\}$
6	3,12	0,6	1	trunc.$\{6^3\} = (3.12^2)$
7	4,8	2,6	∞	4-triakon of case 1
8	4,10	2,6	∞	4-triakon of case 2
9	4,12	2,6	∞	4-triakon of case 3
10	4,14	2,6	∞	4-triakon of case 4
11	4,16	2,6	∞	4-triakon of case 5
12	4,18	2,6	1	4-triakon of case 6
13	4,7	0,5	∞	8-halved (4.8^2)
14	4,8	0,4	1	trunc.$\{4^4\} = (4.8^2)$
15	4,8	1,5	∞	4-halved case 13
16	4,10	1,4	∞	4-halved (4.8^2)
17	5,7	1,3	∞	decorated $\{6^3\}$
18	5,7	2,4	$1+\infty$	decorated $\{6^3\}$
19	5,8	2,2	∞	decorated $\{6^3\}$
20	5,8	3,4	1	decorated $\{6^3\}$
21	5,10	3,2	1	decorated $\{6^3\}$
22	5,11	3,1	1	decorated $\{6^3\}$
23	5,12	3,0	1	decorated $\{6^3\}$

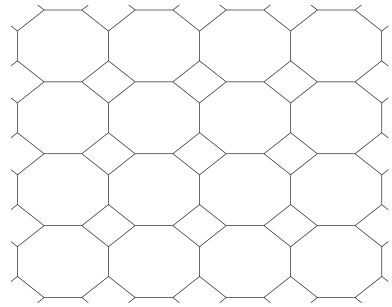
Table 4: All parameters for strictly face-regular (p,q)-planes pR_i and qR_j



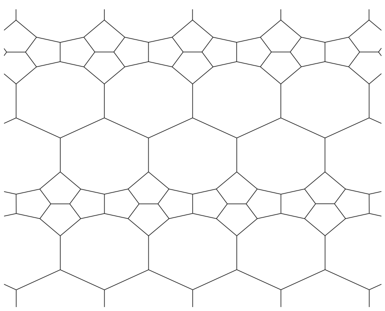
Case 6, $(6, 2, 1)$, $p6mm$



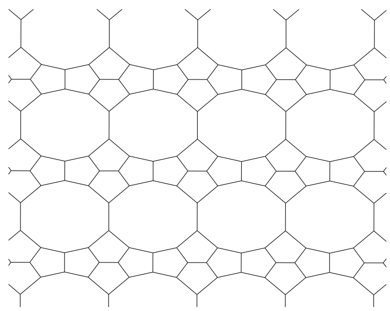
Case 12, $(14, 6, 1)$, $p6mm$



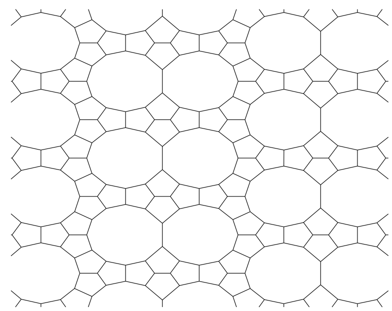
Case 14, $(4, 1, 1)$, $p4mm$



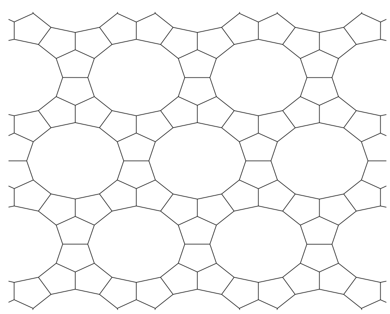
Case 20, $(12, 4, 2)$, $c2mm$



Case 21, $(10, 4, 1)$, $p2mm$



Case 22, $(24, 10, 2)$, $c2mm$



Case 23, $(14, 6, 1)$, $p6mm$

Figure 2: The seven sporadic cases of strictly face-regular tori

there is an infinity of sets Y_q for $7 \leq q \leq 11$ [1]. See Figures 3, 4 and 5 for some $(3,q)$-tori $3R_0$, qR_6 with $q = 7, 8, 9$, respectively.

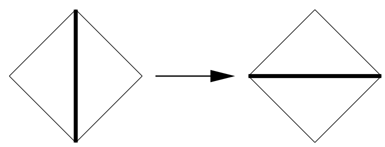
Cases 7–12, i.e., $(4,q)$-tori $4R_2$, qR_6 are obtained from cases 1–6, respectively by 4-*triakon*, i.e., replacing each triangle by triple of adjacent squares.

Case 15, i.e., $(4,8)$-tori $4R_1$, $8R_5$. Given a 6-valent tessellation of the torus by triangle, a *special perfect matching* $\mathcal{SPM}$ is a set of edges, such that it holds:

1. every vertex is contained in exactly one edge of $\mathcal{SPM}$,

2. every vertex is contained in exactly one triangle whose edge, opposite to the vertex, belongs to $\mathcal{SPM}$.

The *flip* of a map with a special perfect matching $\mathcal{SPM}$ consists of changing the edges of the special perfect matching to their opposite according to the diagram below:



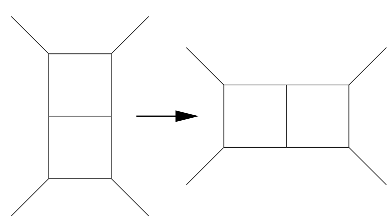
We obtain a new 6-valent tessellation of the torus by triangles with a special perfect matching.

Given a $(4,8)$-torus G, which is $4R_1$ and $8R_5$, consider the map $skel(G)$, whose vertex-set is the set of 8-gonal faces, with two 8-gonal faces being adjacent if they share an edge. Then, given a $(4,3)$-polycycle P_6 of G, its central edge, incident to two squares, can be considered as an edge of $skel(G)$.

Theorem 2.1 *(i) Given a $(4,8)$-map G, which is $4R_1$ and $8R_5$, $skel(G)$ is a 6-valent tessellation of the torus by triangles with a special perfect matching.*

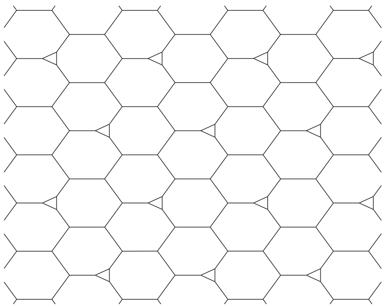
(ii) Given a 6-valent triangulation of the torus with a special perfect matching, it is $skel(G)$ of an unique $(4,8)$-torus G, which is $4R_1$ and $8R_5$.

(iii) The flipping of a map with a special perfect matching corresponds to the following transformation:

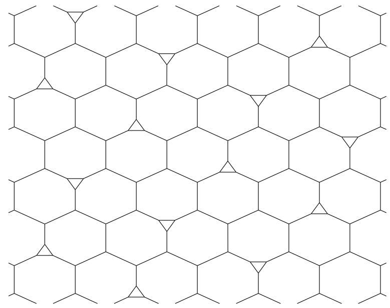


Proof. (i) Given a $(4,8)$-torus, which is $4R_1$, it is clear from the definition of $skel(G)$, that the map obtained is 6-valent and that it has a perfect matching. Clearly, the corona

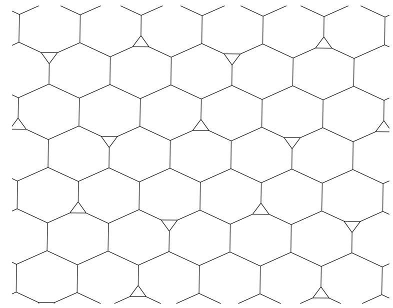
[1] A representation of a set Y_1 can be found in Ban Jelacić square in Zagreb



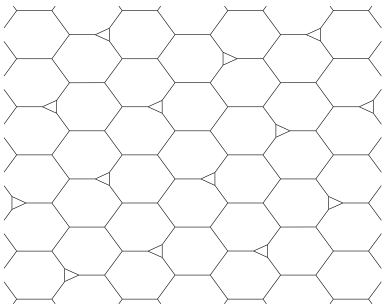
$(8, 1, 3)$, $p31m$



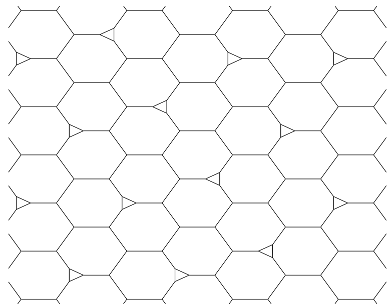
$(16, 2, 6)$, $p2mg$



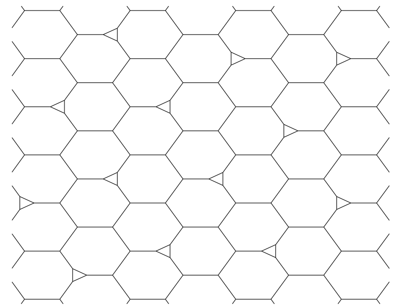
$(16, 2, 6)$, $p2$



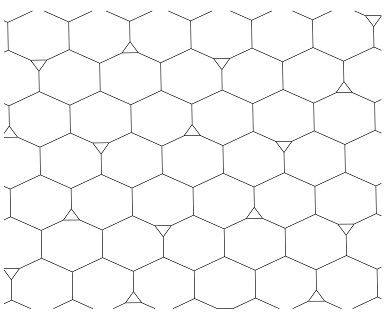
$(24, 3, 9)$, cm



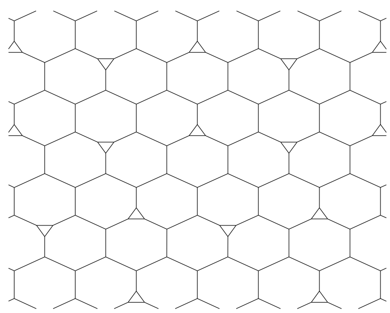
$(32, 4, 12)$, pm



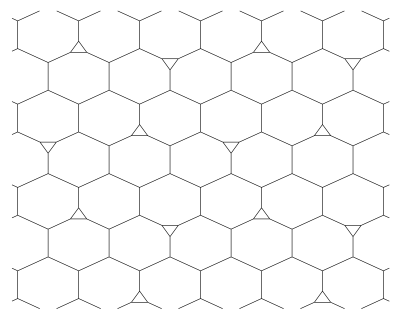
$(32, 4, 12)$, $p2gg$



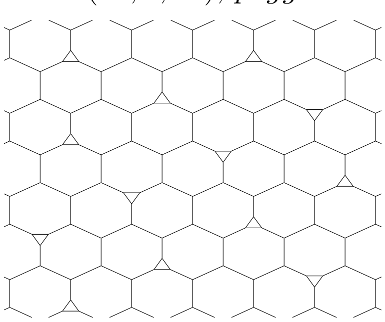
$(32, 4, 12)$, $p2gg$



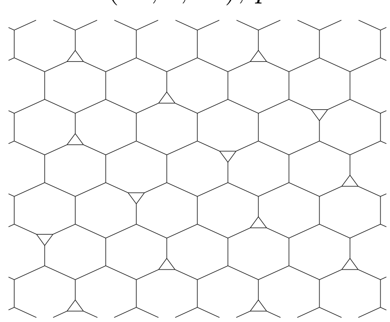
$(32, 4, 12)$, $p2$



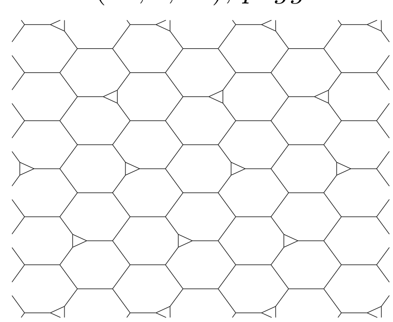
$(32, 4, 12)$, $p2gg$



$(40, 5, 15)$, cm



$(40, 5, 15)$, cm



$(40, 5, 15)$, cm

Figure 3: Some $(3, 7)$-tori, which are $3R_0$ and $7R_6$ (Case 1)

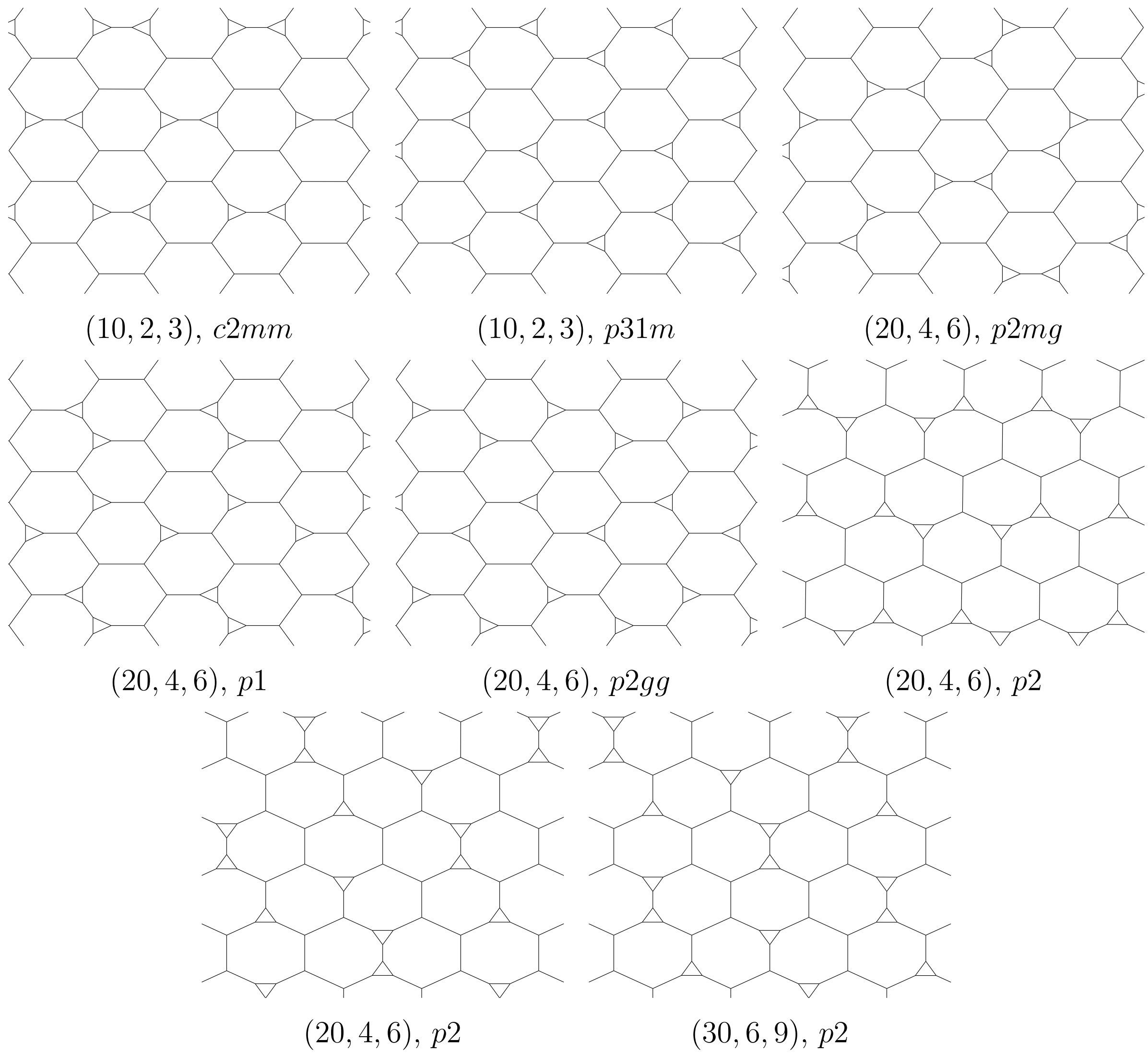
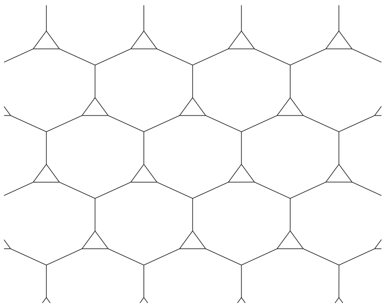
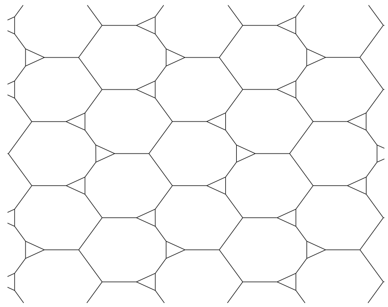


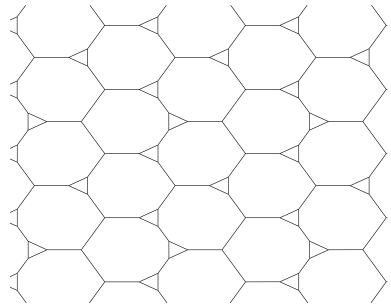
Figure 4: Some $(3, 8)$-tori, which are $3R_0$ and $8R_6$ (Case 2)



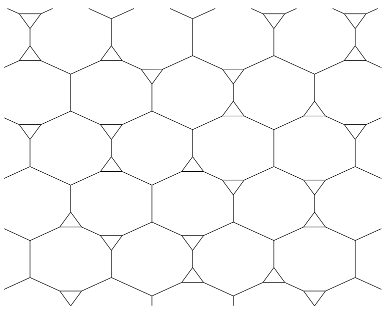
$(4,1,1)$, $p3m1$



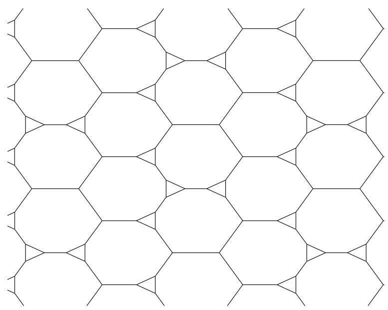
$(12,3,3)$, cm



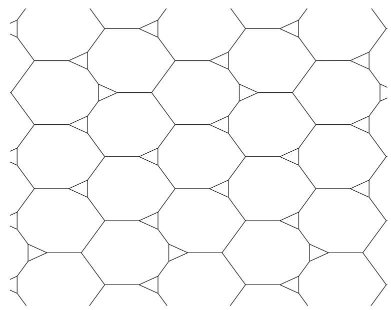
$(16,4,4)$, pm



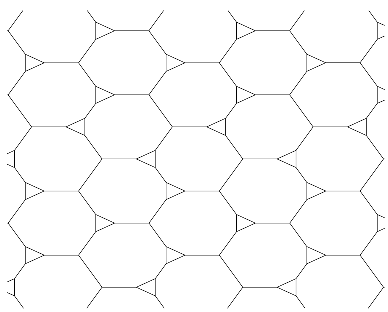
$(16,4,4)$, $p2gg$



$(16,4,4)$, $p3m1$



$(20,5,5)$, cm

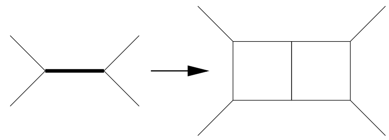


$(20,5,5)$, cm

Figure 5: Some $(3,9)$-tori, which are $3R_0$ and $9R_6$ (Case 3)

of a 8-gon is of the form 84484888 or 84488488. The pattern 848 means that the vertex corresponding to this 8-gonal face is contained into a triangle, whose opposite edge belongs to the perfect matching. Hence, the perfect matching is special.

(ii) If G is a 6-valent torus with a special perfect matching, then take the dual of it and transform every edge, arising from the perfect matching, according to the following scheme:



Clearly, the obtained torus is a $(4, 8)$-torus $4R_1$ and $8R_5$.

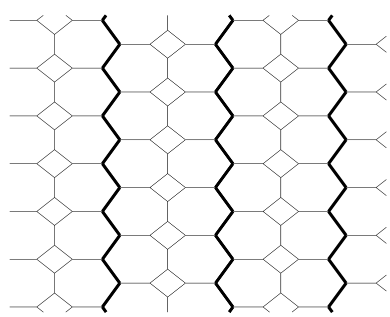
(iii) is obvious. □

See on Figure 7 some examples of such tori.

Case 13, i.e., $(4, 7)$-tori $4R_0$, $7R_5$. Given a $(4, 8)$-torus G, which is $4R_1$ and $8R_5$, the scraping of the central edge, separating the pair of adjacent squares, yields a $(4, 7)$-torus, which is $4R_0$ and $7R_5$. Clearly, the flipping of G does not change the obtained $(4, 7)$-torus.

Take a $(4, 7)$-torus G, which is $4R_0$ and $7R_5$, and call an edge *isolated* if its 4 adjacent edges are not included into 4-gonal faces. By a corona argument, i.e., scanning the possible sequences of gonalities of faces, one sees easily that every 7-gon contains at most 2 isolated edges. Hence, the possible structures for those isolated edges are triples of isolated edges, paths and circuits.

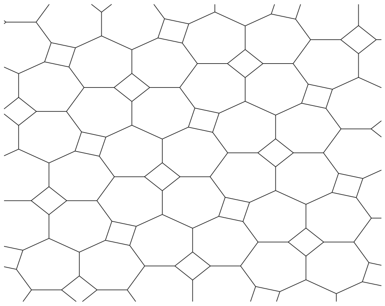
If G contains a circuit of isolated edges, then this set of isolated edges form a zigzag, i.e., every two but not three, consecutive edges are contained in a face. By using local (i.e., corona) arguments, one sees easily that the structure can be completed in a unique way. So, the corresponding $(4, 7)$-plane is:



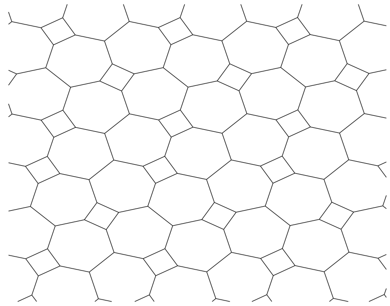
$(6, 1, 2)$, $c2mm$

which we denote by $(4, 7)_{spec}$.

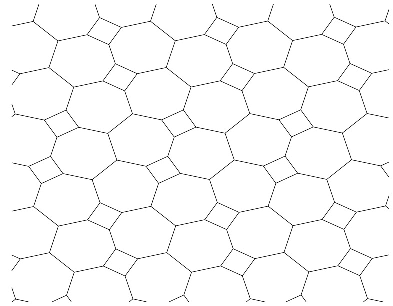
Given a torus G its *squaring* is another torus with 4 times as many edges, vertices which is obtained by replacing a fundamental domain $\mathcal{D}$ on the plane by a domain $2\mathcal{D}$ (i.e., all $2x$ with $x \in \mathcal{D}$; if the fundamental group is generated by v_1 and v_2, then another possibility for the fundamental domain is $\mathcal{D} \cup v_1 + \mathcal{D} \cup v_2 + \mathcal{D} \cup v_1 + v_2 + \mathcal{D}$ for the group generated by $2v_1$ and $2v_2$).



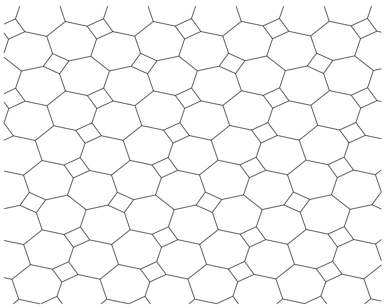
$(12, 2, 4)$, $p4gm$



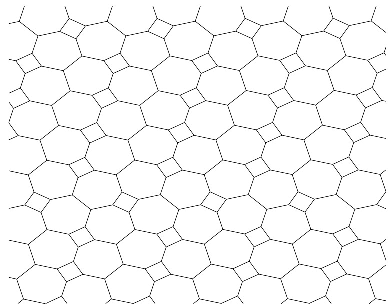
$(18, 3, 6)$, $p2$



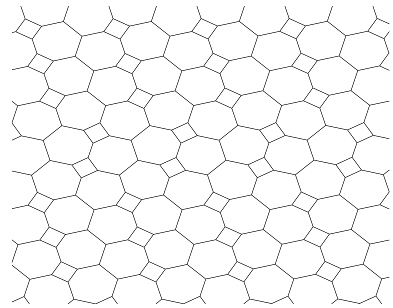
$(24, 4, 8)$, $p2gg$



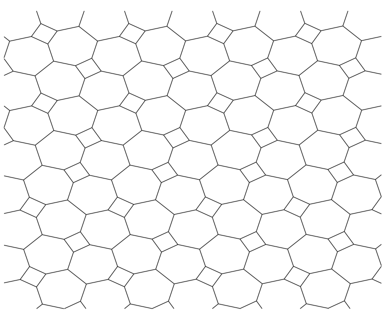
$(24, 4, 8)$, $p4gm$



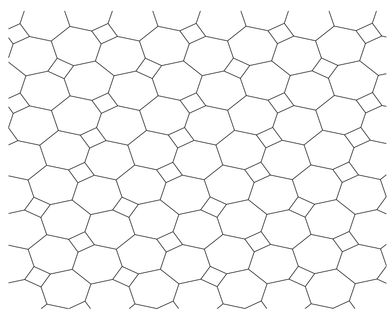
$(30, 5, 10)$, $p2$



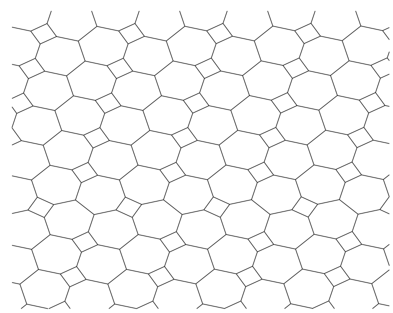
$(30, 5, 10)$, $p2$



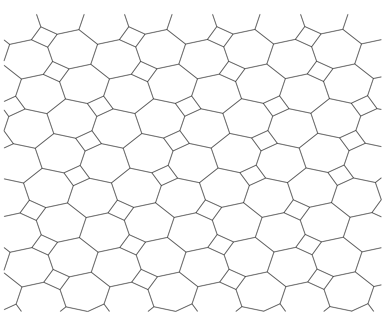
$(30, 5, 10)$, $p2$



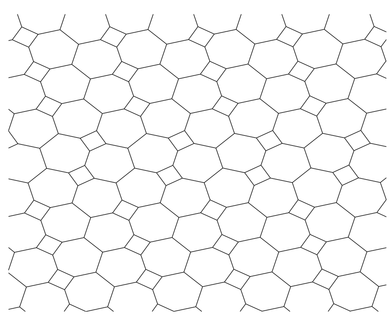
$(36, 6, 12)$, $p2$



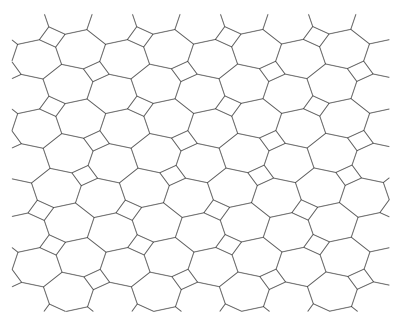
$(36, 6, 12)$, $p2$



$(36, 6, 12)$, $p2gg$



$(36, 6, 12)$, $p2$



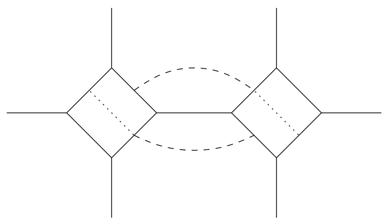
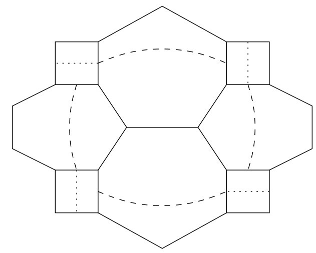
$(36, 6, 12)$, pg

Figure 6: Some $(4, 7)$-tori, which are $4R_0$ and $7R_5$ (Case 13)

Theorem 2.2 *(i) Take a $(4,7)$-torus G; then G or its squaring is obtained by the scraping of central edges of a $(4,8)$-tori G'.*

(ii) If G is different from $(4,7)_{spec}$, then G or its squaring are obtained by the scraping of edges of exactly two $(4,8)$-tori $4R_1$.

Proof. We can assume that G is distinct from $(4,7)_{spec}$. The first step consists in associating to G another 4-valent map $skel(G)$, whose vertex set consists of the set of squares. Every 7-gonal face is adjacent to two 4-gons; hence, it defines an edge of $skel(G)$ and $skel(G)$ is 4-regular. See below some representations of the local structure of G (in straight lines) and $skel(G)$ (in dashed lines):

Clearly, the faces of $skel(G)$ can be triples of edges, paths of isolated edges, or 2-gonal faces enclosing a single edge. Also, since G is different from $(4,7)_{spec}$, $skel(G)$ is connected.

An examination of all possibilities for faces of $skel(G)$ shows that, for every face, one can find two sets of cutting lines of the 4-gons realizing it (one of them is shown on the pictures above).

Now we indicate how one can find a coherent cutting set for G or its squaring. Given a path $v_0, \dots, v_m$ of adjacent vertices in $skel(G)$, if the cutting line of v_0 is chosen, then this defines uniquely the choice of cutting lines of v_i. Assume that there does not exist a coherent cutting. Then there exists a closed path $v_0, \dots, v_m = v_0$, such that the choice of a cutting line on v_0 led us in the end to a different cutting line on $v_m = v_0$. If such a path exist, then we can assume that it does not self-intersect. Assume first that the path is homologous to 0; then, if we lift it to the universal covering plane, it makes, topologically, a closed circle, say, $\mathcal{C}$.

Consider the set of interior faces of this circle. One can find an ordering $F_0, \dots, F_q$, such that, for any $0 \le i \le q$, the graphs determined by $F_0, \dots, F_i$, are enclosed by a path P_i, without self-intersections. The face F_i does admit a coherent cutting of its squares. Hence, by removing the face F_i, one obtains a path P_{i-1}, which does not admit a coherent cutting. But, in the end, we reach a contradiction since P_0 is a single face of $skel(G)$ and such faces admit coherent cutting sets.

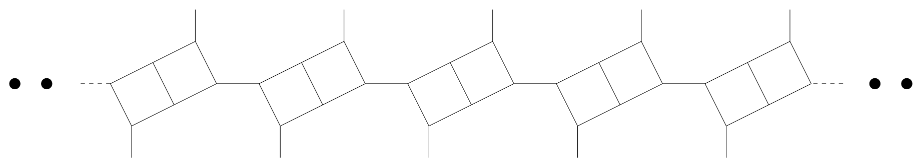
The above argument can be easily generalized to the following result: if P and P' are two closed paths of $skel(G)$, which are homologous, then P admits a coherent cutting set if and only if P' admits a coherent cutting set. The homology group $H_1(G)$ is isomorphic to $\mathbb{Z}^2$, i.e., there are two closed paths P_1 and P_2, such that any other closed path P is homologous to $n_1P_1 + n_2P_2$.

Suppose now that P_1 and P_2 admit coherent cutting set. Then G admits a cutting set. Assume that P_1 or P_2 do not admit coherent cutting set. If P_1 is of the form

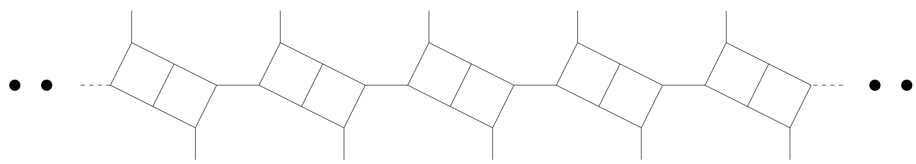
$v_0, \ldots, v_m$, then the path P_1', that correspond to P_1 in the squaring G' of G, is of the form $v_0, \ldots, v_m, \ldots, v_{2m}$. Now, $v_{2m} = v_0$ and $v_0 \neq v_m$. Since there are only two possibilities for the cutting of v_0, the cutting of v_{2m} is coherent with the cutting of v_0. The same argument applies to P_2 and so, the squaring G' does admit a coherent cutting set. The fact, that there are exactly two such cutting sets, is a consequence of the connectivity. □

See on Figure 6 some examples.

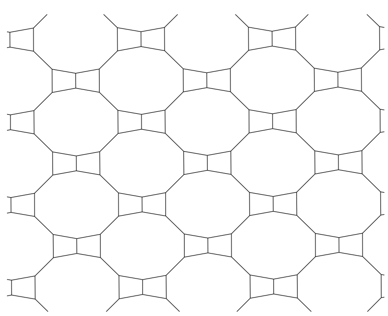
Case 16, i.e., $(4, 10)$-tori $4R_1$, $10R_4$, is described by a continuum. In fact, take two symbols u



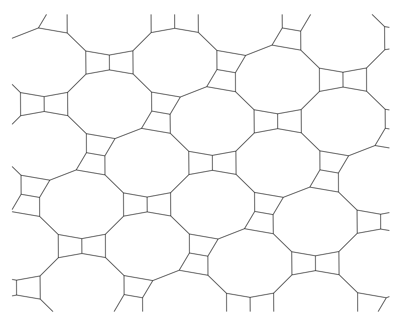
and v



Then those tori correspond to words of the form $(\alpha_0 \ldots \alpha_n)^\infty$ with α_i being equal to u or v. See below the first two examples:

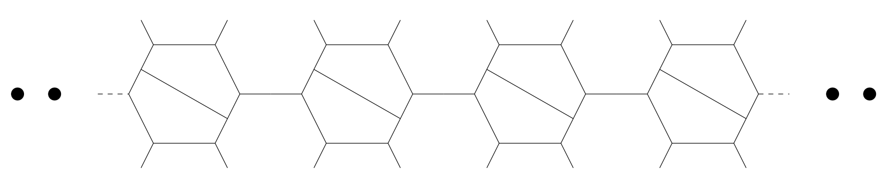


$(6, 2, 1)$, *c2mm*:
word $(u)^\infty$ or $(v)^\infty$

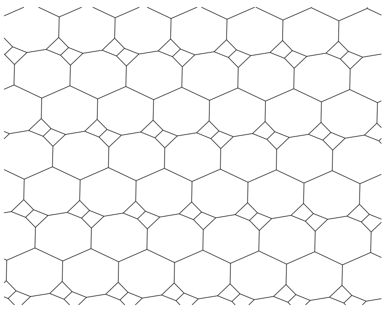


$(12, 4, 2)$, *p2mg*:
word $(uv)^\infty$

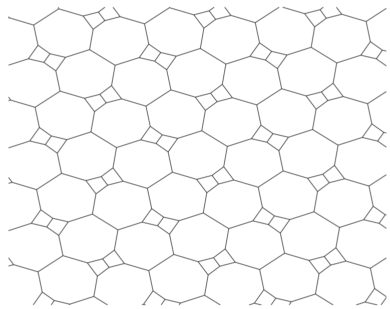
Case 17, i.e., $(5, 7)$-tori $5R_1$, $7R_3$, is described by a continuum. In fact, take the symbols u



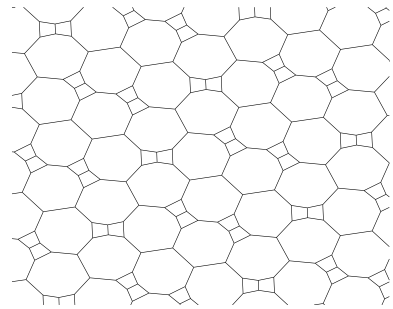
and v



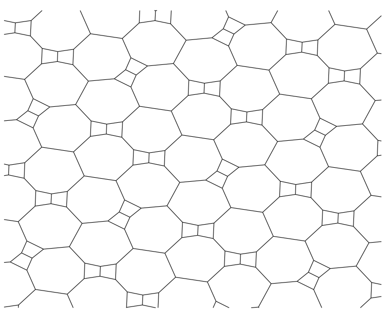
$(8, 2, 2)$, $p2$



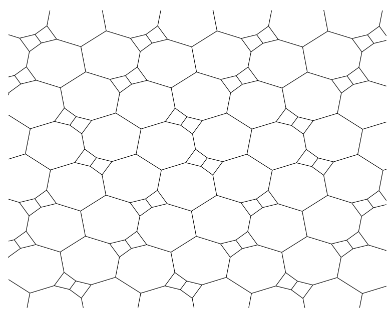
$(16, 4, 4)$, $p2gg$



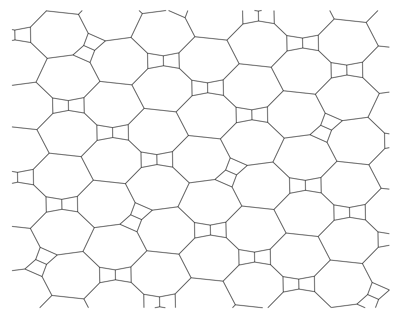
$(24, 6, 6)$, $p2$



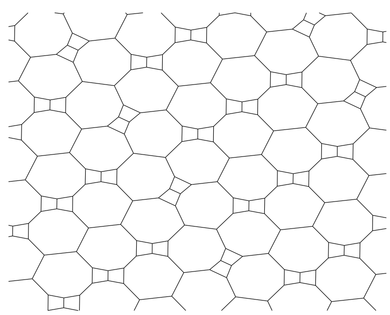
$(24, 6, 6)$, $p2$



$(32, 8, 8)$, $p2gg$

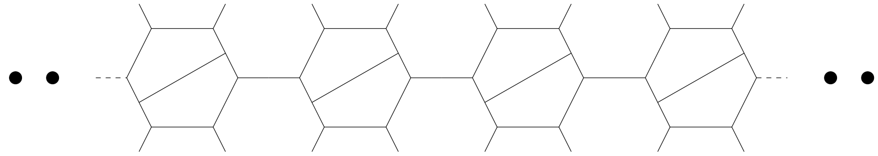


$(32, 8, 8)$, $p2$

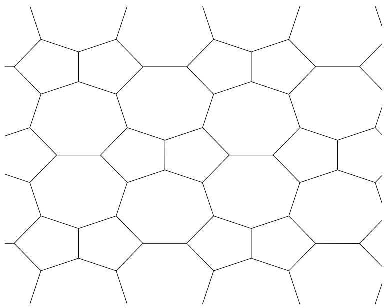


$(32, 8, 8)$, $p2$

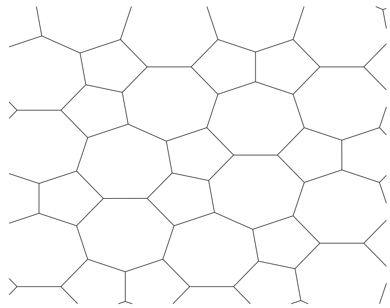
Figure 7: Some $(4, 8)$-tori, which are $4R_1$ and $8R_5$ (Case 15)



Then such tori correspond to words of the form $(\alpha_0 \ldots \alpha_n)^\infty$ with α_i being equal to u or v. See below the first two examples:

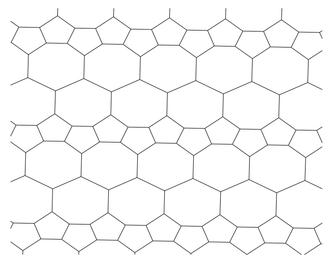


$(8, 2, 2)$, $c2mm$:
word $(u)^\infty$ or $(v)^\infty$



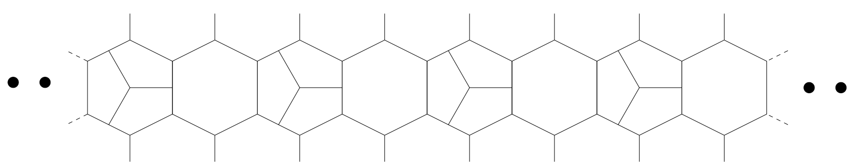
$(16, 4, 4)$, $p2gg$:
word $(uv)^\infty$

Case 18, i.e., $(5, 7)$-tori $5R_2$, $7R_4$; the 5-gons are organized either as infinite lines, or as triples of adjacent pentagons. See below the only possibility for the first case:

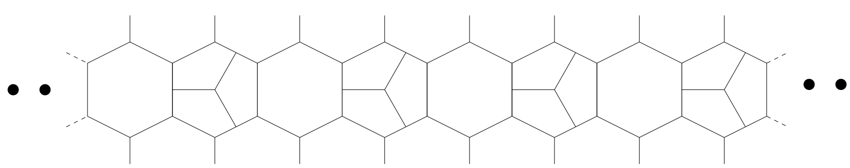


$(8, 2, 2)$, $p2mg$

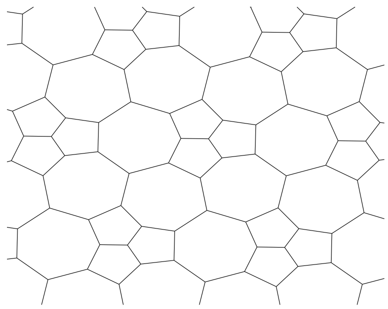
In the second case, a continuum of possibilities appear. Denote by u the following block:



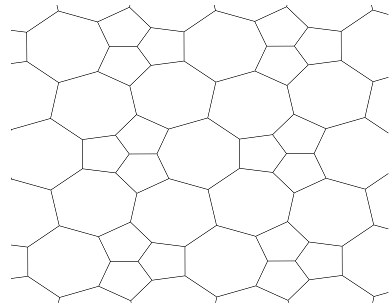
and by v, the block:



The tori correspond to words of the form $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being equal to u or v. See below the first two examples:

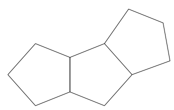


$(12, 3, 3)$, $p31m$:
word $(u)^\infty$ or $(v)^\infty$

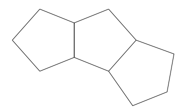


$(24, 6, 6)$, $p2mg$:
word $(uv)^\infty$

Case 19, i.e., $(5, 8)$-tori $5R_2$, $8R_2$; such a torus has its 5- and 8-gons organized in infinite lines. Given a 5-gon, its corona is of the form 58588. Up to orientation of the infinite line of 5-gons, this makes two possible choices, which we write as u or v.

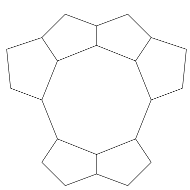


u

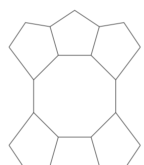


v

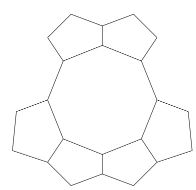
Hence, we can write the infinite word representing the torus in the form of a 5-word (p-words are infinite words describing the orientation of the p-gon in one of the infinite chain of p-gons) $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being u or v. Another viewpoint is possible by considering the infinite sequence of 8-gons. Up to orientation of the infinite lines of 8-gons, this makes three possible choices, which we write as L, S or R:



L



S



R

Hence, we can write the infinite word representing the torus in the form of a 8-word $(\beta_0 \dots \beta_n)^\infty$ with β_i being L, S or R.

Theorem 2.3 *(i) A 5-word is realizable as the sequence of a $(5, 8)$-torus $5R_2$ and $8R_2$ if and only if it is of the form*

$$(\gamma_0 \dots \gamma_n)^\infty$$

with γ_i being uv or vu.

(ii) A 8-word is realizable as the sequence of a $(5, 8)$-torus $5R_2$ and $8R_2$ if and only if it is of the form

$$\{(LS^{m_0}RS^{n_0}) \dots (LS^{m_r}RS^{n_r})\}^\infty$$

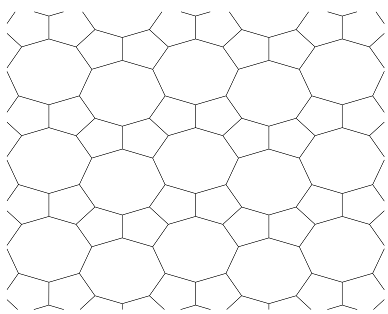
with $m_i, n_i \geq 0$. The corresponding 5-word is $\{(vu)^{m_0}(uv)^{n_0} \dots (vu)^{m_r}(uv)^{n_r}\}^\infty$.

Proof. Let us first prove (ii). It suffices to show that no two L or R can appear in a sequence, even with S between them. Suppose that the pattern LS^mL appear in a sequence. The corresponding infinite line of 8-gons has two adjacent infinite lines of 8-gons. It is easy to see that one contains the pattern $LS^{m-1}L$ and the other contains the pattern $LS^{m+1}L$. By iterating this construction, one finds an infinite line that contains LL. This is impossible, since one 8-gon contains the corona sequence 55555858.

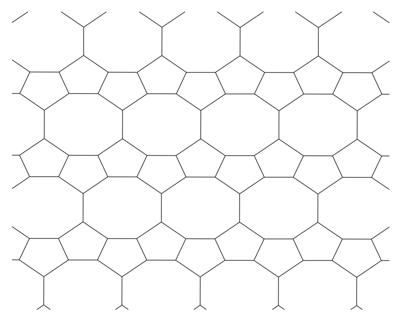
It is easy to see that the 5-word corresponding to the above 8-word is $\{(vu)^{m_0}(uv)^{n_0} \dots (vu)^{m_r}(uv)^{n_r}\}^\infty$. Clearly, any 5-word of the form $(\gamma_0 \dots \gamma_n)^\infty$ with γ_i being uv or vu can be realized in this form. □

Note that there is still another description of those tori. Consider a *step* to be two pentagons being put together; then put the steps together to form an infinite stairway (the infinite line of 5-gons), which can go up or down. The infinite word $(\gamma_0 \dots \gamma_n)^\infty$ with γ_i being uv or vu corresponds to the stairway, for example by assigning uv to "up" and vu to "down".

See below the first two examples:



$(12, 4, 2)$, $p2mg$:
5-word $(uuvv)^\infty$
8-word $(LR)^\infty$



$(6, 2, 1)$, $c2mm$:
5-word $(uv)^\infty$
8-word $(S)^\infty$

2, D_{3h} | 6, D_{3h} | 8, D_{3h} | 14, D_3

Bundle=$GC_{1,0}(Bundle)$ | $GC_{1,1}(Bundle)$ | $GC_{2,0}(Bundle)$ | $GC_{2,1}(Bundle)$

Figure 8: First examples of $(2,6)$-spheres

3 Face-regular $(p,6)$-spheres

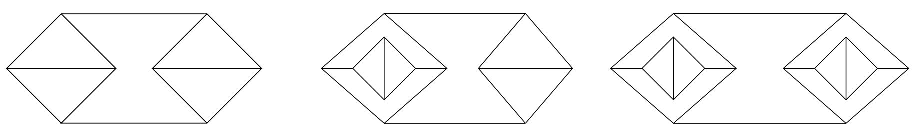
3.1 Face-regular $(2,6)$-spheres

Bundle is a 3-valent graph with 2 vertices, 3 edges and 3 2-gonal faces. See on Figure 8 the first examples of $GC_{i,j}(Bundle)$ (see notation $GC_{i,j}(G)$ in Section 1).

Any $(2,6)$-sphere, which is not Bundle, is $2R_0$. If a $(2,6)$-sphere is $6R_j$, then $(6-j)p_6 = 2p_2 = 6$. So, the cases $j=1,2$ are impossible; the case $j=0$ corresponds to the Bundle. For $j=3$, 4, 5, unique solution is $GC_{1,1}(Bundle)$, $GC_{2,0}(Bundle)$, $GC_{2,1}(Bundle)$, respectively.

3.2 Face-regular $(3,6)$-spheres

It is proved in [DeDu04] that all $(3,6)$-spheres, which are only 2-connected, belong to the infinite series, whose first members are shown below.



G_1 G_2 G_3

Hence, a $(3,6)$-sphere, which is not Tetrahedron, is either $3R_1$ and belongs to this infinite series, or is $3R_0$.

A $(3,6)$-sphere, which is $6R_j$, satisfies to $(6-j)p_6 \leq 3p_3 = 12$. Hence, it is one of the following:

1. the first member G_1 of the infinite series, which is $3R_1$ and $6R_2$,
2. the second member G_2 of the infinite series, which is $3R_1$ and $6R_4$,
3. Nr. 3, i.e., $GC_{1,1}(Tetrahedron)$, which is $3R_0$ and $6R_3$,
4. Nr. 4, i.e., $GC_{2,0}(Tetrahedron)$, or Nr. 5 (twist of Nr. 4), which are $3R_0$ and $6R_4$,
5. Nr. 6, i.e., $GC_{2,1}(Tetrahedron)$, which is $3R_0$ and $6R_5$.

3.3 Face-regular $(4, 6)$-spheres

There is an infinity of $(4, 6)$-spheres, which are $4R_0$; in fact, as the number of vertices goes to infinity, the proportion of such spheres amongst all $(4, 6)$-spheres tends to 1.

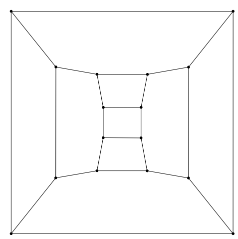
Take a $(4, 6)$-sphere, which is $4R_1$. Insert on every edge, separating two 4-gons, a digon. The resulting map is a $(2, 6)$-sphere with at most one 2-gon being adjacent to each 6-gon. Such spheres are described by the Goldberg-Coxeter construction from the Bundle. Hence, there exists a v-vertex $(4, 6)$-sphere $4R_1$ if and only if $v = 2(k^2 + kl + l^2) - 6$, and it has symmetry D_3 or D_{3h}.

Using decomposition into $(4, 3)$-polycycles, one can see easily that every $(4, 6)$-sphere, which is $4R_2$, is either $Prism_6$, or belongs to the infinite series, members of which are formed by taking two triples of 4-gons and adding t 3-rings of hexagons between them.

The Cube is unique $(4, 6)$-sphere, which is $4R_4$. There is no $(4, 6)$-sphere, which is $4R_3$.

A $(4, 6)$-sphere, which is $6R_j$, satisfies to $(6 - j)p_6 \leq 4p_4 = 24$. Hence, it has at most 56 vertices. For $j = 0$, there is only $Prism_6$. There is no such sphere for $j = 1$.

1. For $j = 2$, there is Nr. 19 and the following sphere:



2. For $j = 3$, there are Nr. 21 and Nr. 22.
3. For $j = 4$, there are Nr. 20, 23, 24, 25.
4. For $j = 5$, there is Nr. 26.

3.4 Face-regular $(5, 6)$-spheres (fullerenes)

There is an infinity of $(5, 6)$-spheres, which are $5R_0$; in fact, as the number of vertices goes to infinity, the proportion of such spheres amongst all $(5, 6)$-spheres tends to 1.

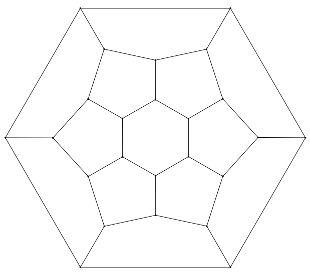
The number of $(5, 6)$-spheres $5R_1$ is also infinite.

The $(5, 6)$-spheres, which are $5R_2$, are enumerated in [DDS04]; see Proposition 3.1 below.

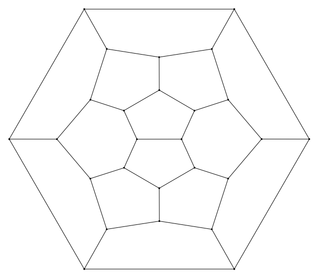
The only $(5, 6)$-spheres, which are $5R_3$, are Nrs. 45 and 46. The $Barrel_6$, i.e., the smallest Nr. 44, is unique $(5, 6)$-sphere, which is $5R_4$.

The Dodecahedron is the unique fullerene, which is $5R_5$.

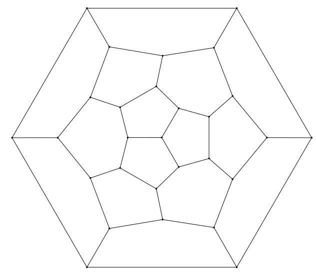
If P is a $(5, 6)$-sphere, which is $6R_j$, then one has, clearly, $(6 - j)p_6 \leq 5p_5 = 60$. For $j = 0, 1, 2, 3, 4, 5$, this yields an upper bound (on the number of vertices $v = 20 + 2p_6$) of 30, 32, 50, 60, 80, 140. The complete enumeration was done by computer for $j \leq 4$. For $j = 5$, such spheres are also $5R_0$ and so, the unique such sphere is Nr. 55.



24, D_{6d} (also $5R_4$, Nr. 44)

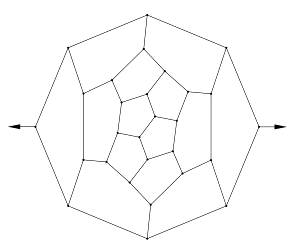


26, D_{3h}

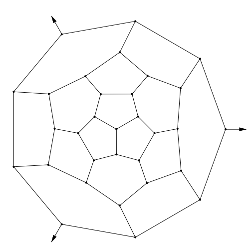


28, T_d (also $5R_3$, Nr. 45)

Figure 9: All $(5,6)$-spheres, which are $6R_0$ (all Frank-Kasper polyhedra, besides Dodecahedron)

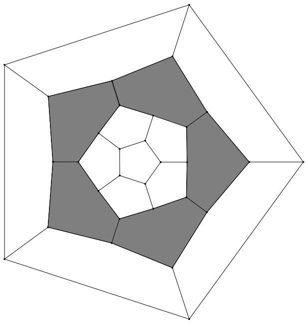


28, D_2

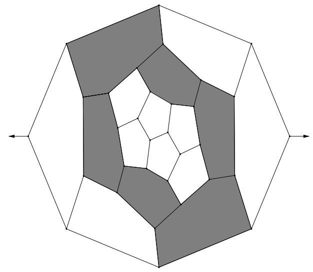


32, D_3

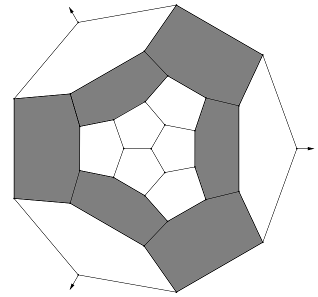
Figure 10: All $(5,6)$-spheres, which are $6R_1$



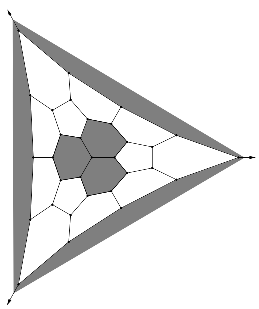
30, D_{5h}



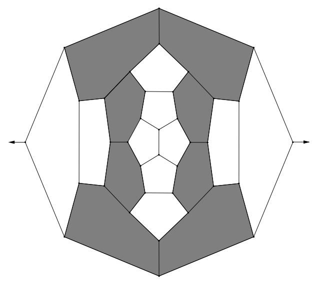
32, D_2



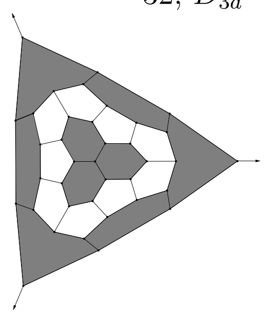
32, D_{3d}



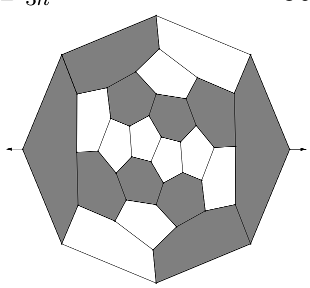
32, D_{3h}



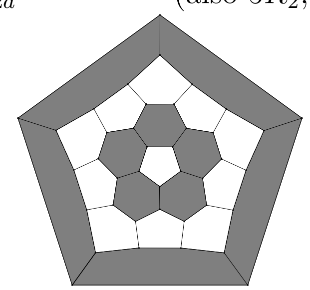
36, D_{2d}



38, C_{3v}
(also $5R_2$, Nr. 47)

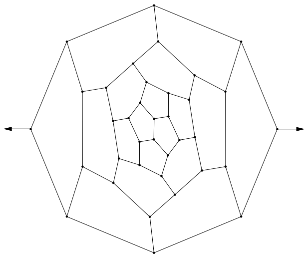


40, D_2

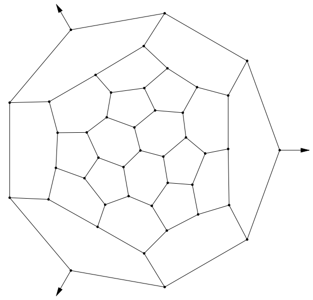


40, D_{5h}

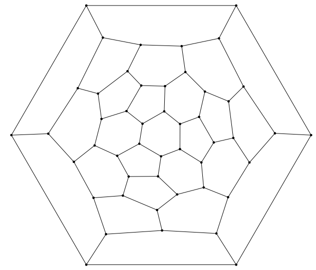
Figure 11: All $(5,6)$-spheres, which are $6R_2$



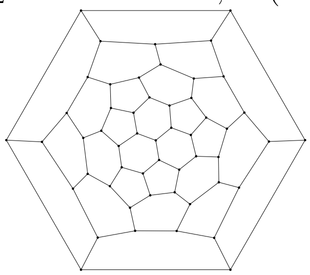
36, D_2



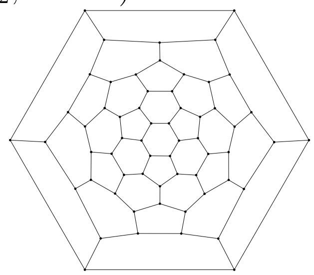
44, T (also $5R_2$, Nr. 48)



48, D_3



52, T (also $5R_1$, Nr. 49)



60, I_h (also $5R_0$, Nr. 51)

Figure 12: All $(5,6)$-spheres, which are $6R_3$

Proposition 3.1 *All $(5,6)$-spheres, which are $5R_2$, are:*

(i) The sporadic ones on Figure 14.

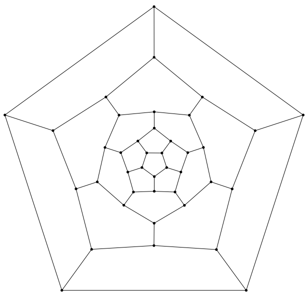
(ii) An infinite series of $(12t+24)$-vertex (for any $t>0$) and of symmetry D_{6d}, if t is even, D_{6h} if t is odd, fullerenes with 5-gons organized into two 6-rings. They are obtained from Barrel$_6$ by inserting t more 6-rings of hexagons.

(iii) An infinite series of (symmetry D_2, D_{2d}, D_{2h}, T or T_d) v-vertex (for any $v \equiv 0 \pmod 4$ with $v \geq 40$) fullerenes with pentagons organized into four 3-rings. They are obtained, by collapsing to the point of all four triangles, from any (3,6)-sphere*, such that no hexagon is adjacent to more than one triangle (such $(3,6)$-spheres are fully characterized in [GrünMo63]; see also [DeDu04] and [FoCr97]). All such spheres with ≤ 50 vertices are $F_{40}(T_d)$, $F_{44}(T)$ (also $6R_3$) and a $F_{48}(D_2)$.*

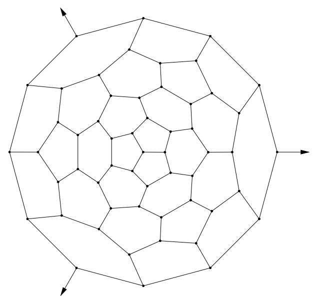
4 General (p,q)-maps

Definition 4.1 *Given a (p,q)-map (i.e., sphere or torus) G, which is qR_j, we associate to it a map (on sphere or torus, respectively) $q(G)$ formed by the q-gonal faces of G and their adjacencies. It is an induced subgraph of the dual graph of G.*

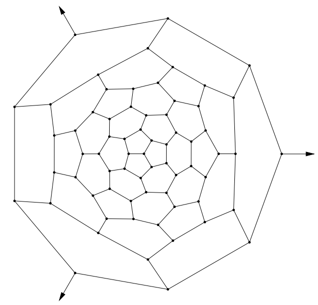
Note that the map $q(G)$ can be non-connected. All $(5,q)$-maps, which are qR_0 or qR_1, are such. All $(5,q)$-spheres qR_2 with more than one cycle also have non-connected $q(G)$. On Figure 81, a $(5,7)$-sphere $7R_3$, with 180 vertices and non-connected $7(G)$, is presented.



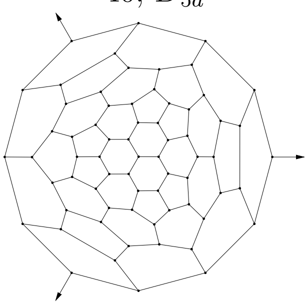
40, D_{5d}



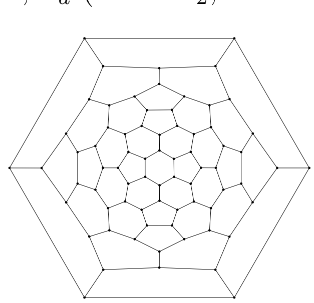
56, T_d (also $5R_2$, Nr. 50)



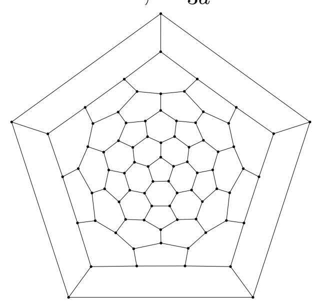
68, D_{3d}



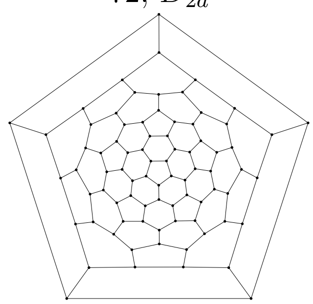
68, T_d (also $5R_1$, Nr. 52)



72, D_{2d}

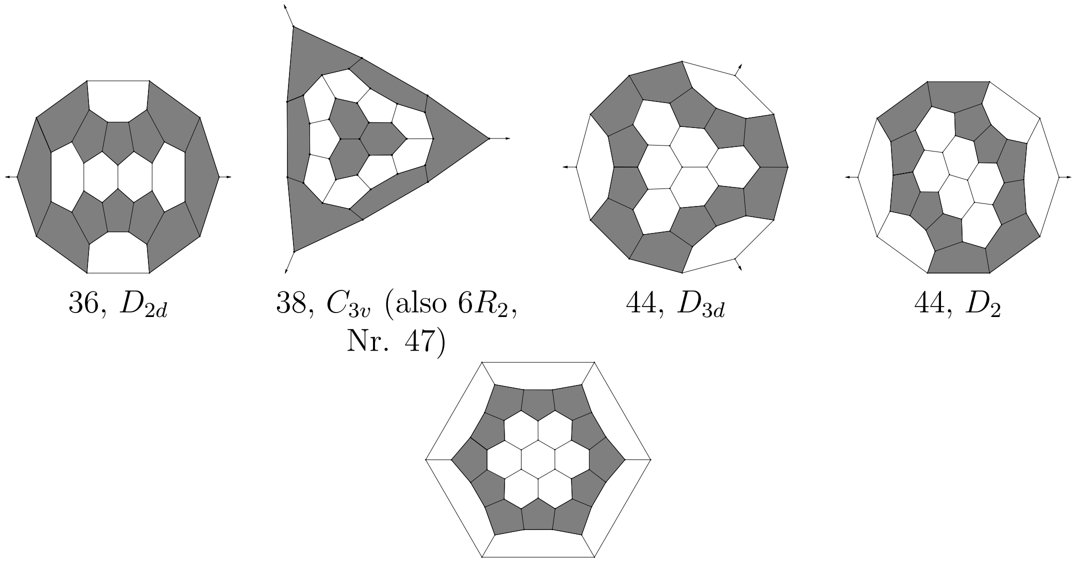


80, D_{5h} (also $5R_0$, Nr. 54)



80, I_h (also $5R_0$, Nr. 53)

Figure 13: All $(5, 6)$-spheres, which are $6R_4$



36, D_{2d} 38, C_{3v} (also $6R_2$, Nr. 47) 44, D_{3d} 44, D_2

48, D_{6d}

Figure 14: All sporadic $(5,6)$-spheres, which are $5R_2$

The infinite series given in Theorems 10.5 and 11.7 for $j=4$ and $j=5$ have non-connected $q(G)$ too.

By Theorem 1.2 the only $(4,q)$-map, which is qR_j and with $q(G)$ non-connected, is $Prism_q$.

Lemma 4.2 *If M is a sphere (respectively, a torus) without 2-gonal faces and 2-valent vertices, then there exist a vertex of degree at most 5 (respectively, 6).*

Corollary 4.3 *(i) If a (p,q)-polyhedron G is qR_j then $j \leq 5$.*

(ii) If a 3-connected (p,q)-torus G is qR_j then $j \leq 6$.

Proof. (i) The sphere $q(G)$ does not contain any 2-gon, since it would imply that G is not 3-connected. Hence, one can apply above Lemma and obtain $j \leq 5$.

(ii) in the torus case, the 3-connectedness implies that the gonalities of faces of $q(G)$ is at most 3. From above lemma, one gets $j \leq 6$. $\square$

If one removes the hypothesis of 3-connectedness, then there is no upper bound on j. For example, by Theorem 6.5, there exist a $(5,5q)$-sphere, which is $5qR_q$ for any $q \geq 2$.

Remind that f_q is the number of q-gonal faces of a map and denote by x_i with $i = 0,1,2,3$ the number of vertices contained exactly in i p-gonal faces.

Theorem 4.4 *Let P be a (p,q)-map.*

(i) If P is qR_0, i.e., q-gons are isolated, then it holds:

$$\begin{cases} (6-p)x_3 + (2(p-q)+(6-p)q)f_q = 4p & \text{on sphere,} \\ (6-p)x_3 + (2(p-q)+(6-p)q)f_q = 0 & \text{on torus.} \end{cases}$$

(ii) If P is qR_1, i.e., q-gons are organized in isolated pairs, then f_q is even and it holds:

$$\begin{cases} (6-p)x_3 + (2(p-q) + (6-p)(q-1))f_q = 4p & \text{on sphere,} \\ (6-p)x_3 + (2(p-q) + (6-p)(q-1))f_q = 0 & \text{on torus.} \end{cases}$$

(iii) If P is qR_2, i.e., q-gons are organized into disjoint simple rings, then it holds:

$$\begin{cases} (6-p)(x_0+x_3) + (4-(4-p)(4-q))f_q = 4p & \text{on sphere,} \\ (6-p)(x_0+x_3) + (4-(4-p)(4-q))f_q = 0 & \text{on torus.} \end{cases}$$

Proof. Clearly, $v = x_0 + x_1 + x_2 + x_3$.

By counting the number e of edges in two different ways, one gets:

$$2e = 3v = qf_q + pf_p \; .$$

The Euler formula for a map of genus g is $2-2g = v - e + f_q + f_p$; it can be rewritten as

$$2 - 2g = -\frac{v}{2} + f_q + f_p \; .$$

Eliminating f_p in above two equations, we obtain:

$$(6-p)v - 2(2-2g)p = 2(q-p)f_q.$$

In case (i) one has $x_0 = x_1 = 0$ and $x_2 = qf_q$. Hence, the above relation takes the form $(6-p)x_3 - 2(2-2g)p = (2(q-p) - (6-p)q)f_q$.

In case (ii) one has $x_0 = 0$, $x_1 = f_q$ and $x_2 = (q-2)f_q$. So, we get $(6-p)x_3 - 2(2-2g)p = (2(q-p) - (6-p)(q-1))f_q$.

The case (iii) is more complicated: we need to distinguish between rings (of q-gons) of length 3 (there are x_0 of them) and rings of length greater than 3. One obtains $x_1 = (q-4)(f_q - 3x_0) + (q-3)3x_0$ and $x_2 = 2(f_q - 3x_0) + 3x_0$, which yield the required formula. $\square$

For qR_0 the above formula yields on sphere:

pair (p,q)	relation	finiteness
$(4,q)$	$x_3 + 4f_q = 8$	always
$(5,q)$	$x_3 + (10-q)f_q = 20$	$q \le 9$

For qR_1 the above formula yields on sphere:

pair (p,q)	relation	finiteness
$(4,q)$	$2x_3 + 6f_q = 16$	always
$(5,q)$	$x_3 + (9-q)f_q = 20$	$q \le 8$

For qR_2 the above formula yields on sphere:

pair (p, q)	relation	finiteness
$(4, q)$	$x_3 + 4f_q = 8$	always
$(5, q)$	$x_3 + (8-q)f_q = 20$	$q \leq 7$

Clearly, in the case $p = 5$ the question is, whether or not finiteness holds for $q > 9$, 8 and 7, respectively. Infiniteness of $(5, q)$-spheres qR_2 for $q > 7$ is proved in Theorem 8.3 (see also [MaSo04] for such spheres with only one cycle of q-gons).

For $j = 0$, the finiteness holds for $q \leq 11$; infiniteness holds for $q = 12$ and, possibly, for all $q \geq 13$. See Section 6 for details.

The case $j = 1$ is more difficult. Finiteness is proved for spheres $9R_1$ in Theorem 7.2 and infiniteness is expected for $q \geq 10$. Also the enumeration of those maps is especially difficult computationally and is not done at present for $q = 8$ and $q = 9$.

Remark 4.5 *The computer enumeration methods, which were used for (p, q)-spheres, are the following:*

(i) Use the program CPF for enumerating all (p, q)-spheres with a given number of vertices. Then select, in the list of generated spheres, those, which are qR_j. If we can find an upper bound on the maximal possible number of vertices, which match the possibilities of the computer, then we are done.

(ii) Take an initial group of p- and q-gonal faces, which could be a part of a (p, q)-sphere qR_j; then consider all possible ways to extend it to a (p, q)-sphere qR_j.

(The initial group of faces is important for the success of the method. For every edge, there is a certain set of possibilities of extension; we choose the edge with the minimal number of such possibilities. Sometimes, we have an upper bound on the number of vertices and so, we know that the program will eventually finish.)

Examples of success of (ii):

- Enumerating all $(5, q)$-spheres, which are qR_0 for $8 \leq q \leq 12$.
- Enumerating all $(4, 7)$-spheres, which are $7R_4$.
- Enumerating all $(4, 8)$-spheres, which are $8R_3$.

The program, which we used in (ii), is much slower and less optimized than the program CPF. However, due to its specialized nature, i.e., the use of the geometry of the problem, it can solve problems, which are out of the range of CPF.

The program CGF (similar to CPF) allows enumeration of (p, q)-tori, but, due to the fact that the existence of a solution implies the existence of solution with arbitrarily large v, this program can only give hints of what should be the results.

Lemma 4.6 *(i) Any 3-connected $(3, q)$-map, except Tetrahedron, is $3R_0$ and (see Theorem 5.1) has $4 \leq q \leq 12$. It is strictly face-regular polyhedron (Nrs. 1, 2) for $q = 4, 5$ and strictly face-regular torus (with parameter sets of Cases 5, 6 from [De02]) for $q = 11, 12$. Unique $(3, 6)$-plane is the graphite $\{6^3\}$ and all (infinity of) $(3, 6)$-spheres are characterized (see Grünbaum-Motzkin [GrünMo63] or [DeDu04]).*

(ii) Any 3-connected $(4,q)$-map $4R_i$, except Cube, has $i \leq 2$.

Any $(5,q)$-map $5R_i$, except $Barrel_q$ and Dodecahedron, has $i \leq 3$.

(iii) Any (p,q)-map qR_j, which is not Platonic, has $j \leq q-1$ with equality only if pR_0 holds.

Proof. (i) follows from Theorem 5.1 below.

(ii) If a map contains a 4-gon adjacent to three 4-gons, then it is not 3-connected. If a $(5,q)$-map is $5R_4$, then it is easy to see that it is $Barrel_q$.

(iii) The only possibilities to complete an q-gon, surrounded by $q-1$ q-gons, to a 3-valent plane graph, are those three Platonic polyhedra or a map pR_0. □

Theorem 4.7 *A 3-connected weakly face-regular (p,q)-map, which is qR_j, satisfies to:*

(i) If it is on torus, then:

either $p=4$, $13-q \leq j \leq q-2$ (so, $q \geq 8$),

or $p=5$, $31-4q \leq j \leq q-2$ (so, $q \geq 7$).

(ii1) If it is on sphere and $q=6$, then:

either unique 16-vertex $(4,6)$-sphere, which is $6R_2$,

or one of 14 fullerenes with (v, Aut, j)=$(26, D_{3h}, 0)$, $(28, D_2, 1)$, $(32, D_3, 1)$, $(30, D_{5h}, 2)$, $(32, D_2, 2)$, $(32, D_{3d}, 2)$, $(36, D_{2d}, 2)$, $(40, D_2, 2)$, $(40, D_{5d}, 2)$, $(36, D_2, 3)$, $(48, D_3, 3)$, $(40, D_{5d}, 4)$, $(68, D_{3d}, 4)$, $(72, D_{2d}, 4)$ shown on Figures 9, 10, 11, 12 and 13.

(ii2) If it is on sphere and $q > 6$, then:

either it has parameters as in the case (i) above (on torus);

or (p,q)=$(4, 7 \leq q \leq 11$, $0 \leq j \leq 11-q$, $f_q \leq \frac{22}{12-q-j}$, or $(p,q) = (5,7)$, $0 \leq j \leq 1$, $f_7 \leq 58/(2-j)$ (2 finite cases);

or $(p,q) = (4, 7 \leq q \leq 12)$, $j = 12-q$, $e_{4-4} = 12$, or $(p,q) = (5,7)$, $j=2$, $e_{5-5} = 30$.

Proof. First, $p=3$ is impossible in our 3-connected case, because at least two p-gons should be adjacent (otherwise, we have pR_0, i.e., strict face-regularity), but Tetrahedron is only 3-valent 3-connected $(3,q)$-map, which can be obtained by completion of 2 adjacent 3-gons.

Now, $j=q$ implies $f_p = 0$. If $j = q-1$, then (by Lemma 4.6) we have pR_0 also, i.e., strict face-regularity. If (p,q)-map satisfies to qR_j, then:

$$e_{p-q} = f_q(q-j) = pf_p - 2e_{p-p} \tag{1}$$

because the right-hand side gives the maximum possible number of such edges. Furthermore, $e_{p-p} > 0$, since at least two p-gons should be adjacent (otherwise, the map is pR_0 and so, strictly face-regular). Above equality (1), together with (1_{tor}), gives $f_q(q-12+j) = 2e_{p-p} > 0$ for $p=4$ and $f_q(4q-30+j) = 2e_{p-p} > 0$ for $p=5$, which implies (i).

(ii), the case $q=6$ on sphere, follows from Theorems 3, 5, 6 of [DeGr01]. Together with (1_{sph}), the equality (1) gives $f_q(q-12+j) = 2e_{4-4} - 24$ for $p=4$ and $f_q(4q-30+j) = 2e_{5-5} - 60$ for $p=5$, implying (ii2). Two finite cases happen, when coefficients of f_q above are positive; two special cases happen, when they are zero. □

Both finite cases in (ii2) of above Theorem are enumerated below.

1. All $(5,7)$-spheres, which are $7R_2$, are enumerated in [DDS04].

2. Unique $(4,12)$-sphere, which is $12R_0$, is $Prism_{12}$.

3. There is no $(4,11)$-sphere, which is $11R_1$.

4. Unique $(4,10)$-sphere, which is $10R_2$, is given in Theorem 4.10.

5. All $(4,9)$-spheres, which are $9R_3$, are enumerated in Theorem 9.1.

6. A classification of $(4,8)$-spheres, which are $8R_4$, is given in Theorem 10.3. In particular, there is an infinity of such spheres.

7. Three examples of $(4,7)$-spheres, which are $7R_5$, are known (see Figure 75) but finiteness or infiniteness is undecided for such spheres.

Corollary 4.8 *Let $q < 8$ and a weakly face-regular 3-connected (p,q)-map is given. Then:*
(i_{tor}) If it is on torus and qR_j, then:
(p,q)=$(5,7)$ and $j = 3, 4$ or 5 (excluding Cases 17, 18 from Table 4.)
(ii_{tor}) If it is on torus and pR_i, then $q = 7$ and:
$(i;p)$=$(0;3)$, $(0;4)$, $(1;4)$, $(2;4)$ or $(2;5)$ (excluding Cases 1, 13, 17, 18 from Table 4.)
(i_{sph}) If it is on sphere and qR_j, then (besides known case $q = 6$):
either (p,q)=$(5,7)$, or (p,q)=$(4,7)$, see Remarks 14.1, 13.1, respectively, below.
(ii_{sph}) If it is on sphere and pR_i, then (besides known case $q = 6$) $q = 7$ and: (i,p)=$(0;3)$, $(0;4)$, $(1;4)$, $(2,4)$, $(2,5)$ (excluding 9 strictly face-regular polyhedra Nrs. 7, 8, 9, 27-31, 57 from Table 3).

Remark 4.9 *For a (p,q)-spheres qR_j the following evenness properties hold:*
(i) If $p = 4$ and f_q is odd, then both, q and j, are even.
(In fact, (1_{sph}) implies $q \equiv 0 \pmod 2$ if f_q is odd; moreover, (4) implies $q \equiv j \pmod 2$ if f_q is odd.)
(ii) If $p = 5$ and f_q is odd, then j is even (it is trivial for $j = 1$); it implies that $v \equiv 0 \pmod 4$ for odd j.
In fact, (4) and (1_{sph}) imply together (for $p = 5$): $f_q(30 - 4q - j) = 60 - 2e_{5-5}$; so, oddness of f_q implies evenness of j.
The only known cases of $(5,q)$-spheres with odd f_q have $(q,j;f_q) = (14,0;3)$, $(15,0;3)$, $(16,0;3)$ and $(8,2;9)$, $(8,4;3)$ (so, there is a case of odd $q = 15$).
Also, the only known cases of (p,q)-spheres qR_j with odd $f_q > 3$ are two $(5,8)$-spheres $8R_2$ with $f_q = 9$, given on Figure 82.

Theorem 4.10 *([DGr02] and [DDS04]; see also Table 1)*
(i) All $(4,q)$-spheres, which are qR_j with $j \le 2$, are Cube, $Prism_q$ and the spheres qR_2, which are classified in [DGr02] and [DDS04], namely, the infinite series $M_4(4,q)$ (see Figure 15 in [DDS04]) for $q \ge 5$ (only for $q = 5$ it is strictly face-regular Nr. 16), unique $M_3(4,6)$ (strictly face-regular Nr. 19) and unique $M_2(4,8)$ (not 3-connected).
(ii) There are no $(4,q)$-tori, which are qR_j, for $j \le 2$.

Proof. (i) The case $j = 0$ is trivial.

Assume $j = 1$ and take two q-gonal faces, which are adjacent along an edge. Those two q-gons are encircled by a circuit $F_1, \dots, F_m$ of 4-gons. Assume F_i is adjacent to both q-gons, then F_{i-1} and F_{i+1} are adjacent and this implies that F_{i-1} and F_{i+1} are both adjacent to another 4-gon, this 4-gon is adjacent to both q-gons, which is an impossibility, since those q-gons can share only one edge.

For the case $j = 2$, see [DDS04].

(ii) The above analysis, which is local, proves that there is no $(4, q)$-torus, which is qR_j for $j \leq 2$. Another, more direct proof can be obtained by using Theorem 4.4. □

Now we consider the following operation on $(5, q)$-spheres, which will be used in Sections 6 and 7.

Given a $(5, q)$-sphere P, call its *tripling*, if it exists, any partition of its q-gons in triples, connected by $K_{1,3}$, i.e., 3 incident edges, which are orthogonal to respective q-gons of the triple.

Denote by $T(P)$ the sphere, obtained from P, with fixed tripling, by replacing each $K_{1,3}$ by the elementary $(5, 3)$-polycycle E_1 (i.e., 3-ring of pentagons). Clearly, we have the following.

Lemma 4.11 *Let P be a $(5, q)$-sphere (with f_q q-gons; so, $v = 2(q-5)f_q + 20$ vertices), which is qR_j and admits a tripling. Then the corresponding $T(P)$ is an $(5, q+1)$-sphere (with f_q $(q+1)$-gons; so, $v' = 2(q-4)f_q + 20$ vertices), which is $(q+1)R_j$.*

Examples of triplings:

1. For qR_0:

   (a) With $f_q = 6$:

T(str. face-reg. Nr. 58, $8R_0$, O_h)	is	68, $9R_0$, D_{3d}
T(68, $9R_0$, D_{3d})	is	80, $10R_0$, T_d
T(80, $10R_0$, D_{2h})	is	92, $11R_0$ D_{3d}
T(92, $11R_0$ D_{3d})	is	104, FK_0, $12R_0$, O_h
T(116, $13R_0$, D_3)	is	128, $14R_0$, D_3
	is	128, $14R_0$, C_2
T(128, $14R_0$, D_3)	is	140, $15R_0$, C_2
T(128, $14R_0$, C_2)	is	140, $15R_0$, C_2 (same as before)
T(128, $14R_0$, C_2)	is	140, $15R_0$, C_2 (another one)
T(140, $15R_0$, C_2)	is	152, $16R_0$, C_2
T(140, $15R_0$, C_2)	is	152, $16R_0$, C_2 (same as before)
	is	152, $16R_0$, D_3

   (b) With $f_q = 12$:

T(str.face-reg. 140, $10R_0$, I_h)	is	164, $11R_0$, T
T(164, $11R_0$, T)	is	188, $12R_0$, D_3
T(188, $12R_0$, D_3)	is	212, $13R_0$, D_3
T(212, $13R_0$, D_3)	is	236, $14R_0$, T
T(236, $14R_0$, T)	is	260, $15R_0$, I_h

(c) With $f_q = 3$:

T(74, $14R_0$, D_{3h})	is	80, $15R_0$, C_{3v}
T(80, $15R_0$, C_{3v})	is	86, $16R_0$, D_{3h}

2. For qR_1 and $f_q = 6$:

T(weak.face-reg. fullerene 32, $6R_1$, D_3)	is	44, $7R_1$, D_3
T(str.face-reg. Nr. 56, 44, $7R_1$, D_{3h})	is	56, $8R_1$, D_{3h}
T(56, $8R_1$, D_3)	is	68, $9R_1$, D_3
T(68, $9R_1$, D_3)	is	80, $10R_1$, D_3
T(68, $9R_1$, D_2)	is	80, $10R_1$, C_2
T(80, $10R_1$, D_3)	is	92, $11R_1$, D_3
T(80, $10R_1$, C_2)	is	92, $11R_1$, C_2

5 Maps pR_i

We start with the following general result.

Theorem 5.1 *Any 3-connected weakly face-regular (p,q)-map, which is pR_i, is in one of following cases (besides strictly face-regular ones Nrs. 1,2)*

(i) $q = 6$, the map is on sphere only and:

either (i1) pR_0 holds with $p = 3$ (i.e., all 3_n, $n > 4$), $p = 4$ or $p = 5$ (so-called IPR fullerenes);

or (i2) pR_1 holds with $p = 4$ or $p = 5$.

or (i3) pR_2 holds with $p = 4$ (one known infinity, see Proposition 2 in [DeGr01]) or $p = 5$ (four maps $M_{12}(6,5)$ and two known infinities, see (i) and (iv) of Theorem 4 of [DeGr01]: with twelve 5-gons organized in four 3-rings or two 6-rings, respectively).

(ii) $q > 6$, map is on sphere or on torus, and:

either (ii1) pR_0 holds with $(p,q) = (3, 7 \le q \le 10)$ or $(p,q) = (4,7)$ (Theorem 5.2 gives infinity of such spheres);

or (ii2) pR_1 holds with $p = 4$, $7 \le q \le 9$ (Theorems 5.3 and 5.4 give infinity of such $(4,q)$-spheres for $q = 7$ and 8, respectively; Proposition 5.5 gives lower bound $v \ge 108$ for the number of vertices in the case $q = 9$);

or (ii3) pR_2 holds with $p = 4$ or $p = 5$ (i.e., p-gons organized into isolated rings, 3-rings permitted);

or (ii4) pR_3 holds with $p = 5$ (see Section 5 for details).

Proof. (i) If $q \leq 6$, then (p,q)-map cannot be on torus by Euler's formula. The remainder of the case (i) follows from [DeGr01] with deleting of strictly face-regular (p,q)-maps, found in [BrDe99], [De02].

(ii) From now, let $q > 6$. Then $p = 3, 4$ or 5 by Euler formula for (p,q)-maps:

$$f_p(6-p) - f_q(q-6) = 12 \quad \text{(on sphere)}, \tag{1_{sph}}$$

$$f_p(6-p) - f_q(q-6) = 0 \quad \text{(on torus)}. \tag{1_{tor}}$$

The relation (1_{sph}) (or (1_{tor})), together with Euler relation $v - 3v/2 + (f_p + f_q) = 2$ or 0, implies:

$$v = 2(q-p)\frac{f_q}{6-p} + \frac{4p}{6-p} \quad \text{(on sphere)}, \tag{2_{sph}}$$

$$v = 2(q-p)\frac{f_q}{6-p} \quad \text{(on torus)}. \tag{2_{tor}}$$

If our (p,q)-map is pR_i, then $i \leq p-2$ by Lemma 4.6. So, besides the case $5R_3$, the only possible cases of pR_i are 9 cases $p = 3, 4, 5$ with $i = 0, 1, 2$. If pR_0 holds, then the number e_{p-q} satisfies to:

$$e_{p-q} = pf_p < f_q\lfloor q/2 \rfloor, \tag{3_0}$$

because the corona of any q-gon can not have p-gons on 2 neighboring edges. If pR_1 holds, then:

$$e_{p-q} = (p-1)f_p < f_q 2\lfloor q/3 \rfloor, \tag{3_1}$$

because the corona of any q-gon can not have p-gons on 3 consecutive edges of the q-gon. (So, the maximum of p-gons in the corona is obtained by $\lfloor q/3 \rfloor$ disjoint pairs plus, for $q \equiv 2 \pmod 3$, one more p-gon.) In both above inequalities we excluded the case of equality, because it corresponds to strict face-regularity.

From now on, we consider the property pR_0 or pR_1 with $p = 3, 4, 5$ for the cases of sphere or torus; clearly, pR_1 is possible only for $p = 4$ or 5, since a pair of adjacent triangles implies non-3-connectedness.

For $p = 3$, (1_{sph}) and (3_0) imply $12 = 3f_3 - f_q(q-6) < f_q(\lfloor q/2 \rfloor - (q-6))$; so, $7 \leq q \leq 10$ and $f_q > 6$ for $q = 7, 8$, while $f_q > 12$ for $q = 9, 10$. But (1_{sph}) implies that $f_q \equiv 0 \pmod 3$ for $p = 3$ and $q = 7, 8, 10$. So, $f_q \geq 9, 9, 13, 15$ for $q = 7, 8, 9, 10$, respectively; it implies, by (3_{sph}), $v \geq 28, 34, 56, 74$, respectively. We should exclude 4 polyhedra (Nrs. 8, 9, 11, 12 from Table 3), which are strictly face-regular $(3,q)$-polyhedra $3R_0$. For $p = 3$, (1_{tor}) and (3_0) imply again $7 \leq q \leq 10$ (the values $q = 11, 12$ are possible only in strictly face-regular case) and we should exclude strictly face-regular tilings with parameter-sets of Cases $1-4$ from Table 4.

For $p = 4$, (1_{sph}) and (3_0) imply $12 = 2f_4 - f_q(q-6) < f_q(\frac{1}{2}\lfloor q/2 \rfloor - (q-6))$; so, $q = 7$ and $f_7 > 24$. But (1_{sph}) implies that f_7 is even for (p,q)=$(4,7)$; so, $f_7 \geq 26$, implying, by (3_{sph}), $v \geq 86$. For $p = 4$, (1_{tor}) and (3_0) imply also $q = 7$; we should exclude the strictly face-regular tiling of Case 13.

For $p=4$, (1_{sph}) and (3_1) imply $12 = 2f_4 - f_b(q-6) < \frac{4}{3} f_q \lfloor q/3 \rfloor - f_q(q-6)$; so, $7 \le q \le 9$ and $f_q \ge 8, 10, 13$ for $q = 7, 8, 9$, respectively. In fact, $f_9 \ge 14$, since (1_{sph}) implies that f_q is even for (p,q)=(4, 9). We get, by (2_{sph}), $v \ge (q-4)f_q + 8$, i.e., $v \ge 32, 48, 78$ for $q = 7, 8, 9$, respectively. We should exclude three strictly face-regular polyhedra Nrs. 27, 28, 34). For $p = 4$, (1_{tor}) and (3_1) imply again $7 \le q \le 9$. We should exclude strictly face-regular tiling of Case 15.

For $p = 5$, (1_{sph}) and (3_0) imply $12 = f_5 - f_q(q-6) < \frac{1}{5} f_q \lfloor q/2 \rfloor - f_q(q-6)$, which is impossible for $q > 6$. Such $(5,q)$-map $5R_0$ is impossible on torus also.

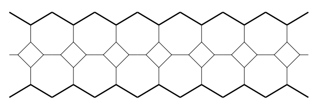
For $p = 5$, (1_{sph}) and (3_1) imply $12 = f_5 - f_q(q-6) < \frac{1}{2} f_q \lfloor q/3 \rfloor - f_q(q-6)$, which is impossible for $q > 6$. Such $(5,q)$-map $5R_1$ is impossible on torus also (a $(5,7)$-map, which gives equality in above inequality, exists, but it is strictly face-regular). □

5.1 Maps $4R_0$

Theorem 5.2 *There exist at least two infinite series of $(4,7)$-spheres, which are $4R_0$. They have $140+42i$ vertices (see two examples on Figure 15). For i even, they are distinct, one is of symmetry D_{7h}, the other of symmetry D_{7d}. For i odd, they are isomorphic and of symmetry D_7.*

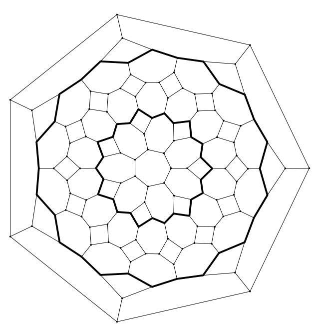
Proof. From the drawing on Figure 15, it is clear that such spheres exist. Now we will show the existence of an infinity of them.

One has the following band structure of 4- and 7-gons:



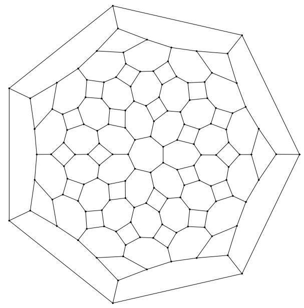
The left and right hand side of this band can be closed, in order to obtain a structure with 14 4-gons.

This structure can be inserted along one of the cutting lines, indicated below, and one gets a $(4,7)$-sphere, which is again $4R_0$ and has 42 more vertices.

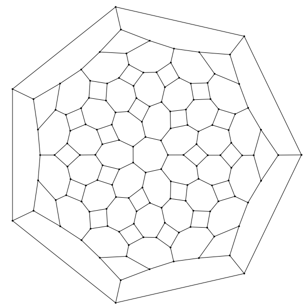


Obviously, the above operation can be repeated as often as one wants. □

Note that, instead of taking a graph with 7-fold symmetry, we could take, as original graph, the strictly face-regular one with 4-fold symmetry; see below their drawing with the cutting lines:

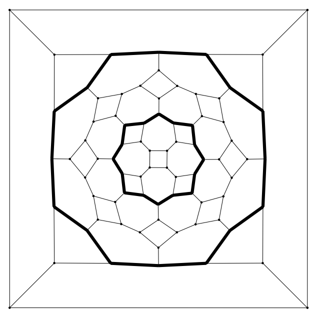


140, D_{7d}

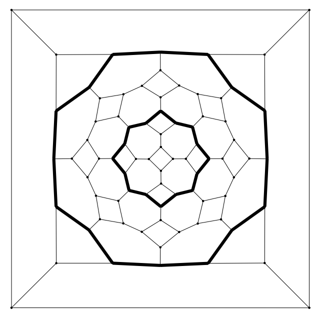


140, D_{7h}

Figure 15: First examples of two infinite series of $(4,7)$-spheres, which are $4R_0$ (see Theorem 5.2)



80, O_h

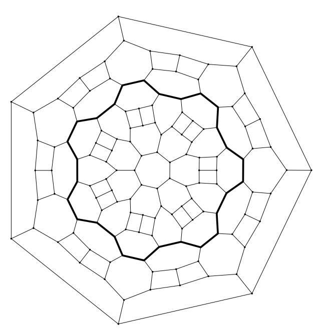


80, D_{4d}

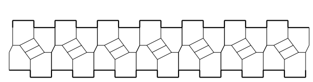
5.2 Maps $4R_1$

Theorem 5.3 *There exist an infinity of* $(4,7)$*-spheres, which are* $4R_1$*.*

Proof. The proof consists of building the following initial example with 140 vertices and symmetry D_{7d}:



We will cut along the over-lined path and insert the following structure, which consists of 7 units.



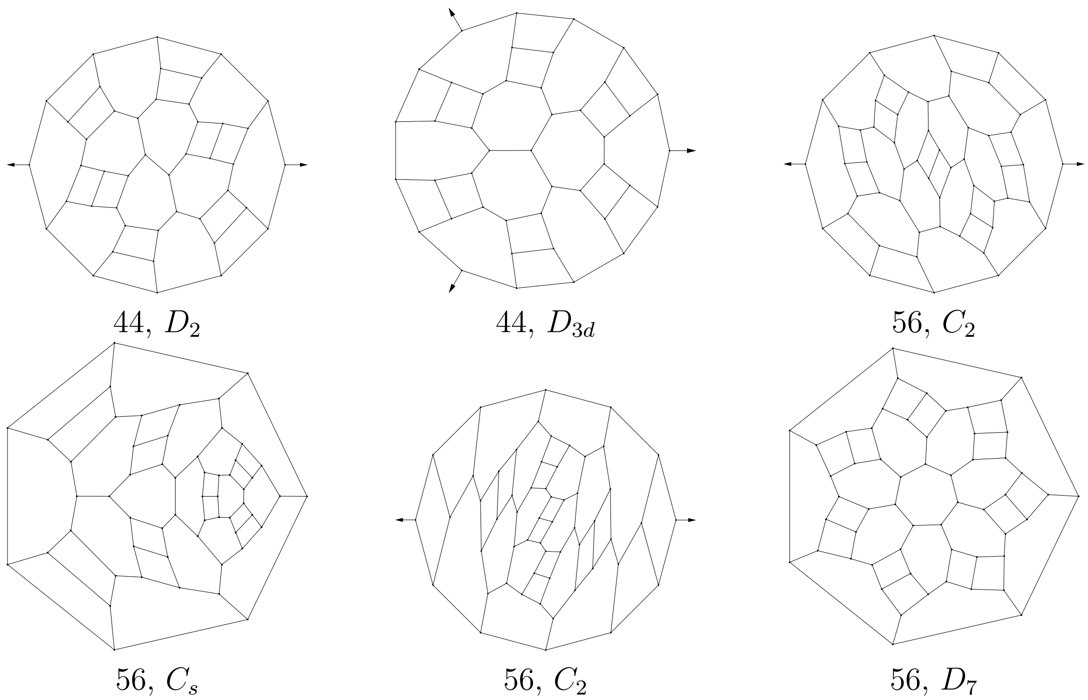
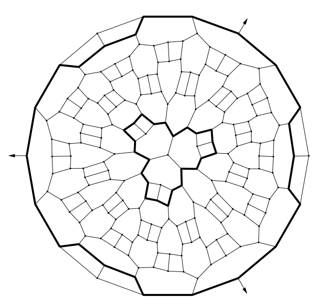


Figure 16: All weakly face-regular $(4,7)$-spheres, which are $4R_1$ and have at most 62 vertices

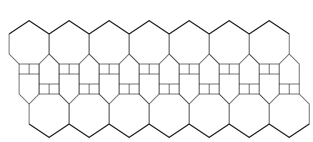
Obviously, the obtained graph is again a $(4,7)$-sphere, which is $4R_1$, and the construction can be repeated. $\square$

Theorem 5.4 *There is an infinity of* $(4,8)$*-spheres, which are* $4R_1$.

Proof. We construct the following example of a $(4,8)$-sphere, which is $4R_1$ (it has 224 vertices):



The idea is to cut along the line, depicted in above drawing, and insert inside the following band structure:



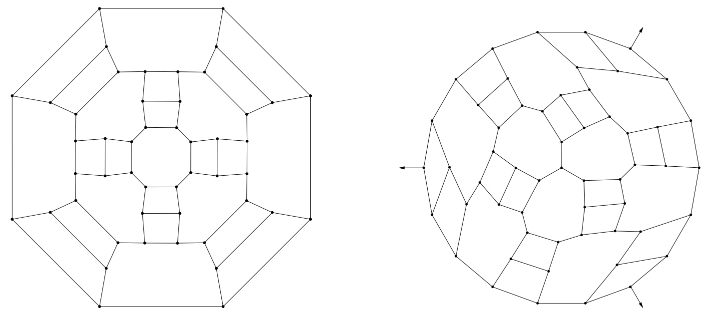
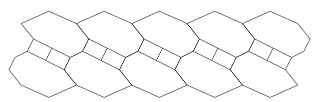


Figure 17: All weakly face-regular $(4,8)$-spheres, which are $4R_1$ and have at most 56 vertices

Obviously, the above operation can be repeated. □

Remark that there are many ways of constructing a $(4,8)$-sphere, using variants of the above construction:

- The first obvious way is to shift the position of the caps above; it gives two more graphs.
- Some other periodic structures are possible, like the following:

  and this leaves more freedom for the construction itself. This reflects the fact that there is an infinity (a continuum) of strictly face-regular $(4,8)$-tilings, which are $4R_1$ and $8R_5$.
- Another interesting fact is that the cutting lines, where one inserts the periodic structure, are *zigzags* (see [DeDu04]). The zigzags form a double covering of the set of edges of any 3-valent plane graph. It turns out that, for $(4,8)$-spheres, those zigzags have no self-intersections. Hence, any $(4,8)$-sphere, which is $4R_1$ and has a zigzag, which does not cut a pair of adjacent 4-gons, can be extended to a larger $(4,8)$-sphere, which is still $4R_1$, along this zigzag.

There exist $(4,9)$-tori, which are $4R_1$, as exemplified by Figure 18 with $(v, f_4, f_9) = (20, 6, 4)$.

Theorem 5.5 *(i) Any* $(4,9)$*-sphere, which is* $4R_1$*, has at least* 108 *vertices.*
(ii) Any $(4,10)$*-torus, which is* $4R_1$ *is also* $10R_4$*.*

Proof. Such maps satisfy to Euler formula $6\chi = 2f_4 - (q-6)f_q$ with χ being equal to 2 for spheres and 0 for tori.

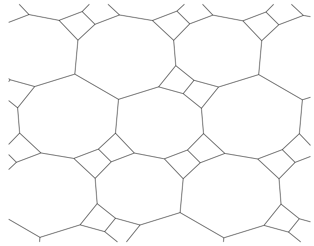


Figure 18: A $(4,9)$-torus, which is $4R_1$

One has $f_4 = 3\chi + \frac{q-6}{2} f_q$.

Every pair of adjacent 4-gons creates pair of adjacent 4-gons, which corresponds to subsequence 9449 in the corona sequence of 9-gons, and pair of isolated 4-gons, which corresponds to the subsequence 949 in corona sequence of 9-gons. So, there are f_4 patterns 9449 and f_4 patterns 949 in the set of corona sequence of 9-gonal faces of the map considered. A packing argument yields the inequality:

$$2f_4 + 3f_4 \leq qf_q,$$

which simplifies to $30\chi \leq (30-3q)f_q$. For $(4,9)$-spheres $4R_1$, this yields $f_9 \geq 20$ and hence, the lower bound. For $(4,10)$-torus $4R_1$, this implies $0 \leq 0$. But this means that in the corona sequence of 10-gons, the pattern 99 does not appear. So, their corona sequence is of the form $\alpha_1 \dots \alpha_r$ with α_u being equal to 94 or 944. Denote by y_2 and y_3 the number of α_i being equal to 94, 944, respectively, for a given q-gonal face F. One has, clearly, $2y_2 + 3y_3 = 10$, whose solutions are (y_2, y_3)=$(5,0)$ or $(2,2)$. So, for all solutions we get $y_3 \leq y_2$. But, on average over all 10-gonal faces, one has $y_3 = y_2$; this is possible only if $y_2 = y_3$ for every 10-gonal faces. So, the map is $10R_4$. □

Take a $(4,9)$-map G, which is $4R_1$, and map every pair of adjacent 4-gons to a single edge. The obtained *reduced map*, denoted by $Red(G)$, is still 3-valent. The number of sides of its faces is between 5 and 9. The set of pairs of adjacent 4-gons of G yields an edge-set $\mathcal{ES}(G)$ in $Red(G)$, which satisfies the following properties:

1. It is a *matching*, i.e., no vertex belong to two edges of $\mathcal{ES}(G)$.

2. For every face F of G, denote by $h(F)$ the number of edges in $\mathcal{ES}(G)$, which are incident to a vertex of F (those edges contain either an edge, or just a vertex of G). One has the equation $h(F) + l(F) = 9$ with $l(F)$ being the gonality of F.

If a set of faces of a graph satisfies the above conditions, then we call it a *special* $(4,9)$-*matching*.

Theorem 5.6 *(i) If G is a v-vertex $(4,9)$-sphere, which is $4R_1$, and $G' = Red(G)$ is its associated graph, then $v = \frac{5}{2}v' + 18$.*

(ii) The smallest $(4,9)$-sphere, which is $4R_1$, is the one with 128 vertices depicted on Figure 19.

(iii) There is an infinity of $(4,9)$-spheres, which are $4R_1$.

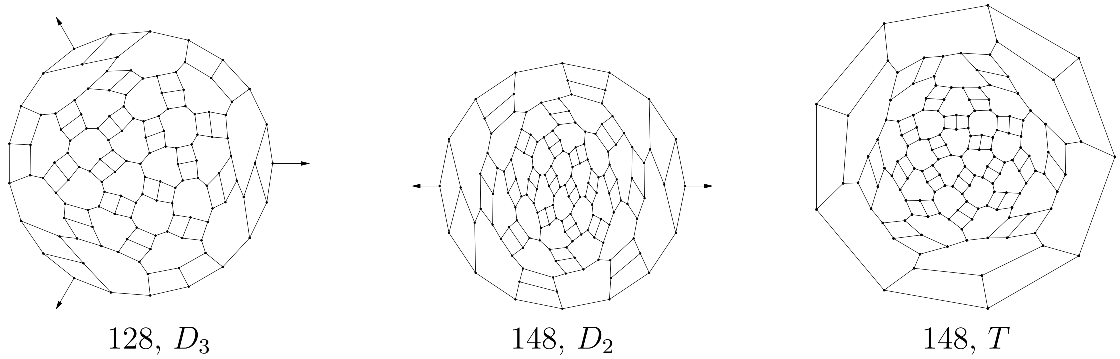
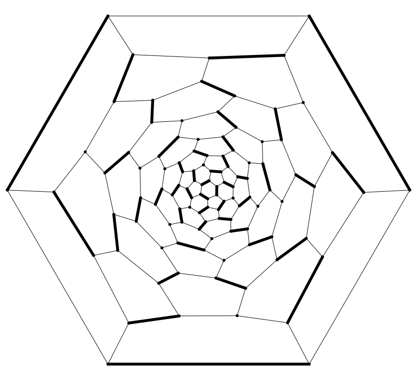


Figure 19: Some $(4,9)$-spheres, which are $4R_1$

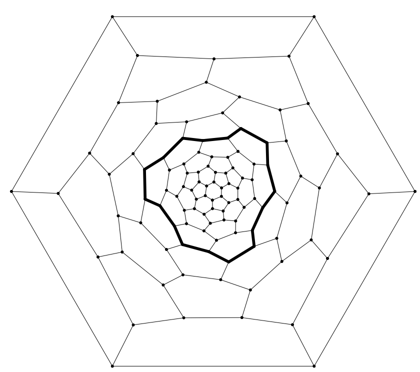
Proof. (i) Take a $(4,9)$-sphere, which is $4R_1$ and has v-vertices. One has the equations $3v = 4f_4 + 9f_9$ and $2f_4 - 3f_9 = 12$. The number of pairs of 4-gons is $n_p = \frac{f_4}{2}$. The sphere $Red(G)$ has $v' = v - 4n_p$ vertices. One obtains easily $v' = 2f_9 - 4$ and $v = 8 + 5f_9$, from which the result follows.

(ii) Take a $(4,9)$-sphere, which is $4R_1$, and consider its reduced map $Red(G)$. $Red(G)$ is a 3-valent map with faces of gonality between 5 and 9. We enumerate those maps up to 44 vertices and for every one of those graphs, we search for special $(4,9)$-matchings. We found one graph with 44 vertices, which is a fullerene and has a unique special $(4,9)$-matching. It defines a $(4,9)$-sphere, which is $4R_1$, and part (i) above proves that it is the smallest one.

(iii) Consider the following $(5,6)$-sphere with its special $(4,9)$-matching:

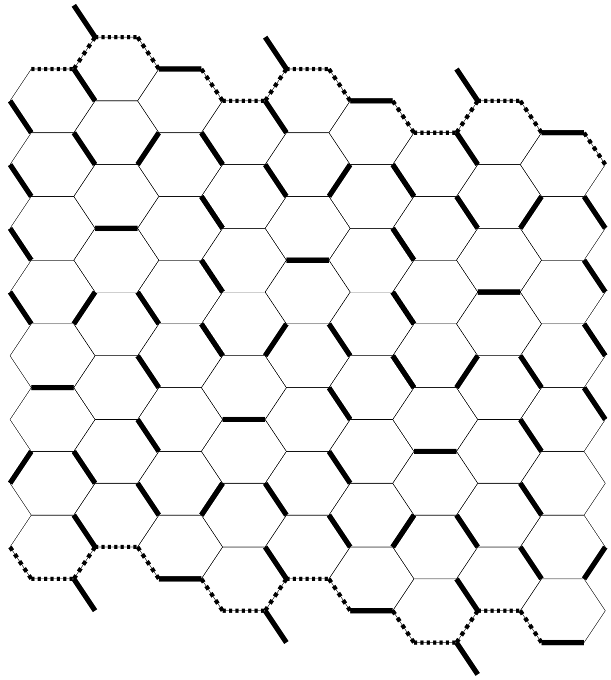


The special $(4,9)$-matching



The cutting lines

This sphere is cutted along the cutting lines and in place is inserted the following structure:



It is easy to see that the obtained structure is still a $(5,6)$-sphere with a special $(4,9)$-matching. Furthermore, the operation can be repeated indefinitely. $\square$

Although the above proof is very easy to check, the way to obtain the example is interesting. First, restrict oneself to fullerenes. Second, search for a cylindrical structures, since almost all infinite series, so far, were of that form. Then one search among the ones of symmetry D_3, since it is the maximal possible symmetry. Then one search among the ones of symmetry D_3, since it is the maximal symmetry possible. The requirement of special symmetry allow us to restrict ourself to *equivariant special* $(4,9)$*-matchings*, i.e., special $(4,9)$-matchings, which have the same symmetry group as the fullerene, so as to prune the search tree. We obtained the fullerene with the special $(4,9)$-matching drawn above, we set a cutting line and consider the problem of finding an insertable structure as a torus problem. We found 14 different possibilities and selected the one of maximal symmetry. All this led us to think that there are many $(4,9)$-spheres, which are $4R_1$.

If one searches for special $(4,9)$-matchings in fullerenes, then this leads to $(4,9)$-spheres with special $(4,9)$-matchings:

1. one with 128 vertices of symmetry D_3,
2. two with 148 vertices $((1,D_2)$ and $(1,T))$,
3. 10 with 168 vertices $((5,C_2)$ and $(5,D_3))$,
4. 23 with 188 vertices $((9,C_1)$, $(10,C_2)$, $(3,D_2)$ and $(1,D_3))$,
5. 66 with 208 vertices $((44,C_1)$, $(19,C_2)$, $(2,C_3)$ and $(1,C_s))$.

5.3 Maps $4R_2$

Theorem 5.7 *(i) A* $(4, q)$*-torus* $4R_2$ *can exist only for* $q = 7 - 16$, 18. *For* $q = 14$, 16, 18 *such tori are strictly face-regular.*

(ii) A $(4, q)$*-sphere* $4R_2$*, which is not* $Prism_q$*, can exist only for* $q = 6 - 13$, 15. *The number of vertices of such spheres should be at least* 20 *(for* $q = 7$*),* 32 *(for* $q = 8$*),* 28 *(for* $q = 9$*),* 44 *(for* $q = 10$*),* 92 *(for* $q = 11$*),* 56 *(for* $q = 12$*),* 116 *(for* $q = 13$*),* 140 *(for* $q = 15$*). If it has this minimal number of vertices, then it is strictly face-regular.*

Proof. A cycle of 4-gons cannot exist neither in case (i), nor in case (ii), due to the exclusion of $Prism_q$. So, all 4-gons are part of triples of 4-gons S_1. Denote by n_t the number of such triples. One has the relations $f_4 = 3n_t$. Furthermore, by a packing argument, one obtains the inequality:

$$6n_t = e_{4-q} \leq f_q 2\lfloor\frac{q}{3}\rfloor\ .$$

The Euler relation is $6\chi = 2f_4 - (q-6)f_q$ with χ being 2 for sphere and 0 for torus. So, one obtains:

$$6\chi = 2f_4 - (q-6)f_q \leq f_q\Psi(q) \text{ with } \Psi(q) = 2\lfloor\frac{q}{3}\rfloor - (q-6).$$

The function Ψ satisfies to:

- $\Psi(q) > 0$ for $q \in \{7, \dots, 13, 15\}$;
- $\Psi(q) = 0$ for $q = 14$, 16, 18;
- $\Psi(q) < 0$ for $q = 17$ or $q \geq 19$.

If $\Psi(q) < 0$, then it excludes the existence of a sphere or a torus. If $\Psi(q) = 0$, then $\chi = 0$. Also, all q-gonal faces should be adjacent to exactly $2\lfloor\frac{q}{3}\rfloor$ 4-gons, i.e., the torus is strictly face-regular.

If $\Psi(q) > 0$, then one has the condition $f_q \geq \frac{6\chi}{\Psi(q)}$, which gives announced lower bounds. □

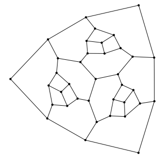
The S_1*-replacement* of a map G by a set S of vertices consists of replacing every vertex in S by a $(4, 3)$-polycycle S_1.

Theorem 5.8 *(i) Every* $(4, 7)$*-map is obtained from a* $(5, 7)$*-map by selecting a set* S*, such that every* 5*-gon is incident to exactly one vertex of this sets and* S_1*-replacing this set.*

(ii) The list of $(4, 7)$*-spheres* $4R_2$ *with* $n \leq 134$ *is obtained from the known* $(5, 7)$*-spheres (up to* 80 *vertices).*

(iii) There exists a $(4, 7)$*-torus* $4R_2$.

(iv) Given a n*-vertex* $(4, 7)$*-sphere* $4R_2$*, one can obtain a* $n + 36$*-vertices* $(4, 7)$*-sphere* $4R_2$*, by replacing the central vertex of this triple by the following structure:*



(v) There is an infinity of $(4,7)$*-spheres, which are* $4R_2$*.*

Proof. (i) Take a $(4,7)$-map and replace every $(4,3)$-polycycle S_1, appearing in it, by a vertex. One gets a map with 3-, 5- and 7-gons. We need to prove that 3-gons cannot appear. If there is a 3-gon, then, in the original map, a 7-gonal face was incident to two $(4,3)$-polycycles S_1. So, one of adjacent 7-gons is also incident to those two S_1. This implies that those two 7-gons are adjacent to two common 7-gonal faces, say, F_1 and F_2. Those two faces are incident to one $(4,3)$-polycycle S_1. One gets the contradiction by seeing that those faces, F_1 and F_2, are adjacent to a 2-gonal face.

(ii) The proof is obtained by computation.

(iii) There exists a $(5,7)$-torus $7R_4$ with 5-gons organized in triples (see Subsection 2.2, Case 18). So, one can apply the operation, given in (i), and get the torus.

(iv) is obvious and (v) follows by repeated applications of (iv). □

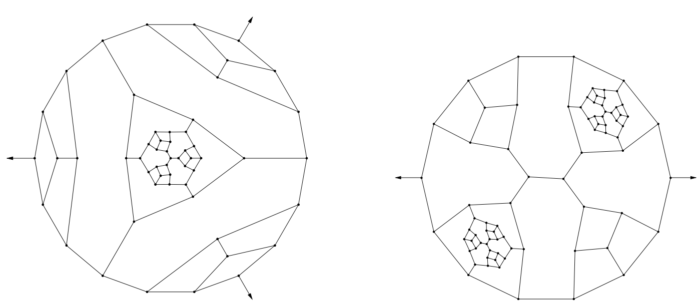
Theorem 5.9 *(i) There is an infinity of* $(4,8)$*-spheres* $4R_2$*.*

(ii) There are at least eight $(4,8)$*-spheres* $4R_2$*, shown on Figure 22, with* 128 *vertices.*

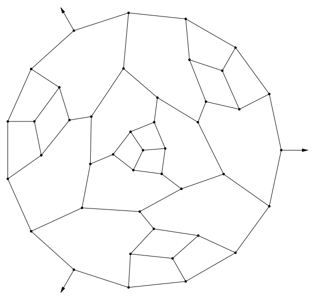
Proof. (i) Take a $(4,3)$-polycycle S_1 and add $2t$ rings of three hexagons around it. Then make the three vertices of degree 2 adjacent to one other vertex (this form another $(4,3)$-polycycle S_1). One obtains a 3-valent plane graph G, which contains two triples of 4-gons. In order to obtain a $(4,8)$-sphere $4R_2$, one should find a subset S of the set of vertices, such that:

- every 4-gon is incident to two vertices of S and
- every 6-gon is incident to one vertex of S.

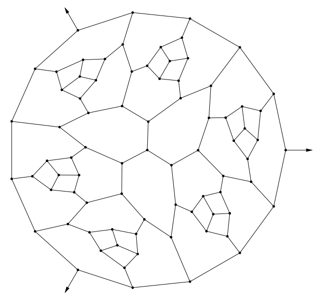
For the elements of S, we take first the vertices of G, which are incident to just one 4-gon of G. Then one needs to find vertices, which are incident to three 6-gons and cover the remaining 6-gons. It is easy to see that this is, indeed, possible. See below first example of the series.



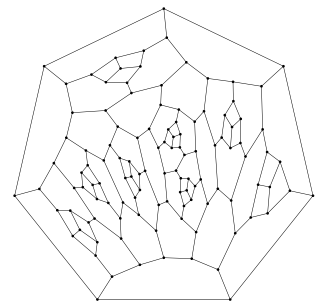
56, D_{3d} 80, C_{2h}



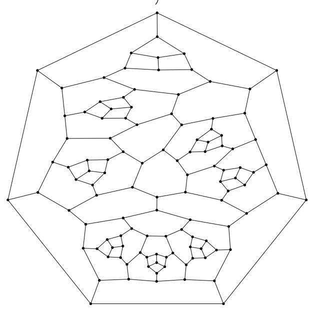
44, T



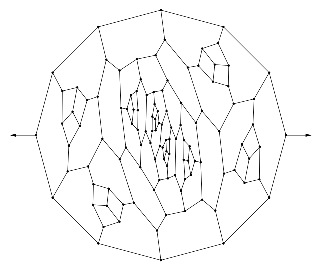
80, D_3



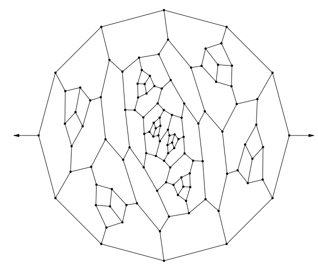
116, C_1



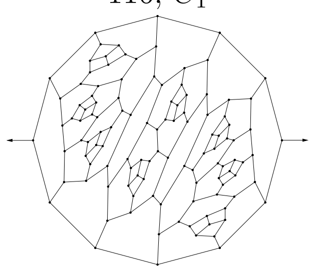
116, C_1



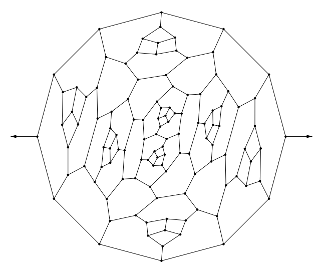
116, C_2



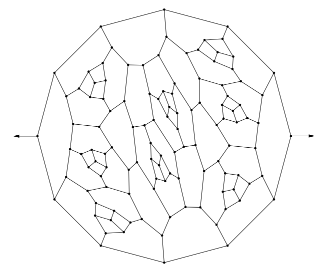
116, C_2



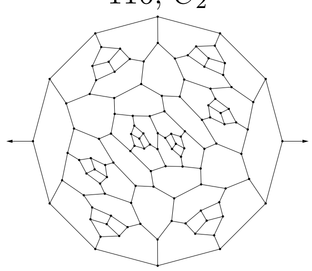
116, C_2



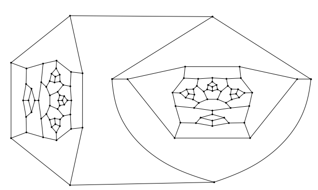
116, C_2



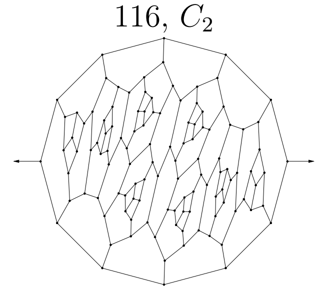
116, C_2



116, C_2

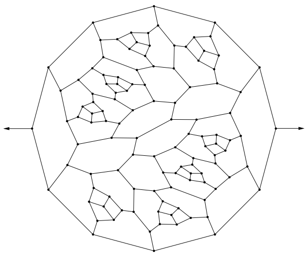


116, C_2

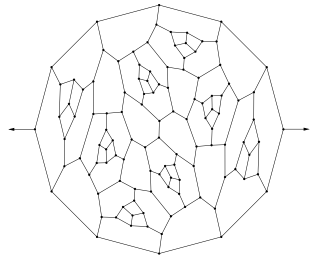


116, C_2

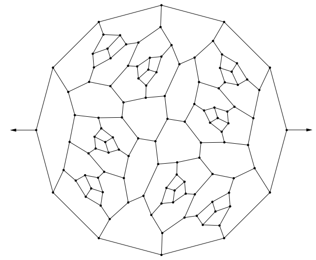
Figure 20: All weakly face-regular $(4, 7)$-spheres, which are $4R_2$, have at most 134 vertices and are not obtained by operation (iv) of Theorem 5.8 (first part)



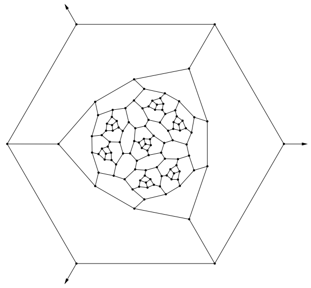
116, D_2



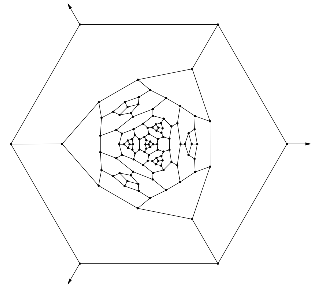
116, D_2



116, D_2



116, D_3



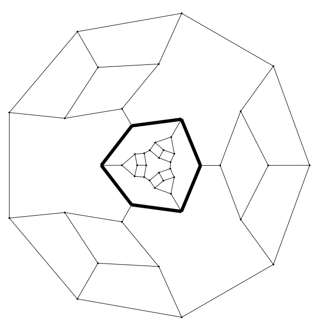
116, D_{3d}

Figure 21: All weakly face-regular $(4,7)$-spheres, which are $4R_2$, have at most 134 vertices and are not obtained by operation (iv) of Theorem 5.8 (second part)

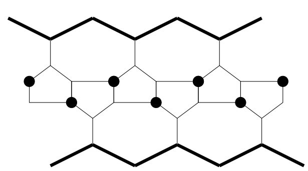
(ii) The $(4,8)$-spheres, which are shown on Figure 22, are constructed according to the same principle of taking a $(4,6)$-sphere and choosing a convenient set S. Take the graph $GC_{2,1}(Cube)$; then, for every 4-gon, there are two diagonals and so, two choices. This makes a total of 64 choices. Reduction by isomorphism yields the result. □

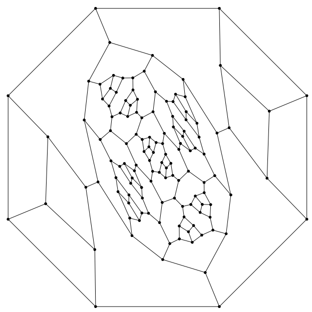
Theorem 5.10 *(i) There is an infinity of* $(4,9)$*-spheres* $4R_2$*.*
(ii) There exists a $(4,9)$*-torus* $4R_2$*.*

Proof. (i) Take the following graph,

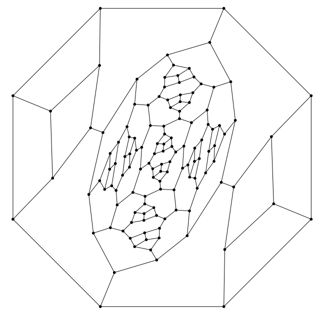


cut it along the overlined edges and insert the following structure

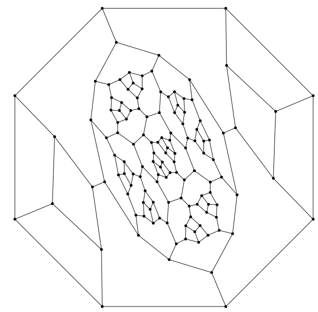




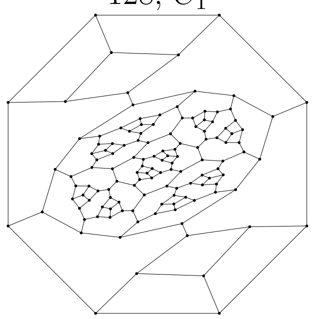
128, C_1



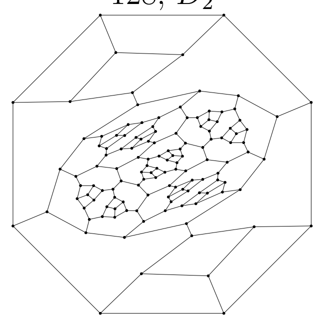
128, D_2



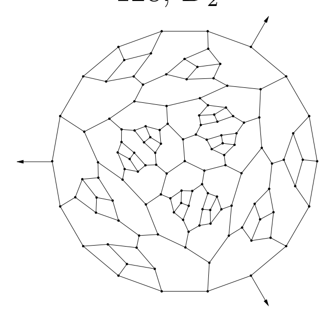
128, D_2



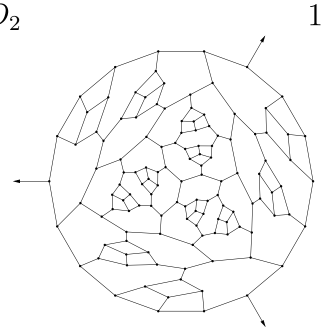
128, D_2



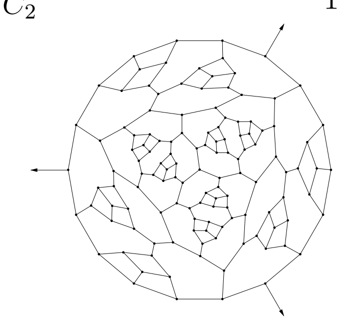
128, C_2



128, D_3



128, D_3



128, T

Figure 22: Some 128-vertex $(4, 8)$-spheres, which are $4R_2$

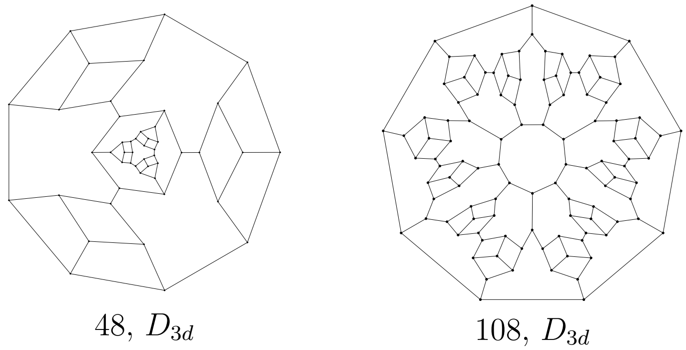


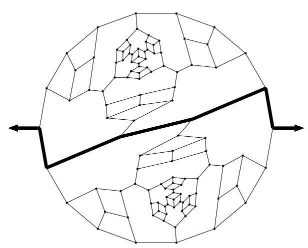
Figure 23: Some weakly face-regular $(4, 9)$-spheres, which are $4R_2$ (including unique such sphere with at most 58 vertices)

with the encircled vertices being S_1-replaced. The operation can, clearly, be repeated and one gets an infinite series.

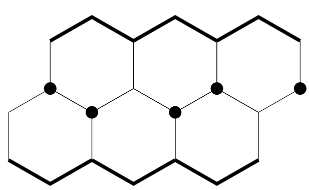
(ii) The above drawing in (i) is, clearly, a part of a plane tiling of such structures. Its quotient is the required torus. □

Theorem 5.11 *There is an infinity of* $(4, 10)$*-spheres, which are* $4R_2$*.*

Proof. Take the following $(4, 10)$-sphere, which is $4R_2$,



and insert, along the overlined edges, the following structure



with the encircled vertices being S_1-replaced. The operation can be repeated and one obtain an infinite sequence of required spheres. □

Theorem 5.12 *There exist an infinity of* $(4, 11)$*-spheres, which are* $4R_2$

Proof. The proof consists of using the infinite families of $(5, 7)$-spheres, which are $7R_4$, constructed in Theorem 10.5. The 5-gons of those polycycles are organized in two polycycles A_3 and bands (of length 6) of pentagons. Define a S_1-replacement set $\mathcal{S}$ by assigning the S_1-vertices in the following way:

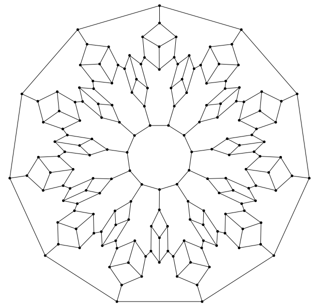
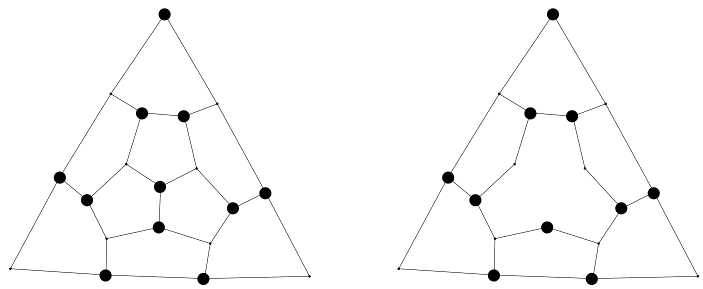


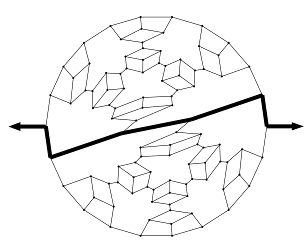
Figure 24: A $(4, 11)$-sphere, which is $4R_2$



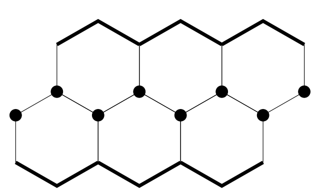
So, every 5-, 7-gon is incident to three, two, respectively, vertices in $\mathcal{S}$. This means that, by doing the S_1-replacement, we obtain a $(4, 11)$-sphere $4R_2$. Since the series of Theorem 10.5 is infinite, we have an infinite series. $\square$

Theorem 5.13 *(i) There is an infinity of* $(4, 12)$*-spheres, which are* $4R_2$*.*
(ii) There is a $(4, 12)$*-sphere* $4R_2$*, which has symmetry* O*.*

Proof. (i) Take the following $(4, 12)$-sphere, which is $4R_2$,



and insert along the overlined edges the following structure



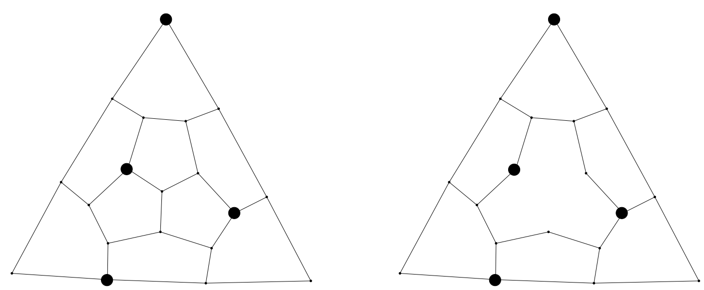
with the encircled vertices being S_1-replaced. The operation can be repeated and one obtains an infinite series of required spheres.

(ii) Take $GC_{2,1}(Cube)$ and triple it along the set of vertices, which are incident to a 4-gon or to a 3-fold axis of symmetry. □

Theorem 5.14 *There exist an infinity of* $(4, 13)$*-spheres, which are* $4R_2$*.*

Proof. The proof consists of using the infinite families of $(5, 7)$-spheres, which are $7R_4$ constructed in Theorem 10.5. As in Theorem 5.12, we will define a set $\mathcal{S}$, which defines the S_1-replacement. Every 5-gonal face should be incident to 4 elements of $\mathcal{S}$ and every 7-gonal face should be incident to 3 elements of $\mathcal{S}$. Hence, it is easier to use the complement $\overline{\mathcal{S}} = \{1, \dots, v\} - \mathcal{S}$ with v being the number of vertices of the plane graph.

The 5-gons of those graphs are organized in two polycycles A_3 and bands, of length 6, of pentagons. The 7-gons are organized in a series of two parallel rings of length 3. There exist a simple zigzag that separate those two rings. We assign all those vertices to $\overline{\mathcal{S}}$. For the 5-gons we assign vertices in the following way:



So, every 5-, 7-gon is incident to one, four vertices in $\overline{\mathcal{S}}$. This means that by doing the S_1-replacement on $\mathcal{S}$, we obtain a $(4, 13)$-sphere $4R_2$. Since the series of Theorem 10.5 is infinite, we have an infinite series. □

Theorem 5.15 *(i) Given a* $(5, 7)$*-map, which is* $7R_4$ *and such that* 7*-gons have the corona* 777555*, one can obtain a* $(4, 15)$*-map, which is* $4R_2$*, by* S_1*-replacement of the set of vertices incident to* 5*-gonal faces.*

(ii) There exists a $(4, 15)$*-torus, which is* $4R_2$*.*

(iii) There exists an infinity of $(4, 15)$*-spheres, which are* $4R_2$*.*

Proof. (i) is obtained by considering the local structure.

(ii) Take the unique $(5, 7)$-torus, which is $5R_2$ and $7R_4$, and use (i).

(iii) The $(5, 7)$-spheres, constructed in Theorem 10.5, give, using (i), an infinite series. □

Both, Theorem 5.12 and 5.14, were obtained by computer enumeration. More precisely the work was done along the following lines:

Proposition 5.16 *(i) If G is a* $(4, 11)$*-sphere, which is* $4R_2$ *and different from* $Prism_{11}$*, then it is obtained by the* S_1*-replacement of a set* $\mathcal{S}$ *in a* 3*-valent plane graph* G'*.* G' *has* $-4 + 12x$ *vertices,* G *has* $8 + 42x$ *vertices and* $\mathcal{S}$ *has* $2 + 5x$ *vertices, for some* $x \geq 1$*.*

(ii) If G is a $(4, 13)$-sphere, which is $4R_2$ and different from $Prism_{13}$, then it is obtained by the S_1-replacement in a set $\mathcal{S}$ of a 3-valent plane graph G'. G' has $-4 + 12x$ vertices, G has $8 + 54x$ vertices and $\mathcal{S}$ has $2 + 7x$ vertices, for some $x \geq 1$.

Proof. We prove only (i), since the proof of (ii) is very similar. Denote by n_1 the number of vertices of G', which are not tripled and n_2 the number of vertices of G', which are tripled. Then one has $f_4 = 3n_2$ and $-5f_{11} + 2f_4 = 12$. Denote by v the number of vertices of G. One has $v = n_1 + 7n_2$ and $3v = 11f_{11} + 4f_4$. Eliminating the unknown f_{11}, v and f_4, we obtain the relation $n_1 = \frac{7n_2-44}{5}$. So, one can write n_2 in the form $2 + 5x$ with $x \in \mathbb{Z}$. This yield $n_1 = 7x - 6$, $v = 8 + 42x$ and $n_1 + n_2 = -4 + 12x$. $\square$

The above proposition give us a strategy for finding some $(4, 11)$-spheres, which are $4R_2$:

1. Take a 3-valent graph with faces of gonality 5, 7, 8, 9 and 11.
2. Do an exhaustive search for S_1-replacement sets $\mathcal{S}$ such that every 5-, 7-, 9- and 11-gonal face is incident to 3, 2, 1 and 0 vertices in $\mathcal{S}$ respectively.

If one consider all 3-valent graphs G' up to $-4 + 12N$ vertices, and manage to do the exhaustive search, then one can get the complete list of $(4, 11)$-spheres up to $8 + 42N$ vertices.

The same applies for $(5, 13)$-spheres. But in that case, more than half of the vertices are in the S_1-replacement set $\mathcal{S}$. Hence, it is better, from the computational viewpoint, to do an exhaustive enumeration of their complement.

The 3-valent plane graphs with faces of gonality 5, 7, 9 or 11 can be enumerated up to 56 vertices. By applying the exhaustive enumeration procedure, we obtained 87 spheres. This means that the enumeration of $(4, 11)$-spheres $4R_2$ has been completed up to 218 vertices. Besides two strictly face-regular ones, the remaining spheres have 176 vertices. The repartition by symmetry is the following: $(38, C_1)$, $(31, C_2)$, $(4, C_{2h})$, $(1, C_3)$, $(10, C_i)$, $(2, C_s)$ and $(1, D_{11d})$. If we limit ourselves to $(5, 7)$-spheres, then one can extend the enumeration up to 68 vertices. Using the exhaustive enumeration technique, we found 27276 graphs, whose repartition by symmetry is the following: $(26299, C_1)$, $(895, C_2)$, $(3, C_{2h})$, $(1, C_{2v})$, $(9, C_3)$, $(16, C_i)$, $(28, C_s)$, $(16, D_2)$, $(8, D_3)$ and $(1, S_4)$.

The 3-valent plane graphs with faces of gonality 5, 7, 9, 11 or 13 can be enumerated up to 56 vertices. By applying the exhaustive enumeration procedure, we obtain 12 spheres. This means that the enumeration of $(4, 13)$-spheres $4R_2$ has been completed up to 278 vertices. Besides two strictly face-regular ones, the remaining spheres have 224 vertices. The repartition by symmetry is the following: $(3, C_1)$, $(4, C_2)$, $(2, C_i)$, $(2, D_3)$ and $(1, S_6)$. If we limit ourselves to $(5, 7)$-spheres, then we can extend the enumeration up to 68 vertices. Using the exhaustive enumeration technique, we found 805 graphs, whose repartition by symmetry is the following: $(707, C_1)$, $(86, C_2)$, $(4, C_3)$, $(1, C_s)$, $(3, D_2)$ and $(4, D_3)$.

5.4 Maps $5R_2$

Theorem 5.17 *(i) A $(5,q)$-sphere, which is $5R_2$, has $q = 7$.*
(ii) A $(5,q)$-torus, which is $5R_2$, has $q = 7$ or 8.

Proof. Euler formula in Theorem 4.4(iii) implies the relation:

$$(6-q)(x_0+x_3)+(8-q)f_5 = 4q\chi$$

with χ being 2 and 0 for sphere and torus, respectively.

If $q > 8$, one gets an impossibility. If $q = 8$, then $-2(x_0+x_3) = 2\chi$, which implies $\chi = 0$ (i.e., a torus) and $x_0 = x_3 = 0$. □

Note that an infinity of $(5,7)$-spheres $5R_2$ is constructed in [HaSo04]. It will be proved, that a $(5,8)$-torus is $5R_2$ if and only it is $8R_2$, in Theorem 8.1 below.

Any $(5,7)$-plane with isolated pairs of adjacent pentagons, is decorated graphite, but this is not the case if pentagons are organized into isolated *triples* of adjacent ones.

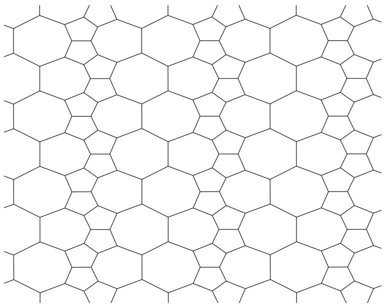
Theorem 5.18 *Any $(5,7)$-torus, which is decorated graphite and such that the pentagons are organized into isolated triples (so, $5R_2$), is $7R_4$. Moreover, corresponding $(5,7)$-plane belongs to Case 18 from Table 4 but it is not the sporadic tiling, where 5-gons are organized as infinite bands.*

Proof. We have $e_{5-5} = f_5$, i.e., 3 for each isolated triple of pentagons. So, $e_{5-7} = 5f_5 - 2f_5 = 3f_7$. Again, no 7-gon is adjacent to 3 pentagons in a row and so, any 7-gon is adjacent to at most 4 pentagons. Only two coronae of a 7-gon, 5575577 and 5575757, give adjacency to exactly 4 pentagons. The first corona is impossible, since deleting all $(5-5)$-edges will transform the 7-gon into a 5-gon, which contradicts the decorated graphite property. The second corona is also impossible, because in this case a 7-gon, which is adjacent to such 7-gon, will have the first corona. So, any 7-gon is adjacent to at most three pentagons. In order to get $e_{5-7} = 3f_7$, counting by 7-gons, we need that *each* 7-gon is adjacent to exactly three pentagons, i.e., we have $7R_4$. Moreover, only coronae 5575777 and 5577577 are possible. The remainder of Theorem is clear from Figure 10 in [DFSV00] and description of the continuum of Case 18. □

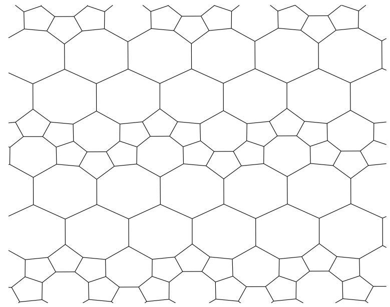
See on Figure 25 some $(5,7)$-tori, which are $5R_2$. See on Figure 68 some $(5,7)$-tori, which are $7R_4$.

5.5 Maps $5R_3$

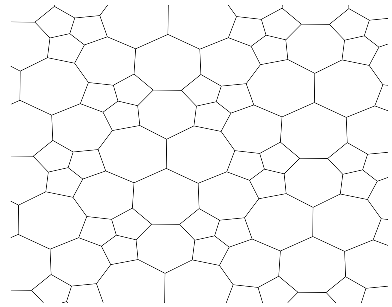
Theorem 5.19 *Let G be a $(5,q)$-torus, which is $5R_3$; then it holds:*
(i) $q \leq 12$.
(ii) If $q = 12$, then it is also $12R_0$.
(iii) If $q = 11$, then it is also $11R_1$.
Let G be a $(5,q)$-sphere, which is $5R_3$; then it holds:
(i) $q \leq 10$.
(ii) If $q = 7$, then there are exactly two spheres (the first being strictly face-regular Nr. 56), shown on Figure 26.



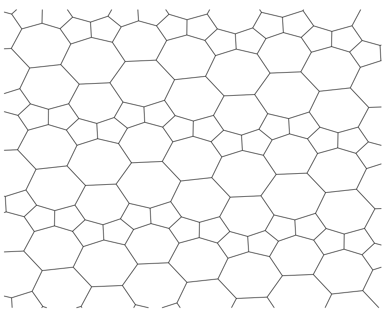
$(16, 4, 4)$, $p2mg$



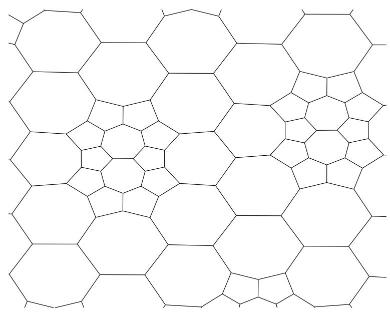
$(24, 6, 6)$, $p2$



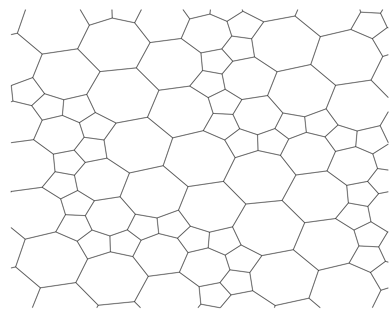
$(24, 6, 6)$, $p31m$



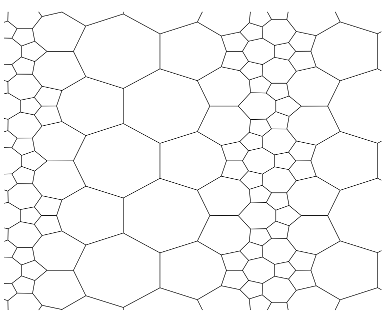
$(24, 6, 6)$, $p2$



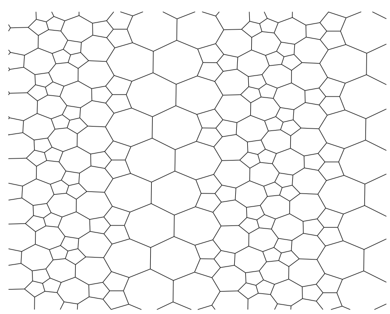
$(40, 10, 10)$, $p2$



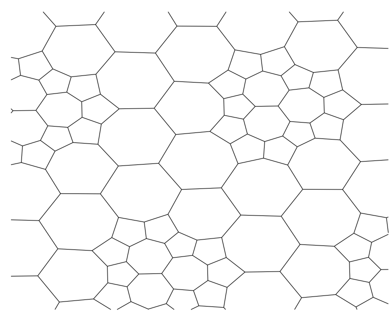
$(48, 12, 12)$, pg



$(48, 12, 12)$, cm



$(48, 12, 12)$, pg



$(48, 12, 12)$, $p31m$

Figure 25: Some $(5, 7)$-tori, which are $5R_2$

The proof is a combination of three lemmata below:

Lemma 5.20 *The set of* 5*-gonal faces of a* $(5, q)$*-map, which is* $5R_3$*, is partitioned into polycycles* E_1 *and* E_2*.*

(i) There is no $(5, q)$*-torus, which is* $5R_3$*, for* $q > 12$*; for* $q = 12$*, such torus is also* $12R_0$*.*

(ii) There is no $(5, q)$*-sphere, which is* $5R_3$*, for* $q \geq 12$*.*

Proof. Take a $(5, q)$-map $5R_3$ and a pentagon of this map. This pentagon is adjacent to pentagons on either three consecutive edges, or two consecutive edges and one isolated edge. Easy to see, that the set of 5-gons is partitioned into $(5, 3)$-polycycles E_1 and E_2. Denote by n_1 and n_2 their respective numbers. We will first treat the simpler toric case. First, one has the relation $f_5 = 3n_1 + 4n_2$. By Euler formula, one also has $f_5 = (q-6)f_q$, which implies the relation:

$$e = 3(q-5)f_q = 3\frac{q-5}{q-6}f_5 \ .$$

By direct counting, one has:

$$e_{5-5} = (3 + \frac{3}{2})n_1 + 6n_2 \quad \text{and} \quad e_{5-q} = 6n_1 + 8n_2 \ .$$

We then obtain the relation:

$$\begin{array}{rcl} e_{q-q} & = & 3\frac{q-5}{q-6}(3n_1 + 4n_2) - \{(3 + \frac{3}{2})n_1 + 6n_2\} - \{6n_1 + 8n_2\} \\ & = & \{\frac{9(q-5)}{q-6} - (9 + \frac{3}{2})\}n_1 + \{\frac{12(q-5)}{q-6} - 14\}n_2 \\ & = & \frac{18 - \frac{3}{2}q}{q-6}n_1 + \frac{24-2q}{q-6}n_2 \\ & = & \frac{12-q}{q-6}\{\frac{3}{2}n_1 + 2n_2\} \ . \end{array}$$

If $q > 12$, then $e_{q-q} = n_2 = n_1 = 0$, which is impossible. If $q = 12$, then $e_{12-12} = 0$, i.e., the torus is $12R_0$.

The computation for $(5, q)$-spheres is, essentially, a remake, with some additional constant, of the toric case. One obtains first $f_5 = 12 + (q-6)f_q$, then:

$$e = 3(10 + (q-5)\frac{f_5 - 12}{q-6}) \quad \text{and} \quad v = 2(10 + (q-5)\frac{f_5 - 12}{q-6})$$

and, finally:

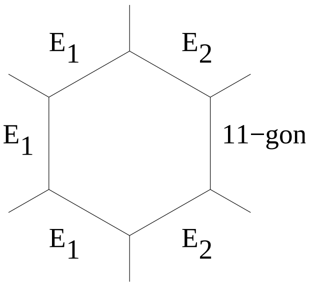
$$e_{q-q} = \{30 - 36\frac{q-5}{q-6}\} + \frac{12-q}{q-6}\{\frac{3}{2}n_1 + 2n_2\} \ .$$

If $q \geq 12$, then e_{q-q} becomes negative, which is impossible. □

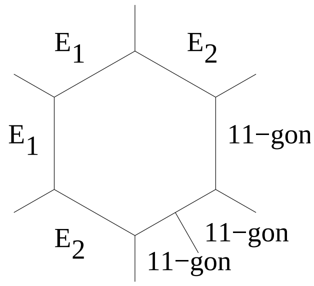
Lemma 5.21 *(i) A* $(5, 11)$*-torus, which is* $5R_3$*, is also* $11R_1$*.*

(ii) There is no $(5, 11)$*-sphere, which is* $5R_3$*.*

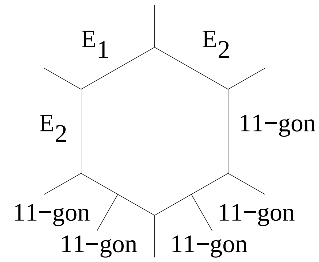
Proof. Given an 11-gonal face, which is incident to a $(5,3)$-polycycle E_1, it is clear, that on every side it is incident also to an E_1 or E_2, and that this can terminate only with a $(5,3)$-polycycle E_2. Hence, we decompose the set of 11-gonal faces into the following 6 *types*:



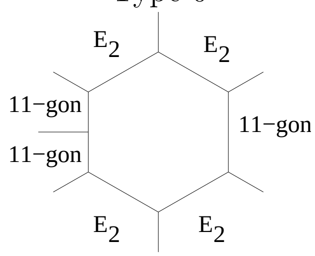
Type 0



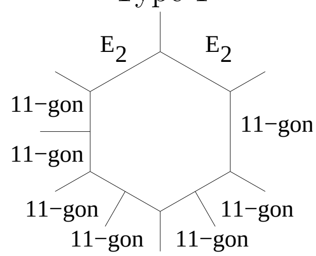
Type 1



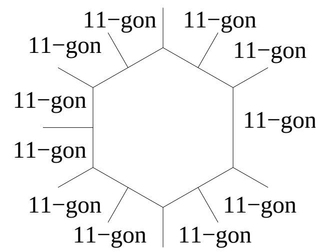
Type 2



Type 3



Type 4



Type 5

Denote by $f_{11,i}$ with $0 \le i \le 5$ the number of 11-gonal faces of type i.

Consider the number x_i of vertices, which are contained in exactly i q-gonal faces. One has, clearly:

$$x_0 = n_1 + 2n_2, \quad x_1 = 6n_1 + 6n_2 \quad \text{and} \quad x_2 = 2n_2\ .$$

Let us consider first the toric case. The number of vertices of our torus is equal to $2\frac{q-5}{q-6}(3n_1 + 4n_2)$. From this we get:

$$\begin{array}{rcl} x_3 & = & 2\frac{6}{5}(3n_1+4n_2) - (n_1+2n_2) - (6n_1+6n_2) - (2n_2) \\ & = & \{\frac{36}{5} - 7\}n_1 + \{8\frac{6}{5} - 10\}n_2 \\ & = & \frac{1}{5}n_1 - \frac{2}{5}n_2\ . \end{array}$$

By direct counting, one gets also:

$$\left\{\begin{array}{rcl} 3x_3 & = & 2f_{11,1} + 4f_{11,2} + f_{11,3} + 6f_{11,4} + 11f_{11,5} \\ 3n_1 & = & 3f_{11,0} + 2f_{11,1} + f_{11,2} \\ 4n_2 & = & 2f_{11,0} + 2f_{11,1} + 2f_{11,2} + 4f_{11,3} + 2f_{11,4}. \end{array}\right.$$

Using the preceding equation, one obtains:

$$\left\{\begin{array}{rcl} x_3 & = & \frac{2}{3}f_{11,1} + \frac{4}{3}f_{11,2} + \frac{1}{3}f_{11,3} + 2f_{11,4} + \frac{11}{3}f_{11,5} \\ \frac{1}{5}n_1 - \frac{2}{5}n_2 & = & \{-\frac{1}{5} + \frac{1}{5}\}f_{11,0} + \{-\frac{1}{5} + \frac{1}{5}\frac{2}{3}\}f_{11,1} + \{-\frac{1}{5} + \frac{1}{5}\frac{1}{3}\}f_{11,2} + \{-\frac{2}{5}\}f_{11,3} + \{-\frac{1}{5}\}f_{11,4} \\ & = & -\frac{1}{15}f_{11,1} - \frac{2}{15}f_{11,2} - \frac{2}{5}f_{11,3} - \frac{1}{5}f_{11,4}, \end{array}\right.$$

which, clearly, implies $f_{11,1} = f_{11,2} = f_{11,3} = f_{11,4} = f_{11,5} = 0$. Hence, (i) is true.

Furthermore, for spheres one has:

$$\begin{array}{rcl} x_3 & = & 2(10 + 6\frac{f_5-12}{5}) - x_0 - x_1 - x_2 \\ & = & \frac{-22}{5} + \frac{1}{5}n_1 - \frac{2}{5}n_2 \ . \end{array}$$

Using the same expression of x_3, n_1 and n_2 in terms of $f_{11,i}$, we obtain the equality:

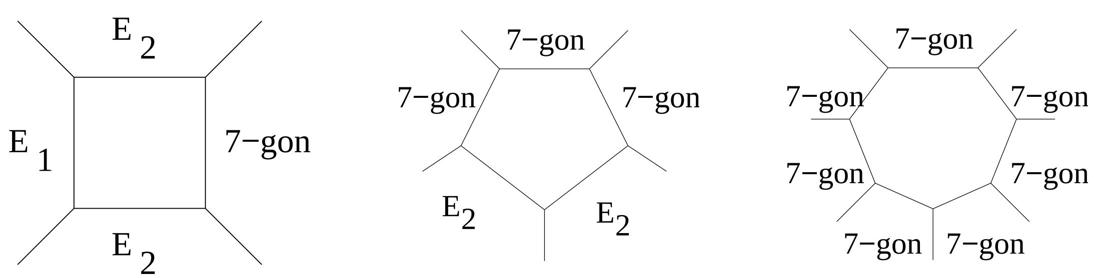
$$x_3 = \frac{-22}{5} - \frac{1}{15}f_{11,1} - \frac{2}{15}f_{11,2} - \frac{2}{5}f_{11,3} - \frac{1}{5}f_{11,4},$$

i.e., x_3 is negative, an impossibility. $\square$

Lemma 5.22 *(i) A* $(5,7)$*-sphere, which is* $5R_3$*, is one of two spheres (the first being strictly face-regular Nr. 56), shown on Figure 26.*

(ii) There is no $(5,7)$*-torus, which is* $5R_3$*.*

Proof. By analogy with the above lemma, one can split the set of 7-gonal faces into the following types:



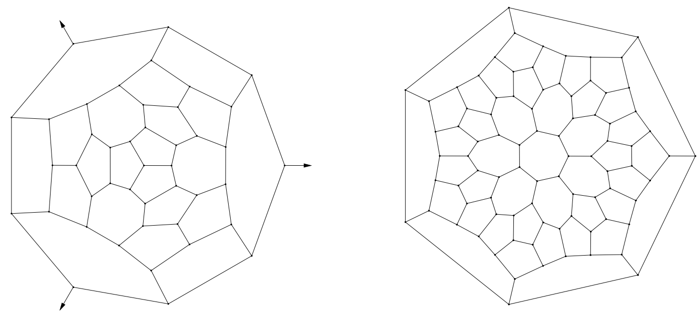
Suppose that a sphere contains a face of type 0; then this face is necessarily adjacent to another face of type 0. Those two faces are bordered by four $(5,3)$-polycycles: two E_1 and two E_2. Each of two $(5,3)$-polycycles E_1 has two vertices of degree 2. Since the polycycle E_1 is adjacent only to 7-gons of type 0, we have only one way of filling the structure: by adding faces of type 0. Hence, the obtained sphere is strictly face-regular.

Assume now that there is no face of type 0; then $n_1 = 0$. Hence, the only appearing $(5,3)$-polycycles are E_2. Those polycycles are adjacent by pairs; so, they make cycles in the sphere. Since this is a $(5,7)$-sphere, there exist at least one $(7,3)$-polycycle, say P_7, bordered by a ring of $(5,3)$-polycycles E_2.

From the structure of the polycycle E_2, we know that P_7 has boundary sequence $(32^3)^h$ for some h. If one removes the exterior 7-gons on the boundary, then one obtains the boundary sequence 2^h, i.e., a simple h-gon. Hence, $h = 7$. On the other side of the structure, we do the same analysis and obtain the second $(5,7)$-sphere, which is $5R_3$.

On the other hand, the above proof shows that there is no $(5,7)$-torus, which is $5R_3$. $\square$

We consider now, for $q = 8, 9, 10$, $(5,q)$-maps, which are $5R_3$ and require additional analysis. The set of 5-gonal faces of such maps admits a partition into $(5,3)$-polycycles



44, D_{3h} (also $7R_1$) 84, D_{7h}

Figure 26: All $(5,7)$-spheres, which are $5R_3$

E_1 and E_2. E_1 has two open edges and E_2 has three open edges. If one considers the graph formed by all those polycycles, then it has 2- and 3-valent vertices. This graph is not necessarily connected. The connected components of $q(G)$ (i.e. the generalized $(q,3)$-polycycles formed by the q-gonal faces) are bounded by 5-gons on one or several boundaries. We will be able to obtain some classification results of those generalized $(q,3)$-polycycles, when they have only one boundary. This will allow us to obtain some examples of $(5,8)$-maps $5R_3$ and $(5,10)$-maps $5R_3$. Also, we will be able to completely classify the $(5,9)$-maps $5R_3$.

It is known (see [DDS04]) that the boundary sequence of a $(q,3)$-polycycle does not characterize the polycycle; however, given a boundary sequence all $(q,3)$-polycycles having this sequence can be enumerated.

Take such a $(q,3)$-polycycle; it is bounded by elementary $(5,3)$-polycycles E_1 and E_2, which we represent, in a symbolic way, by $E_1E_2^{n_1}\dots E_1E_2^{n_u}$. This *symbolic sequence* of E_1 and E_2 corresponds to the boundary sequence:

$$b(n_1,\dots,n_u) = 22(2232)^{n_1}\dots 22(2232)^{n_u}.$$

Lemma 5.23 *(the* $(8,3)$*-case)*
(i) Consider the symbolic sequence $(n_1,\dots,n_u)$*.*

- *If* $n_i = 0$*,* 1 *or* 2 *for some* i*, then* $b(n_1,\dots,n_u)$ *is* $(8,3)$*-fillable if and only if* $(n_1,\dots,n_u)$ *is equal to* $(0,0,0,0)$*,* $(1,1,1)$ *or* $(2,2,2)$*, respectively.*
- *If* $n_i \geq 3$ *for all* i*, then the boundary sequence* $b(n_1,\dots,n_u)$ *is* $(8,3)$*-fillable if and only if the boundary sequence*

$$22(23)^{x_1}\dots 22(23)^{x_u} \text{ with } x_i = n_i - 3$$

is $(8,3)$*-fillable.*

(ii) For any given u*, there is a finite number of symbolic sequences* $(n_1,\dots,n_u)$*, such that* $b(n_1,\dots,n_u)$ *is* $(8,3)$*-fillable. Up to isomorphism, the list consists, for* $t \leq 11$*, of:*

u	*symbolic sequences*	f_8
3	$(1,1,1)$ *and* $(2,2,2)$	3 *and* 6
4	$(0,0,0,0)$ *and* $(3,3,3,3)$	1 *and* 13
6	$(3,4,3,4,3,4)$	24
8	$(3,4,3,5,3,4,3,5)$	35
9	$(3,4,4,3,4,4,3,4,4)$	39
10	$(3,4,3,5,3,5,3,4,3,6)$	46
11	$(3,4,3,5,4,3,4,4,3,4,5)$	50

(iii) The boundary sequence $b(n_1,\dots,n_u)$ is $(8,3)$-fillable if and only if the boundary sequence $b(3,4^{n_1},\dots,3,4^{n_u})$ is $(8,3)$-fillable.

(iv) There is an infinity of boundary sequences of the form $b(n_1,\dots,n_u)$, which are $(8,3)$-fillable.

Proof. (i) Clearly, if some $n_i=0$, then the only way to close the structure, is by obtaining an isolated 8-gon and its symbolic sequence is $(0,0,0,0)$.

So, assume $n_i\geq 1$. If $n_i=1$ for some i, then there is an unique way of closing the structure and one obtains a triple of 8-gons associated to the symbolic sequence $(1,1,1)$. Hence, assume $n_i\geq 2$. If $n_i=2$ for some i, then there is an unique way of closing the structure and one obtains six 8-gons associated to the symbolic sequence $(2,2,2)$. Hence, assume $n_i\geq 3$. Clearly, the set of edges of the boundary, which are incident to an 8-gonal face of a possible $(8,3)$-filling, is a path or the empty set, i.e., the face cannot be incident to two different segments of the boundary. So, the boundary sequence $b(n_1,\dots,n_u)$ admits an $(8,3)$-filling if and only if the boundary sequence, which is obtained by filling all faces incident to the boundary, i.e., $22(23)^{x_1}22(23)^{x_2}\dots 22(23)^{x_u}$ with $x_i=n_i-3$, also admits a $(8,3)$-filling.

(ii) By using Proposition 1.4, one can see that the numbers f_8 and x (of 8-gonal faces and, respectively, of interior vertices of an possible $(8,3)$-filling) are:

$$\left\{\begin{array}{rcl} f_8 &=& u-3\\ x &=& 2u-8-\sum_{i=1}^{u} x_i. \end{array}\right.$$

So, for a fixed u one has $x_i\leq 2u-8$ and there is a finite number of possible boundary sequences and so, a finite number of possible $(8,3)$-fillings. The enumeration is then done by computer.

(iii) By taking the initial boundary sequence $b(3,4^{n_1},3,4^{n_2},\dots,3,4^{n_u})$ and applying the transformation of the second item of (i), one obtains the boundary sequence $b(n_1,\dots,n_u)$.

(iv) By using the transformation in (iii), one can, from a given boundary sequence $b(n_1,\dots,n_u)$, obtain another one. So, we get an infinity of such boundary sequences. □

From the above analysis, it seems likely that, for any $u\geq 4$, there exists at least one boundary sequence $b(n_1,\dots,n_u)$, which is $(8,3)$-fillable.

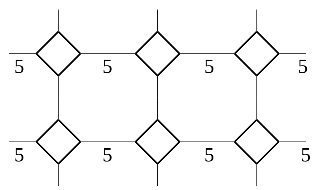
Theorem 5.24 *(i) There exists a $(5,8)$-torus, which is $5R_3$ and not $8R_4$.*

(ii) There exists a sequence $(F_i)_{i\geq 0}$ of $(5,8)$-spheres $5R_3$ with $1640+1152i$ vertices. Their symmetry is O_h if $i=0$ and D_{4h}, otherwise.

(iii) The following $(5,8)$-spheres $5R_3$ exist:

1. *A sphere with 56 vertices and symmetry O_h (also $8R_0$).*
2. *A sphere with 92 vertices and symmetry T_d (also $8R_2$).*
3. *A sphere with 164 vertices and symmetry T_d.*
4. *A sphere with 488 vertices and symmetry O_h.*
5. *A sphere with 3944 vertices and symmetry C_{2v}.*
6. *A sphere with 4196 vertices and symmetry T_d.*
7. *A sphere with 6248 vertices and symmetry D_{4h}.*

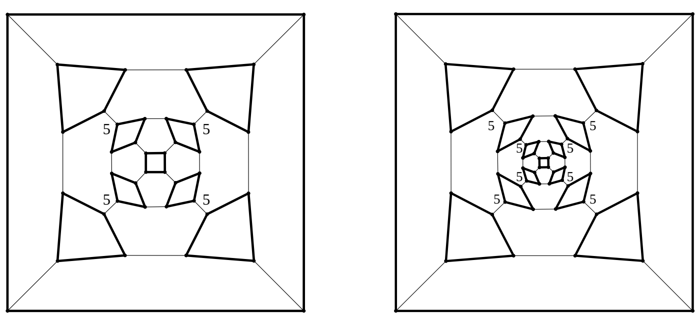
Proof. (i) The $(5,8)$-torus $5R_3$ is obtained by taking the $(8,3)$-polycycles, whose symbolic sequence is $(3,3,3,3)$ and $(3,4,3,5,3,4,3,5)$, and gluing them together according to the drawing below:



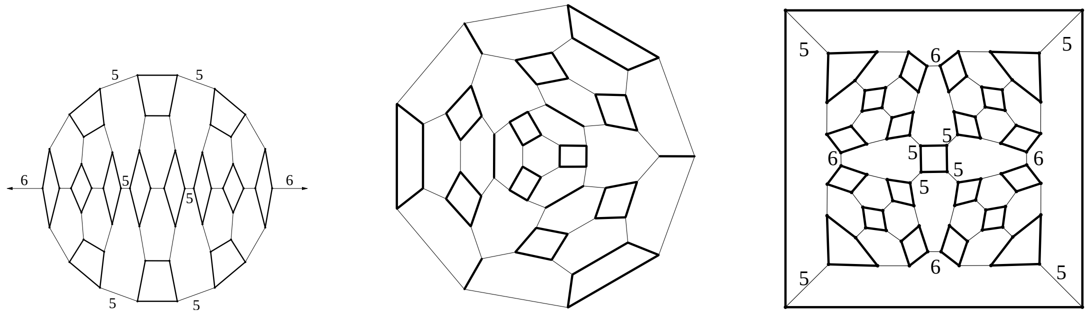
value 3 assigned to overlined, other non-notated edges are assigned the value 4.

Clearly, one gets the needed torus from this structure.

(ii) An infinity of $(5,8)$-spheres $5R_3$ is obtained by a variation of (i). This time one takes the symbolic sequences $(3,3,3,3)$, $(3,4,3,4,3,4)$ and $(3,4,3,5,3,4,3,5)$. For $i=0$, we form truncated Octahedron, which has symmetry O_h. For $i=1$, 2, we form the structure according to the following drawings.



Overlined edges have value 3, non-notated edges have value 4.



$(3944), C_{2v}$ $(4196), T_d$ $(6248), D_{4h}$

Figure 27: Some 3-valent spheres, which we used as skeletons of $(5,8)$-spheres $5R_3$; the overlined edges are assigned the value 3, while non-notated edges are assigned the value 4

For $i \geq 3$, we have an obvious generalization.

(iii) The relative wealth of above examples of $(8,3)$-polycycles, allow us to build a variety of examples of $(5,8)$-spheres, which are $5R_3$. They are described by the assignment of values to edges of a 3-valent plane graph such that every circuit $(n_1, \dots, n_t)$ appearing on a face is $(9,3)$-fillable. Namely, one gets:

1. Take $(0,0,0,0)$ and form Cube from it. It will be unique strictly face-regular $(5,8)$-sphere $5R_3$, $8R_0$ of symmetry O_h with 56 vertices.

2. Take $(1,1,1)$ and form Tetrahedron from it. It will be unique strictly face-regular $(5,8)$-sphere $5R_3$, $8R_2$ of symmetry T_d with 92 vertices.

3. Take $(2,2,2)$ and form Tetrahedron from it. It will be a $(5,8)$-sphere $5R_3$ of symmetry T_d with 164 vertices.

4. Take $(3,3,3,3)$ and form Cube from it. It will be a $(5,8)$-sphere $5R_3$ of symmetry O_h with 488 vertices.

5. Take three graphs on Figure 27 and assign values to their edges accordingly. One gets three $(5,8)$-spheres with 3944, 4196 and 6248 vertices.

See on Figure 69 some example of $(5,8)$-tori, which are $8R_4$ but not $5R_3$.

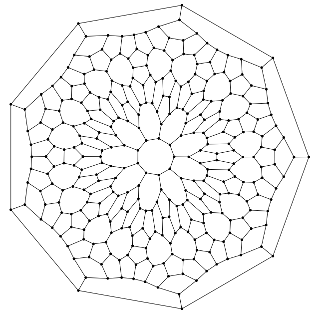
Lemma 5.25 *(the* $(9,3)$*-case)*

(i) Consider the symbolic sequence $(n_1, \dots, n_u)$.

- *If* $n_i = 0$ *for some* i*, then the boundary sequence* $b(n_1, \dots, n_u)$ *is* $(9,3)$*-fillable if and only if* $(n_1, \dots, n_u) = (0,1,0,1)$.

- *If* $n_i \geq 1$ *for all* i*, then the boundary sequence* $b(n_1, \dots, n_u)$ *is* $(9,3)$*-fillable if and only if the boundary sequence*

$$2(233)^{x_1} \dots 2(233)^{x_u} \text{ with } x_i = n_i - 3$$



486, D_{9h}

Figure 28: The only $(5,9)$-sphere, which is $5R_3$

is $(9,3)$-fillable.

(ii) For any given u, there is a finite number of symbolic sequences $(n_1,\dots,n_u)$, such that $b(n_1,\dots,n_u)$ is $(9,3)$-fillable. Up to isomorphism, the list consists, for $u \leq 30$, of two following sequences:

u	*symbolic sequences*	f_9
4	$(0,1,0,1)$	2
9	$(1,1,1,1,1,1,1,1,1)$	10

Proof. (i) If $n_i = 0$, then the boundary sequence contains at least 7 consecutive 2. In order to be fillable, it should contain exactly 7 consecutive 2. The only possibility then is, clearly, $(n_1,\dots,n_u) = (0,1,0,1)$.

If $n_i \geq 1$, then, as for the case of $(8,3)$-polycycles, one can see that a face, which is incident to the boundary, is incident on only one segment of edges. Hence, there is an unique way of adding 9-gons on the boundary, so as to form a ring. The boundary sequence of this filling is:

$$2(233)^{x_1}2(233)^{x_2}\dots 2(233)^{x_u} \quad \text{with} \quad x_i = n_i - 1.$$

(ii) By using Euler formula (see 1.4), one gets that the numbers f_9 and x (of 9-gonal faces and interior vertices of an possible $(9,3)$-filling) are:

$$\begin{cases} f_9 &= \frac{1}{3}(u - 6 - \sum_i x_i) \\ x &= \frac{2}{3}(u - 9 - 4\sum_i x_i). \end{cases}$$

So, for a fixed u, one has $x_i \leq \frac{u-9}{4}$ and there is a finite number of possible boundary sequences, i.e., a finite number of possible $(9,3)$-fillings. The enumeration is then done by computer. $\square$

It seems likely that the only symbolic sequences $(n_1,\dots,n_u)$, such that $b(n_1,\dots,n_u)$ is $(9,3)$-fillable, are: $(0,1,0,1)$ and $(1,1,1,1,1,1,1,1,1)$.

Given a closed orientable map G, one assign an orientation on any one of its edges and form a $\mathbb{Z}$-module $C_1(G)$ having this set of oriented edges as basis. The $\mathbb{Z}$-module $Z_1(G)$ is the submodule of $C_1(G)$ generated by the set of closed cycles of G. Given any face of G, one associate to it the set of incident edges in clockwise orientation; the generated $\mathbb{Z}$-module is called $B_1(G)$. Easy to see that $B_1(G)$ is a subset of $Z_1(G)$.

The homology group $H_1(G)$ is the quotient of $Z_1(G)$ by its subgroup $B_1(G)$. One can prove that $H_1(G)$ is isomorphic to $\mathbb{Z}^{2g}$ with g being the genus of the map G.

Theorem 5.26 *(i) The only* $(5,9)$*-sphere, which is* $5R_3$*, is the one with* 486 *vertices and symmetry* D_{9h}*; it is obtained by taking* $Prism_9$ *and assigning the values* 1 *to the edges, which are incident to the* 9*-gons, and* 0*, otherwise (see Figure 28).*

(ii) There is no $(5,9)$*-torus, which is* $5R_3$*.*

Proof. If the $(9,3)$-polycycle with boundary sequence $b(0,1,0,1)$ appear in the decomposition of the set of 9-gonal faces, then we are done. This is so, since an edge of value 0 can belong only to a $(9,3)$-polycycle with boundary sequence $b(0,1,0,1)$. Hence, a path of such faces appear. By considering the adjacent 9-gons, one sees that the structure should close and obtains the announced graph.

(i) Take such a sphere G and consider the graph $E_1(G)$, whose vertex-set consists of the $(5,3)$-polycycles E_1 of G with two vertices being adjacent if they are linked by a sequence of $(5,3)$-polycycles E_2. The graph $E_1(G)$ can be considered as a sphere, except that it is not necessarily connected, i.e., some "faces" of $E_1(G)$ are bounded by several cycles. Denote by $C_1, \dots, C_t$ the connected components of $E_1(G)$, which are plane graphs in the original sense, and by $F_1, \dots, F_l$ the faces of $E_1(G)$, which have several cycles. Denote by $Conn(G)$ the graph, whose vertex-set consists of both, C_i and F_j, and whose edge $C_i - F_j$ corresponds to a cycle of C_i belonging to F_j. The sphericity condition then implies that $Conn(G)$ is a tree. Hence, it has at least one vertex of degree at most 1.

If the vertex has degree 0, this means that $E_1(G)$ is connected. Denote by p_i the number of faces of gonality i of $E_1(G)$. By Lemma 5.25 and Euler formula, only p_4, p_9 or p_i with $i \geq 30$ can be non-zero. So, Euler formula:

$$2p_4 - 3p_9 - \sum_{i \geq 30}(i-6)p_i = 12$$

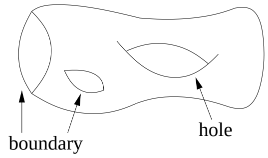
implies $p_4 > 0$ and the conclusion.

Assume that the vertex, say C_i, has degree 1. The connected component C_i is incident to the face F_j along a cycle of length h. Consider now the plane graph formed by C_i only. So, its faces are only 4-, 9- and i-gons with $i \geq 30$ and the h-gon. Euler formula then reads:

$$(6-h) + 2p_4 - 3p_9 - \sum_{i \geq 30}(i-6)p_i = 12.$$

This implies $p_4 > 0$ and the conclusion.

(ii) The proof for torus uses the same principle, as for spheres with additional complications. First, we cannot exclude, from the beginning, that several boundary sequences can be filled by the same map, which contains a hole, as shown below.



But if this happen, then one can consider the graph $Conn(G)$, whose vertex-set consists of connected components $C_1, \ldots, C_t$ of $E_1(G)$ and of "faces" $F_1, \ldots, F_s$ having several cycles. The graph $Conn(G)$ is a tree, since its homology is captured by the boundaries filled by 9-gons with a hole. So, the proof for spheres works just as well and one obtains that it cannot be a torus.

Suppose now, that one of the maps $E_1(G)$ is a torus. Then the graph $Conn(G)$ is a tree and the proof for spheres applies. So, this is impossible.

From now on, all connected components of the map $E_1(G)$ are plane graphs and the set of 9-gons is partitioned into generalized $(9,3)$-polycycles with one or more boundaries. The graph $Conn(G)$ is no longer a tree and the fact, that it is a torus, is encapsulated in the cycles of $Conn(G)$.

Denote by $DE(G)$ the set of directed edges of $Conn(G)$. Denote by $V(G)$ the vector space with canonical basis e_d and $d \in DE(G)$. For every cycle c of $Conn(G)$, choose an orientation of it and denote by $f(c)$ its representation in $V(G)$. Denote by $H(G)$ the vector space of $V(G)$, generated by all cycles c of $Conn(G)$. It is easy to see that the dimension of the homology group $H_1(G)$ of our map G is equal to $2dim(H(G))$. Since our map is a torus, we should have $dim(H(G)) = 1$. So, there is a single cycle in $Conn(G)$.

Suppose that $Conn(G)$ contains a vertex of degree 3 (either a component C_i, or a face F_j), then there is a vertex of degree 1, which should be a component $C_{i'}$ and so, we reach a contradiction by the same method as in the proof for spheres. So, vertices of $Conn(G)$ are of degree 2. This means that $Conn(G)$ is of the form:

$$\cdots - C_1 - F_1 - C_2 - \cdots - C_t - F_t - C_1 - \ldots$$

for some t.

Every connected component C_i is incident to two faces F_i and $F_{i+1 \pmod t}$ along cycles of length l_i and k_i (they are the number of $(5,3)$-polycycles E_1 present on those cycles). The Euler formula for the plane graph C_i reads:

$$(6 - l_i) + (6 - k_i) + 2p_4 - 3p_9 - \sum_{i \geq 30} (i-6)p_i = 12,$$

or, in other terms:

$$2p_4 = l_i + k_i + 3p_9 + \sum_{i \geq 30} (i-6)p_i.$$

If $p_4 > 0$, then we have the pattern $(0,1,0,1)$ and a contradiction is reached. So, $p_4 = 0$. This implies $l_i = k_i = 0$ and also $p_i = 0$. It means that C_i are just rings of $(5,3)$-polycycles E_2.

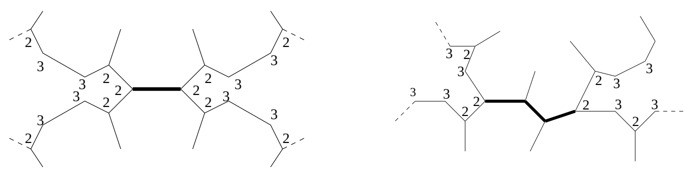
Now consider the generalized $(9,3)$-polycycles F_i with two boundary sequences of the form $(2223)^{h_i}$ and $(2223)^{g_i}$. As for the case of $(9,3)$-polycycles, there is an unique way of filling the faces on the boundaries. But the presence of two boundaries creates several complications, which were not present in the case of $(9,3)$-polycycles.

We use the same strategy, as for $(9,3)$-polycycles, whose boundary is of the form $b(\dots)$: we fill all faces, which are not incident. The first case is when the added 9-gonal faces do not share an edge in common. In that case the generalized $(9,3)$-obtained, obtained by filling previous boundaries, has two boundaries of the form $(323)^{h_i}$ and $(323)^{g_i}$. Denote by f_9 the number of interior 9-gons and by x the number of interior vertices. Application of proposition 1.4 yields:

$$\begin{cases} f_9 &= \frac{v_2-v_3}{3} = -\frac{h_i+k_i}{3}, \\ x &= \frac{2v_2-5v_3}{3} = -\frac{8}{3}(h_i+k_i), \end{cases}$$

which are both strictly negative, an impossibility.

The other case is when the added 9-gons share some edges in common. The possibilities are, locally, the following:



Let p be the number of appearances of such common sequence of edges. This means that the problem of filling the generalized $(9,3)$-polycycles with two boundaries is splitted into the problem of filling p $(9,3)$-polycycles given by their boundary sequences.

Those boundary sequences are of the form $(323)^{h_i}\alpha(323)^{k_i}\beta$ with α,β being equal to 223, 322 or 32223. By applying again Proposition 1.4, one obtains:

$$\begin{cases} f_9 &= \frac{v_2-v_3}{3} - 2, \\ x &= \frac{2v_2-5v_3}{3} - 6. \end{cases}$$

Those formulae give:

$$x = -\frac{8}{3}(h_i+k_i) + u_\alpha + u_\beta - 6$$

with u_α, u_β being equal to $\frac{-1}{3}$ or $\frac{-4}{3}$, according to the value of α or β. In any case, $x<0$ and this is impossible. So, there is no $(5,9)$-torus, which is $5R_3$. $\square$

In the case $q=10$, the only known symbolic sequences $(n_1,\dots,n_t)$, such that $b(n_1,\dots,n_t)$ is $(10,3)$-fillable, are, up to isomorphism:

t	symbolic sequences	f_{10}
5	$(0,0,0,0,0)$	1
6	$(0,1,0,1,0,1)$	3
9	$(0,1,1,0,1,1,0,1,1)$	6

Theorem 5.27 *The following $(5,10)$-spheres $5R_3$ exist:*

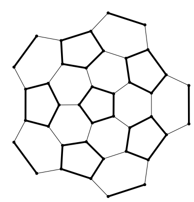
1. *A sphere with 140 vertices and symmetry I_h (also $10R_0$).*

2. A sphere with 740 *vertices and symmetry* I_h*.*

3. A sphere with 7940 *vertices and symmetry* I_h*.*

Proof. The first example can be seen on Figure 30. The second example is constructed using truncated Icosahedron (the smallest $(5,6)$-sphere with isolated pentagons): its set of edges is partitioned into two classes: edges $5-6$ (pentagon-hexagon) and edges $6-6$ (hexagon-hexagon). By assigning the value 0 to the first kind of edges and the value 1 to the second class, all pentagons have type (0^5) and all hexagons have type $(0,1,0,1,0,1)$; so, one obtains a sphere of symmetry I_h and 740 vertices.

The third example is obtained by taking the following 3-patch:

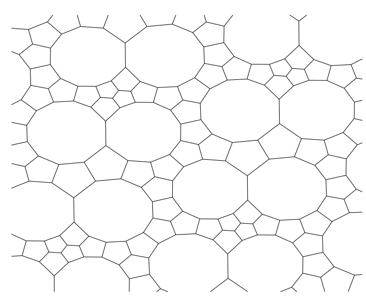


Overlined edges are assigned the value 0, while other edges are assigned the value 1 (this graph is an isometric subgraph of truncated Icosahedron). If one takes Icosahedron and substitute every 5-valent vertex with the above patch and glue along the open edges of value 0, then one obtains a graph with 5-, 6- and 9-gonal faces. The symbolic sequence of the 9-gonal faces is $(1,0,1)^3$. So, the structure can be filled. □

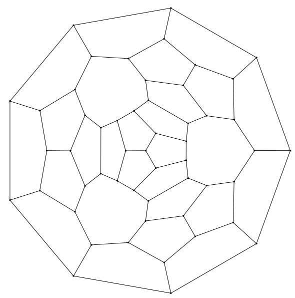
Conjecture 5.28 *A* $(5,10)$*-torus, which is* $5R_3$*, is also* $10R_2$*.*

See on Figures 43 and 44 some examples of $(5,10)$-tori, which are $10R_2$ but not $5R_3$.

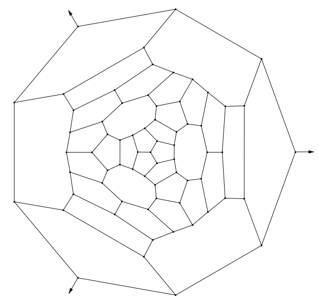
See below an example of a $(5,11)$-torus, which is $11R_1$ but not $5R_3$:



$(48,20,4)$, $p2$

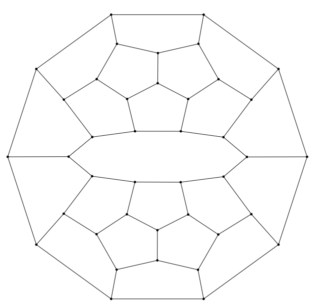


52, T_d

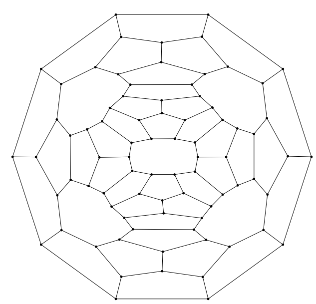


68, D_{3d}

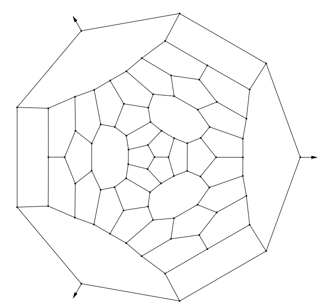
Figure 29: All weakly face-regular $(5, 9)$-spheres, which are $9R_0$ (2nd is a tripling of strictly face-regular Nr. 58)



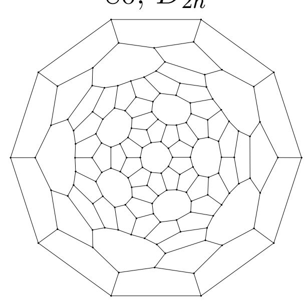
40, D_{2h}



80, D_{2h}

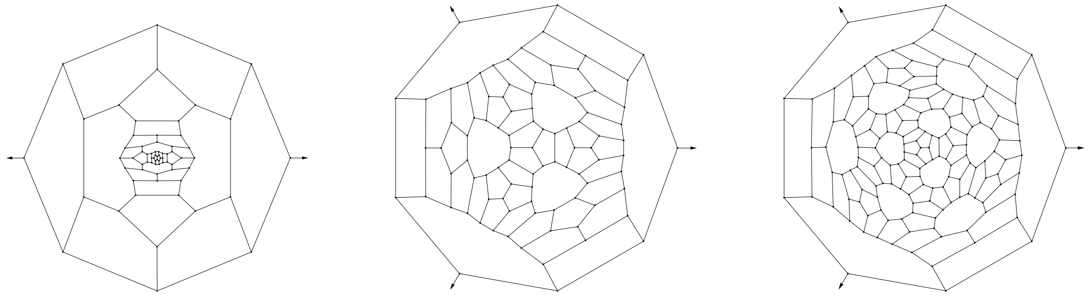


80, T_d



140, I_h (also $5R_3$)

Figure 30: All three weakly face-regular $(5, 10)$-spheres, which are $10R_0$, and the strictly face-regular 140-vertex $(5, 10)$-sphere $10R_0$ (3rd is a tripling of 2nd on above Figure)



68, D_{2d} 92, D_{3d} 164, T

Figure 31: All weakly face-regular $(5, 11)$-spheres, which are $11R_0$ (last two are triplings of, respectively, 2nd and 4th on above Figure)

6 Frank-Kasper spheres and tori

We call *Frank-Kasper-*$(5, q)$*-map* any $(5, q)$-map, which is qR_0 (in Chemistry and Crystallography Frank-Kasper polyhedra are just four polyhedra, dual to all four $(5, 6)$-polyhedra), which are $6R_0$.

A $(5, 3)$-polycycle is called 0-*elementary* if it is elementary and if its boundary sequence is of the form $223^{q_1} \dots 223^{q_m}$. By inspection of the list of finite elementary $(5, 3)$-polycycles (see Figure 1), one obtains that there are exactly three 0-elementary polycycles: E_1, C_1 and C_3.

The classification of all Frank-Kasper $(5, q)$-spheres can be done using following notions.

Fix a q-gonal face F; an edge e is said to be *pending to F* if it shares exactly one vertex with F. A *bridge* is an edge, which is pending to exactly two q-gonal faces.

Lemma 6.1 *Take a Frank-Kasper $(5, q)$-map, which is not $Barrel_q$. Then the set of all bridges, together with edges incident to q-gonal faces, establish a partition of the set of 5-gonal faces into 0-elementary $(5, 3)$-polycycles.*

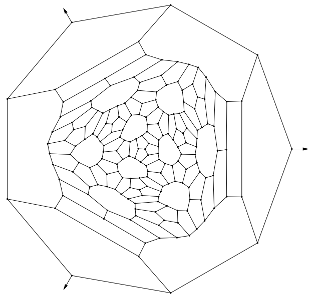
Proof. Suppose that a q-gonal face is not pending to any bridge. It means that this q-gonal face is bounded by two concentric rings of pentagonal faces, i.e., that the map is $Barrel_q$. So, the generalized $(5, 3)$-polycycles, appearing from the partition by bridges, are $(5, 3)$-polycycles. Those $(5, 3)$-polycycles are necessarily 0-elementary. □

Now, given a Frank-Kasper map, consider the map, defined by taking, as vertices, all 0-elementary $(5, 3)$-polycycles forming it. The q-gonal faces and bridges of the Frank-Kasper map correspond to faces and, respectively, edges, of this map. Then one removes vertices of degree 2 and obtain a 3-valent map, which will be called *major skeleton.*

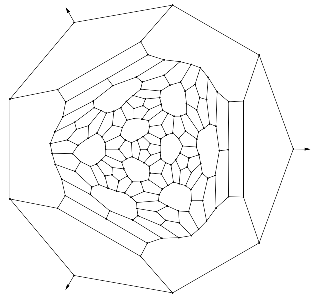
Theorem 6.2 *(i) There exist $(5, q)$-tori, which are qR_0, if and only if $q \geq 12$.*

(ii) $(5, 12)$-tori, which are $12R_0$, are also $5R_3$.

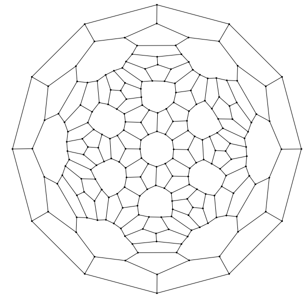
(iii) $(5, 13)$-tori, which are $13R_0$, are in one-to-one correspondence with $(3, 7)$-tori, which are $3R_0$ and $7R_6$.



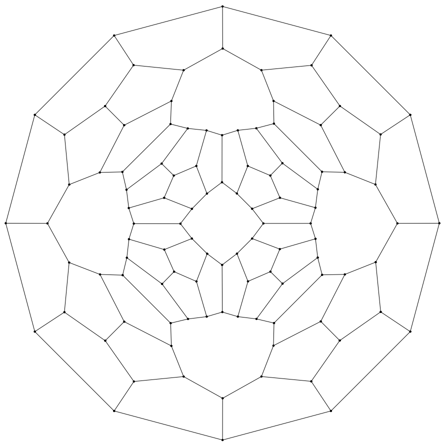
188, D_3



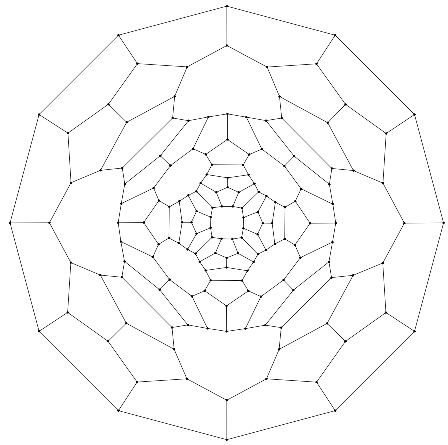
188, T_h



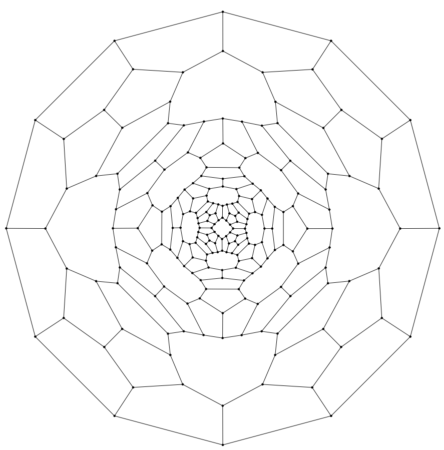
216, D_{2d}



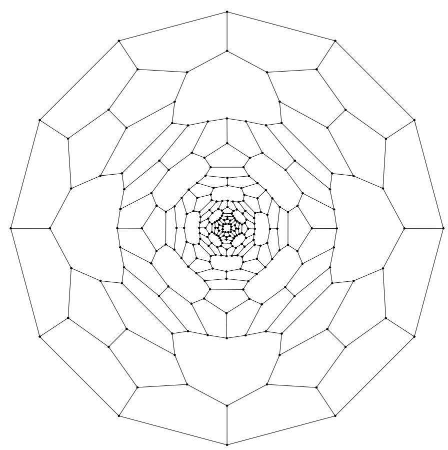
104, O_h, FK_0



160, D_{4d}, FK_1

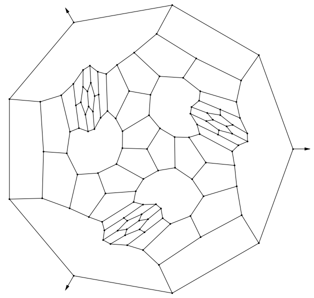


216, D_{4h}, FK_2

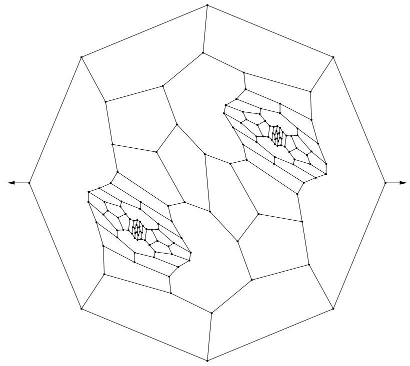


272, D_{4d}, FK_3

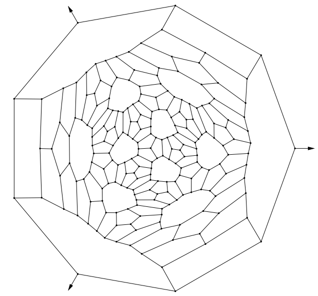
Figure 32: All weakly face-regular $(5, 12)$-spheres, which are $12R_0$: 3 sporadic cases and the series FK_i, illustrated here for $0 \leq i \leq 3$ (1st and 4th are triplings of, respectively, 3rd and 2nd on above Figure)



116, D_3

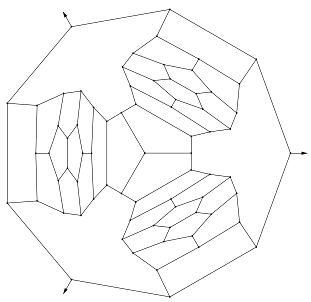


148, D_{2d}

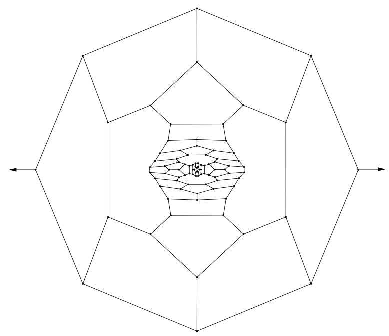


212, D_3

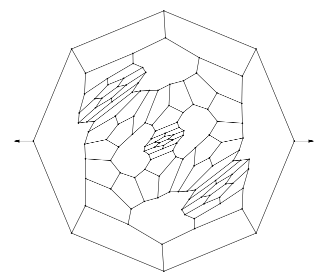
Figure 33: All two weakly face-regular $(5,13)$-spheres, which are $13R_0$ and have at most 148 vertices, and the one arising by tripling of 1st on above Figure



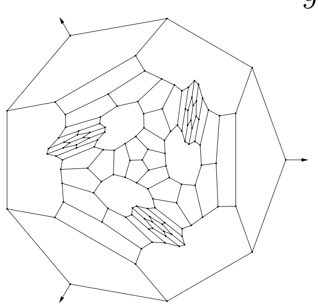
74, D_{3h}



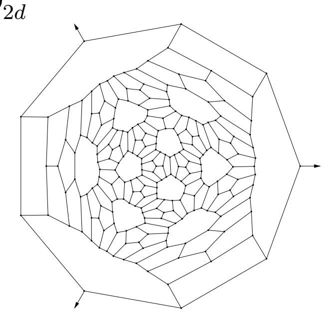
92, D_{2d}



128, C_2

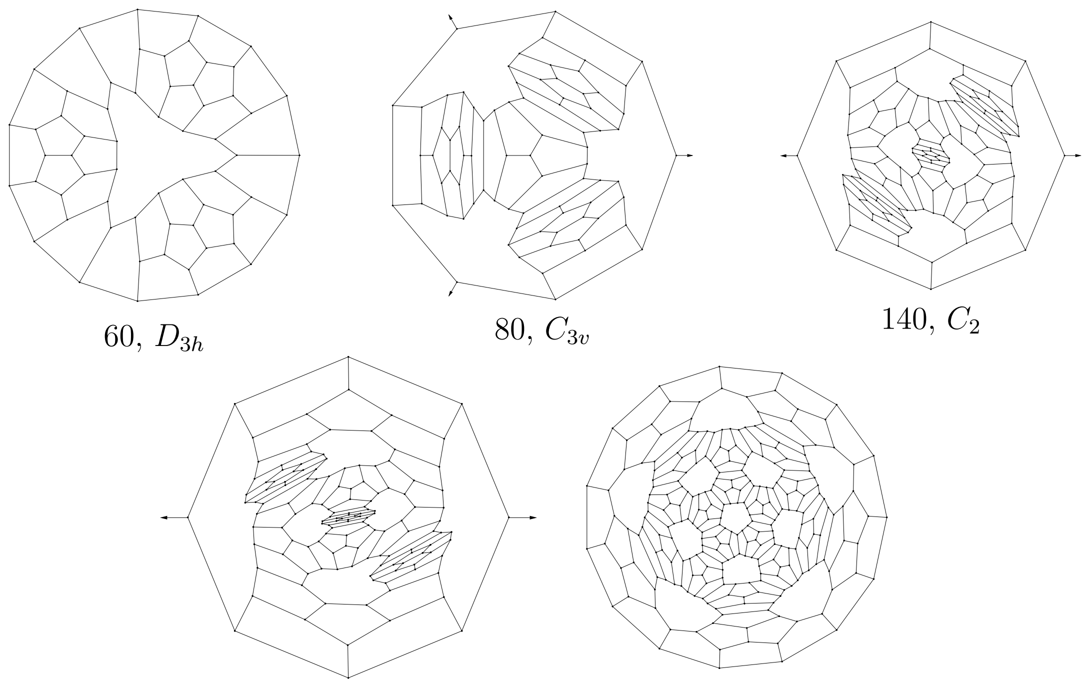


128, D_3



236, T

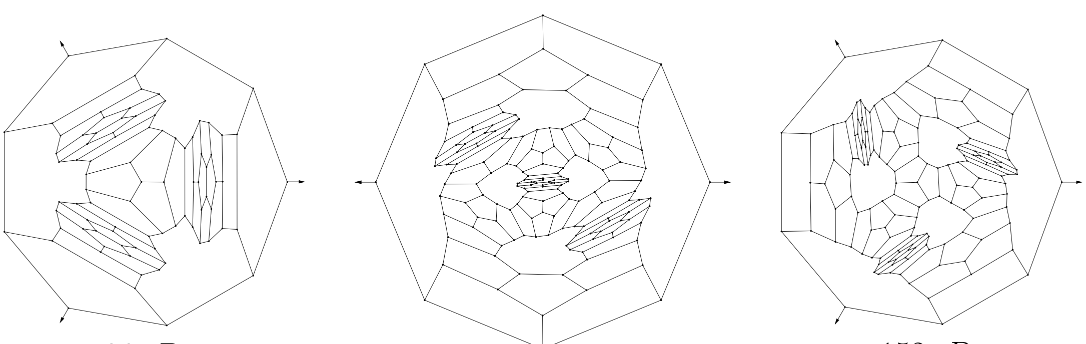
Figure 34: Both weakly face-regular $(5,14)$-spheres, which are $14R_0$ and have at most 100 vertices, and those arising by tripling (two triplings of 1st and a tripling of 3rd on above Figure)



60, D_{3h} 80, C_{3v} 140, C_2

140, C_2 260, I_h

Figure 35: Both two weakly face-regular $(5, 15)$-spheres, which are $15R_0$ and have at most 100 vertices, and those arising by tripling of last three on above Figure

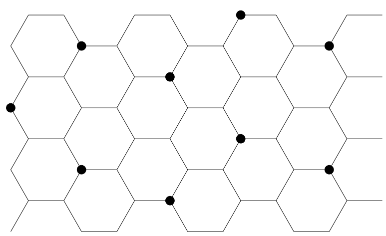


86, D_{3h} 152, C_2 152, D_3

Figure 36: Three weakly face-regular $(5, 16)$-spheres, which are $16R_0$ (1st is unique with at most 100 vertices); they are triplings of, respectively, 2nd, 3rd and 4th on above Figure

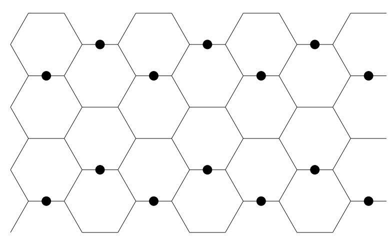
Proof. In order to prove the existence of $(5, q)$-tori, which are qR_0, for $q \geq 12$, it suffices to give example of periodic $(5, q)$-planes, which are qR_0.

Our basic example is the graphite lattice sheet, i.e., the 3-valent tiling $\{6^3\}$ of the plane by hexagons. At every vertex of this tiling, one can substitute a 0-elementary $(5, 3)$-polycycles, either E_1 or C_3. If one substitutes only E_1, we obtain a $(5, 12)$-plane, which is $12R_0$. In order to obtain a $(5, 13)$-plane, one needs to substitute a part of the E_1, by some C_3, such that every hexagon is incident to exactly one C_3. It is easy to see that this is, indeed, possible; see below an example of such a choice.



Furthermore, one can partition the set of vertices of the graphite lattice $\{6^3\}$ into 6 orbits O_i, such that every hexagon contains exactly one vertex in the orbit O_i. So, by putting C_3 into vertices of orbits O_1,...,O_i, one obtains $(5, 12+i)$-plane, which is $12+iR_0$ and again has a graphite-like structure.

If one inserts the $(5, 3)$-polycycle C_1 into edges of the graphite lattice according to the drawing below,



then from a $(5, q)$-plane obtained by the above procedure, one obtains a $(5, q + 5)$-plane, which is still $(q + 5)R_0$ and still have this graphite-like structure. This procedure can, obviously, be repeated; so, one gets the existence result for $q \geq 12$.

Assume $q \leq 11$. Given a $(5, q)$-plane, the gonality of a face in the major skeleton is equal to the number of $(5, 3)$-polycycles E_1 and C_3, to which it is incident. Clearly, there are at most 5 such incidences for each face. Since the major skeleton is 3-valent, we reach a contradiction by Euler formula and (i) holds.

(ii) If M is a $(5, 12)$-plane, which is $12R_0$, and if F is a 12-gonal face, then F is incident to 0-elementary $(5, 3)$-polycycles. Clearly, the gonality of F in the major skeleton is equal to the number of 0-elementary $(5, 3)$-polycycles E_1 and C_3, in which it is contained. So, this gonality is at most 6. But a tiling of the plane by faces, whose gonality is at most 6, is possible only if all faces have gonality 6, i.e., if all 12-gonal faces are adjacent only to polycycles E_1. Such a structure is unique and it is $5R_3$.

(iii) If M is a $(5,12)$-torus, which is $13R_0$, and if F is a 12-gonal face, then F is incident to 0-elementary $(5,3)$-polycycles. Clearly, the gonality of F in the major skeleton is equal to the number of 0-elementary $(5,3)$-polycycles E_1 and C_3, in which it is contained. So, this gonality is at most 6 and this corresponds to a face, which is incident to five polycycles E_1 and one polycycle C_3. Since a 3-valent torus with faces of gonality at most 6 is made of faces of gonality exactly 6; this means that all 13-gonal faces are adjacent to five polycycles E_1 and one polycycle C_3. So, the torus is described by the graphite lattice, together with a set $\mathcal{S}$ of marked vertices, where every hexagon is incident to exactly one vertex in $\mathcal{S}$.

On the other hand, if one take a strictly face-regular $(3,7)$-torus $3R_0$ and $7R_6$ and change the triangles to vertices, then one obtains exactly the same combinatorial object. $\square$

For $q = 14$, the major skeleton can have some faces of gonality 7 and 5; so, the above reduction is not possible.

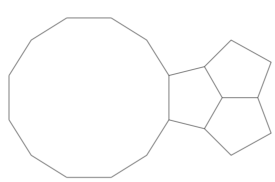
Theorem 6.3 *(i) For $q \leq 11$, the number of Frank-Kasper $(5,q)$-polyhedra is finite; they are:*

- *For $q = 6$, four classical ones,*
- *For $q = 7$, Dodecahedron and Barrel$_7$,*
- *For $q = 8$, Dodecahedron, Barrel$_8$ and strictly face-regular Nr. 58,*
- *For $q = 9$, 10 and 11, Dodecahedron, Barrel$_q$ and ones indicated on Figures 29, 30 and 31, respectively.*

(ii) For $q = 12$ (besides Dodecahedron and Barrel$_{12}$), three sporadic spheres and one infinite series $(FK_i)_{i\geq 0}$ with $104 + 56i$ vertices (the symmetry is O_h if $i = 0$, D_{4d} if i is odd and D_{4h}, otherwise), indicated on Figure 32.

Proof. Take a $(5,q)$-sphere G for $q \leq 11$. Then every face of the major skeleton is incident to at most 5 vertices corresponding to E_1 or C_3. So, the major skeleton is a 3-valent sphere with faces of gonality at most 5. There is a finite number of possibilities, which can be dealt with by computer and hence, we have (i).

If $q = 12$, then, by the same reasoning, the gonality of faces of the major skeleton is at most 6. Since it is a plane graph, this means that there exists a face of gonality at most 5. Such a face is incident to a vertex v, corresponding to a $(5,3)$-polycycle C_3 or C_1. So, the original $(5,12)$-sphere contains the pattern below.



We then run the enumeration procedure by taking the above patch as starting point. This enumeration procedure, after creating three spheres, indicated above, goes into a infinite loop, which creates the infinite sequence of maps; hence, (ii) follows. $\square$

Remark 6.4 *Given a* $(5,6)$*-polyhedron (i.e., a* fullerene*), one can generate a* $(5,12)$*-sphere in the following way:*

Take a set $\mathcal{S}$ *of vertices of* G*, such that no* 6*-gonal face is incident to an element of* $\mathcal{S}$ *and every pentagon is incident to exactly two vertices of* $\mathcal{S}$*. Then the map, obtained by substituting to elements of* $\mathcal{S}$ *a* $(5,3)$*-polycycle* C_3 *and to other vertices a* $(5,3)$*-polycycle* E_1*, is a Frank-Kasper* $(5,12)$*-sphere.*

Only two fullerenes admit such sets $\mathcal{S}$*:*

1. *Dodecahedron admits two such sets* $\mathcal{S}$*, up to isomorphism; they yield the polyhedra* 188*,* D_3 *and* 188*,* T_h*.*

2. *Barrel*$_6$ *admits only one such set, which yields the polyhedron* 216*,* D_{2d}*.*

Theorem 6.5 *All* $(5,q)$*-spheres with* $f_q=2$ *have* $4q$ *vertices.*

Besides Barrel$_q$*, all such spheres have* $q\equiv 0 \pmod 5$*; they are:*

(i) A sphere formed by splitting $q/5$ *Dodecahedra with one edge splitted into two edges and gluing them together; it is* qR_0 *and has symmetry* $D_{q/5h}$*.*

(ii) A sphere formed by splitting $q/5$ *Dodecahedra with one edge splitted into two half-edges and gluing them; it is* $qR_{q/5}$ *and has symmetry* $D_{q/5h}$*.*

7 $(5,q)$-spheres and tori, which are qR_1

Take a $(5,q)$-map, which is qR_1; then one can define, in the same way as for maps qR_0, the notion of *bridge*.

A $(5,3)$-polycycle is called 1-*elementary* if it is elementary and if its boundary sequence is of the form $2^{n_1}3^{m_1}\dots 2^{n_t}3^{m_t}$, where each n_i is 1 or 2 and, moreover, if $n_i=1$, then $n_{i-1}=n_{i+1}=2$. So, every 0-elementary $(5,3)$-polycycle is also 1-elementary. We obtain that the list of 1-elementary $(5,3)$-polycycles, which are not 0-elementary, consists of C_2, D and any E_{2n} with $n\geq 1$.

Theorem 7.1 *Given a* $(5,q)$*-map, which is* qR_1*, then the set of all bridges, together with edges, incident to* q*-gonal faces, establish a partition of the set of* 5*-gonal faces into* 1*-elementary* $(5,3)$*-polycycles.*

Proof. Suppose that a pair of adjacent q-gonal faces is not pending to any bridge. This means that those q-gons are bounded by two concentric rings of pentagonal faces, but, clearly, such a structure do not exist.

Hence, the decomposition of the set of 5-gonal faces by bridges creates only $(5,3)$-polycycles and no 3-patches with embedded pairs of q-gons inside. In order to prove that the elementary $(5,3)$-polycycles, which can appear, are only the 1-elementary ones, we will examine the list of elementary $(5,3)$-polycycles.

An admissible $(5,3)$-polycycle need to have the pattern 22 in its boundary sequence, since, otherwise, it would be bounded by a ring of q-gons, which are adjacent to at least two q-gons. This eliminate A_i, $1 \le i \le 5$.

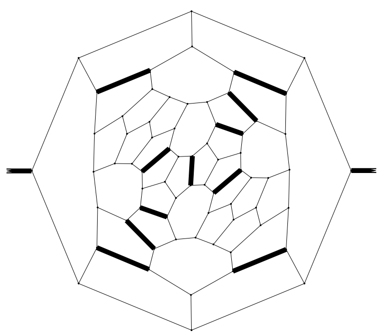
A pattern $3^{h_1}23^{h_2}23^{h_3}$ with $h_i \ge 1$ corresponds to a q-gon with $q = 2 + h_2$; we will prove that this is not possible. Clearly, the pattern $3^{h_1}23^{h_2}23^{h_3}23^{h_4}$ with $h_i \ge 1$ cannot appear, since it would imply that one of the q-gons of the pair has a vertex of degree 2, which cannot be matched by a polycycle. This eliminates B_3.

The $(5,3)$-polycycle B_2 is not possible, since the closure of the two vertices of degree 2 in 32323, would yield a q-gon with $q = 3$ and such structure do not exist.

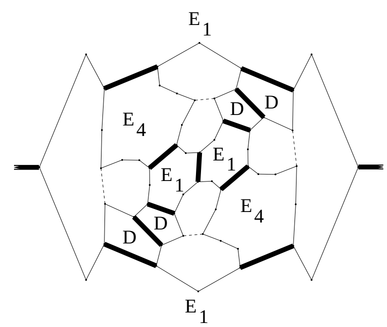
Now consider the case of $(5,3)$-polycycle E_{2n-1}; the closure of the two isolated vertices of degree 2, would yield a $(n+1)$-gon. But opposite side of the $(5,3)$-polycycle E_{2n-1} has the sequence 23^n2; so, after uniting with other $(5,3)$-polycycle, it would yield a q-gon with $q > n+1$, which is impossible. The only remaining admissible $(5,3)$-polycycles are the 1-elementary ones. □

All 1-elementary $(5,3)$-polycycles appear in decompositions of $(5,q)$-maps, which are qR_1.

See below an example of such a decomposition:



bridges of the
decomposition



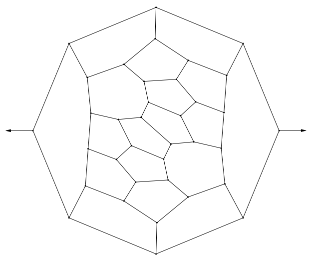
1-elementary $(5,3)$-polycycles of the
decomposition

Theorem 7.2 *(i) The number of* $(5,9)$*-spheres, which are* $9R_1$*, is finite.*

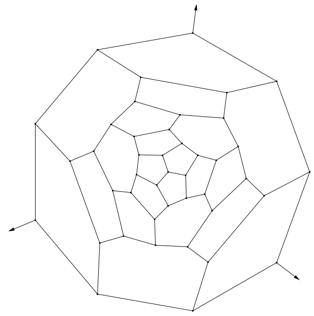
(ii) There are no $(5,q)$*-tori, which are* qR_1*, for* $q \le 9$*.*

Proof. By Theorem 4.4, one obtains $x_3 = 20$, where x_3 is the number of vertices contained in 3 pentagons. Since the gonality of faces is at most 9, the 1-elementary $(5,3)$-polycycles, forming its decomposition, are D, C_1, C_2, C_3, E_1, E_2, E_4, E_6, E_8, E_{10}, E_{12}. Denote by f_D, f_{C_1}, …, the number of those polycycles. By counting the number of interior vertices, one obtains:

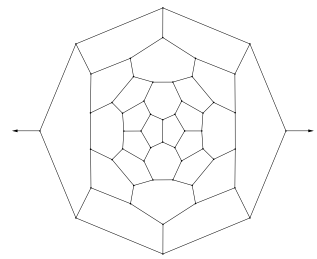
$$20 = 10f_{C_1} + 7f_{C_2} + 4f_{C_3} + f_{E_1} + \sum_{i=1}^{6} 2if_{E_{2i}} .$$



36, D_2

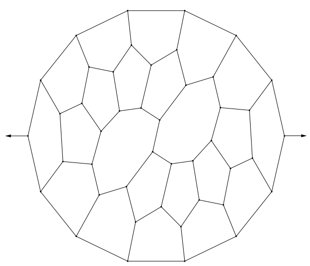


44, D_3

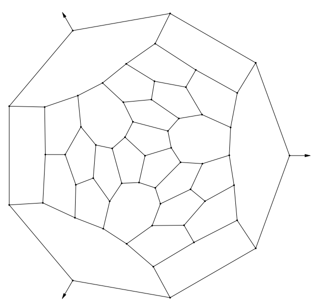


52, D_{2d}

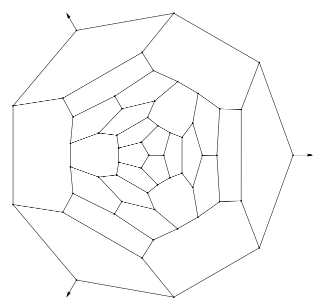
Figure 37: All weakly face-regular (5, 7)-spheres $7R_1$ (2nd is a tripling of weakly face-regular (5, 6)-sphere $32, D_3$, which is $6R_1$)



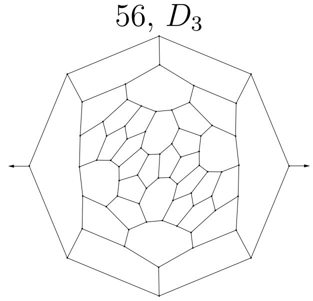
44, D_2



56, D_3



56, D_{3h}



68, D_2

Figure 38: All weakly face-regular (5, 8)-spheres, which are $8R_1$ and have at most 74 vertices (3rd is a tripling of strictly face-regular Nr. 56)

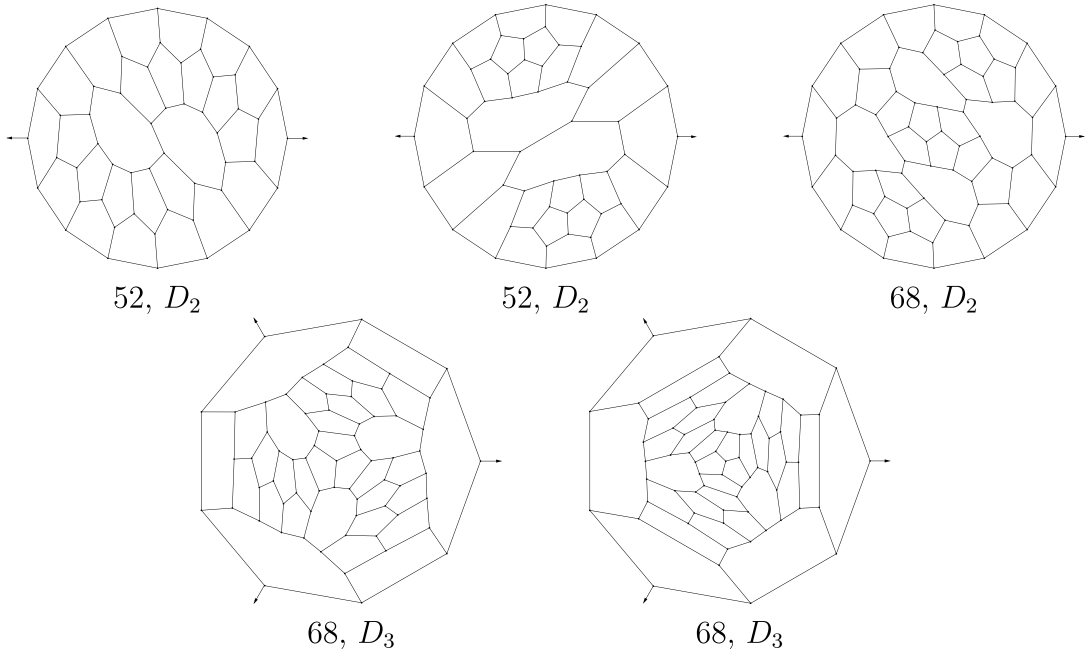


Figure 39: All weakly face-regular $(5,9)$-spheres, which are $9R_1$ and have at most 68 vertices (4th is a tripling of 2nd on above Figure)

On the other hand, one has the equations:

$$\begin{cases} e_{9-9} = \frac{1}{2}f_9 & = & \frac{1}{2}f_D + \frac{1}{2}f_{C_2} + \sum_{i=1}^{6} f_{E_{2i}}, \\ e_{5-9} = 8f_9 & = & 3f_D + 10f_{C_1} + 10f_{C_2} + 9f_{C_3} + 6f_{E_1} + \sum_{i=1}^{6}(6+2i)f_{E_{2i}}\ . \end{cases}$$

The two above inequalities imply:

$$e_{5-9} - 6e_{9-9} = 5f_9 = 7f_{C_1} + 10f_{C_2} + 9f_{C_3} + 6f_{E_1} + \sum_{i=1}^{6} 2if_{E_{2i}}\ .$$

It is clear, that the linear programming problem:

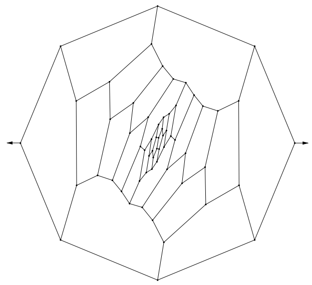
$$\begin{array}{rl} \textbf{maximize} & \sum_i a_i x_i \\ \textbf{subject to} & \sum_i b_i x_i = b, \\ & x_i \geq 0 \text{ with } a_i, b_i > 0 \end{array}$$

has the solution $b \max_i \frac{a_i}{b_i}$. Hence, one obtains $f_9 \leq 20\frac{6}{5} = 24$. So, $n \leq 212$ and (i) is true.

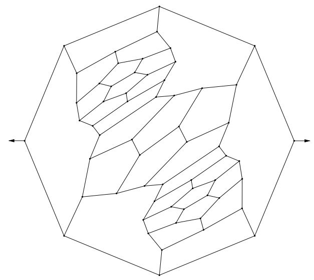
Let us consider (ii). First, one obtains by Theorem 4.4 $x_3 = 0$, then $f_{C_1} = f_{C_2} = f_{E_1} = f_{E_{2i}} = 0$. Subsequently, one obtains the relations:

$$e_{9-9} = \frac{1}{2}f_9 = \frac{1}{2}f_D, \quad \text{and} \quad e_{5-9} = 8f_9 = 3f_D;$$

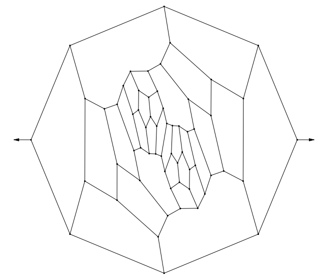
so, $f_9 = f_D = 0$. Hence, there are no $(5,9)$-tori, which are $9R_1$. If $q \leq 8$, then the proof comes directly from application of Theorem 4.4(ii). $\square$



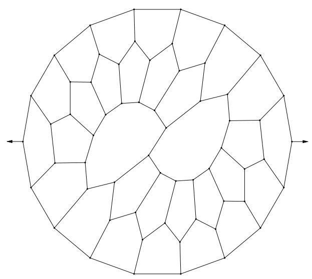
60, C_2



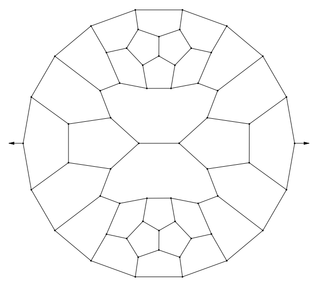
60, C_2



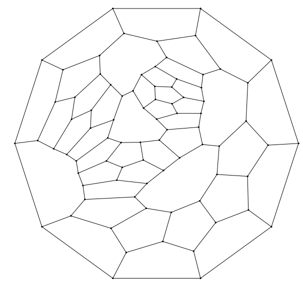
60, C_2



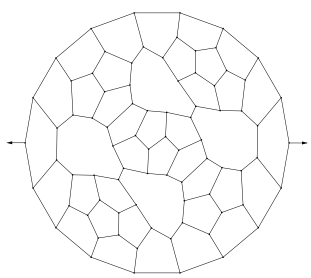
60, D_2



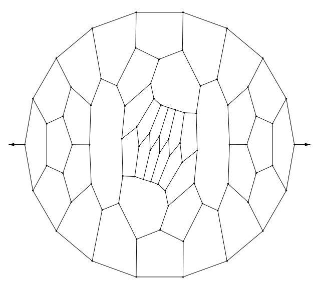
60, D_{2h}



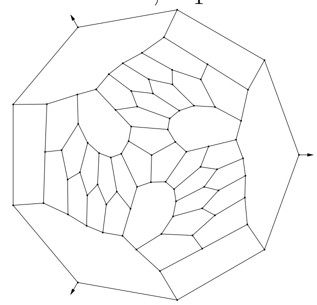
80, C_1



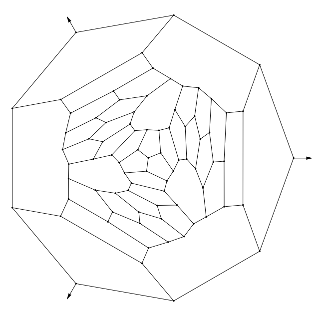
80, C_2



80, C_2



80, D_3



80, D_3

Figure 40: All weakly face-regular $(5,10)$-spheres, which are $10R_1$ and have at most 80 vertices (8th and 10th are triplings of, respectively, 3rd and 4th on above Figure)

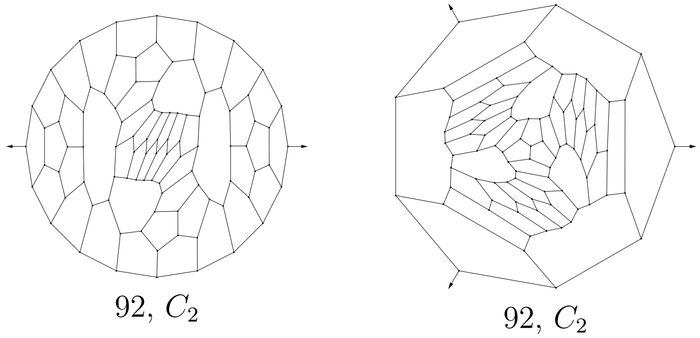


Figure 41: All weakly face-regular $(5, 11)$-spheres, which are $11R_1$ and arising by tripling from $(5, 10)$-spheres on above Figure

Recall that a special perfect matching of a 6-valent tessellation by triangles is a perfect matching such that every vertex is contained in exactly one vertex whose face opposite to it belongs to the perfect matching.

Theorem 7.3 *(i) The elementary* $(5, 3)$*-polycycles, appearing in the decomposition of a* $(5, 10)$*-torus, which is* $10R_1$*, are* D *and* E_1*.*

(ii) Every $(5, 10)$*-torus, which is* $10R_1$*, corresponds, in a following way, to a 6-valent tessellation of the torus by triangles, together with a special perfect matching:*

- 10*-gonal faces correspond to vertices,*
- $(5, 3)$*-polycycles* D *and* E_1 *correspond to triangular faces,*
- *bridges between* $(5, 3)$*-polycycles make one part of the edge-set, while edges between two* 10*-gons make the perfect matching part.*

Proof. A priori, the $(5, 3)$-polycycles, appearing in the decomposition of such a $(5, 10)$-torus are D, C_1, C_2, C_3, E_1 and E_{2i} with $1 \le i \le 7$. We use the same notation as in the preceding theorem. By Theorem 4.4, one has $x_3 = f_{10}$; so, in the same way as in the above theorem, one obtains:

$$\left\{\begin{array}{rcl} f_{10} = x_3 & = & 10f_{C_1} + 7f_{C_2} + 4f_{C_3} + f_{E_1} + \sum_{i=1}^{7} 2if_{E_{2i}}, \\ e_{10-10} = \frac{1}{2}f_{10} & = & \frac{1}{2}f_D + \frac{1}{2}f_{C_2} + \sum_{i=1}^{7} f_{E_{2i}}, \\ e_{5-10} = 9f_{10} & = & 3f_D + 10f_{C_1} + 10f_{C_2} + 9f_{C_3} + 6f_{E_1} + \sum_{i=1}^{7}(6+2i)f_{E_{2i}} \ . \end{array}\right.$$

Subtracting those equations, one obtains successively:

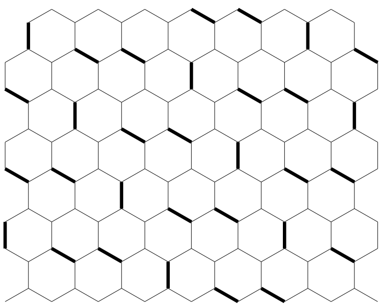
$$\left\{\begin{array}{rcl} e_{5-10} - 6e_{10-10} = 6f_{10} & = & 10f_{C_1} + 7f_{C_2} + 9f_{C_3} + 6f_{E_1} + \sum_{i=1}^{7} 2if_{E_{2i}}, \\ e_{5-10} - 6e_{10-10} - 6x_3 = 0 & = & -50f_{C_1} - 35f_{C_2} - 15f_{C_3} - \sum_{i=1}^{7} 10if_{E_{2i}} \ . \end{array}\right.$$

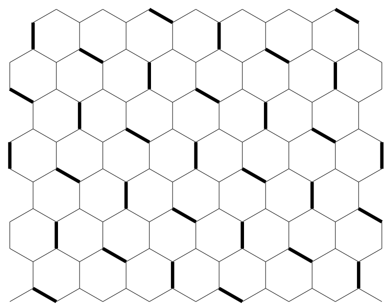
The second equation implies $f_{C_1} = f_{C_2} = f_{C_3} = f_{E_{2i}} = 0$; hence, (i) follows.

The above equalities yield $f_{10} = x_3 = f_{E_1} = f_D$. We say that a $(5, 3)$-polycycle is *incident* to a 10-gonal face if it shares a sequence of edges with it. A $(5, 3)$-polycycle D or E_1 is incident to exactly three 10-gonal faces. Hence, every 10-gonal face is incident to exactly three polycycles D and three polycycles E_1. Consider now the toric map, whose vertices are 10-gonal faces and faces are $(5, 3)$-polycycles D or E_1. Its edges are bridges, which are common to two adjacent elementary $(5, 3)$-polycycles, or the edges linking two polycycles D and separating two 10-gonal faces.

Clearly, this torus is 6-valent and the set of edges between two adjacent 10-gons define a special perfect matching. From this special perfect matching, one can get the position of polycycles D and E_1 and get the original $(5, 10)$-torus. So, (ii) follows. □

Remark 7.4 *In fact, there is a huge number of possibilities for perfect matchings. See on picture below two such possibilities.*

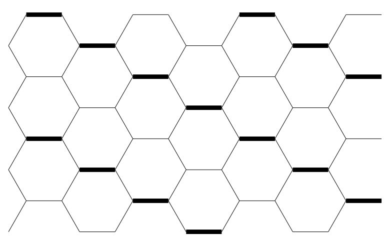




Remark also that Theorem 7.3 combined with Theorem 2.1 implies that there exist a one-to-one correspondence between $(5, 10)$*-torus* $10R_1$ *and* $(4, 8)$*-torus* $4R_1$ *and* $8R_5$.

Theorem 7.5 *For any* $q \geq 10$*, there exist a* $(5, q)$*-torus, which is* qR_1.

Proof. Such tori can be obtained as quotients of a $(5, q)$-plane. We will get again such a plane from the graphite lattice with added structure on it. On any vertex of the structure below, which is incident to an overlined edge, we put a $(5, 3)$-polycycle D, while on other vertices we put a $(5, 3)$-polycycle E_1. The obtained $(5, q)$-plane is $10R_1$.



Remark that the $(5, 3)$-polycycles D are adjacent between themselves by pairs; every pair of $(5, 3)$-polycycles D can be substituted by a $(5, 3)$-polycycle E_{2n} with $n \geq 1$. It is easy to see that the structure is a $(5, 10 + n)$-plane, which is $(10 + n)R_1$. □

Note also that there are many other possibilities for creating maps qR_1 (for example, substituting E_1 by C_3 or inserting $(5, 3)$-polycycles C_1).

Conjecture 7.6 *For any* $q \geq 10$*, there exist an infinity of* $(5, q)$*-spheres, which are* qR_1.

8 Maps qR_2

Remind that those maps were considered in [DDS04] on the sphere. However, the weakly face-regular case for plane tilings and tori was not considered there; all strictly face-regular (p,q)-planes were given in [De02].

Theorem 8.1 *A $(5,8)$-torus is $8R_2$ if and only if it is $5R_2$.*

Proof. Take a $(5,8)$-torus and assume that it is $8R_2$. One obtains, by Theorem 4.4(iii), the relation $x_0 + x_3 = 0$, i.e., $x_0 = x_3 = 0$.

Also, by Euler formula, one obtains $f_5 = 2f_8$. The total number of edges is equal to:

$$\frac{1}{2}(5f_5 + 8f_8) = 9f_8 \ .$$

Moreover, the property $8R_2$ implies $e_{8-8} = f_8$, $e_{5-8} = 6f_8$ and $e_{5-5} = 2f_8$.

Denote by $f_{5,i}$ the number of 5-gonal faces, which are adjacent to exactly i 5-gonal faces. By direct counting, it follows:

$$f_5 = \sum_{i\geq 0} f_{5,i} \quad \text{and} \quad 2f_8 = \frac{1}{2}\sum_{i\geq 0} i f_{5,i} \ .$$

Subtracting those two equations, one obtains $0 = \sum_{i\geq 0}(\frac{i}{2} - 1)f_{5,i}$. Suppose that $f_{5,i} \neq 0$ for $i > 2$. Then at least one vertex contained into exactly three 5-gonal faces, which is impossible. So, the equation reduces to:

$$0 = -f_{5,0} - \frac{1}{2}f_{5,1} \ .$$

This implies $f_{5,i} = 0$ for $i < 2$. So, our $(5,8)$-torus is strictly face-regular $5R_i$ with $i = 2$.

If the map is $5R_2$, then again Theorem 4.4(iii) implies $x_0 + x_3 = 0$, i.e., $x_0 = x_3 = 0$.

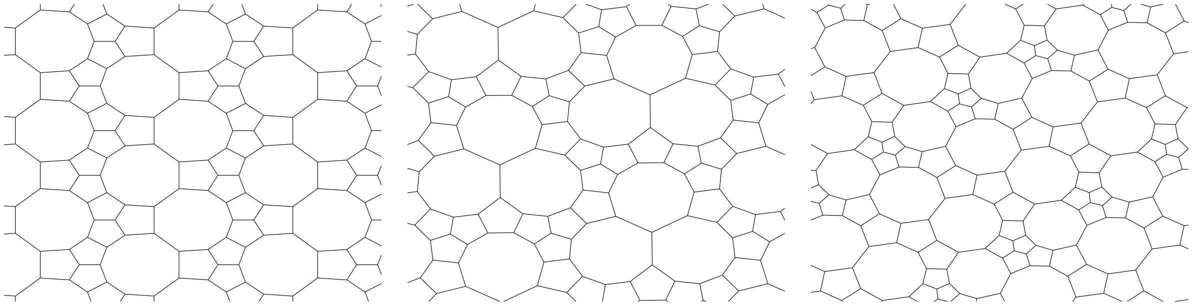
Also, by the same computation, one gets $e_{8-8} = f_8$, $e_{5-8} = 6f_8$ and $e_{5-5} = 2f_8$.

Take an 8-gonal face F; since $x_0 = x_3 = 0$, the corona sequence of F does not contains the pattern 88. Assume that F contains the pattern 858; by $5R_2$ property, the 5-gonal faces should have the corona 88855. This implies that $x_0 + x_3 > 0$.

So, the corona 858 is impossible and this implies that 8-gonal faces are adjacent to at most two 8-gonal faces. Since, on average, 8-gonal faces are adjacent to two 8-gonal faces, this implies property $8R_2$. $\square$

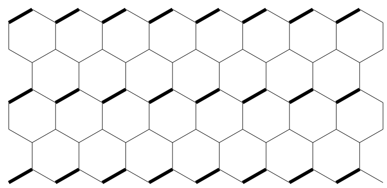
Theorem 8.2 *A $(5,q)$-torus qR_2 exists if and only if $q \geq 8$.*

Proof. Consider the graphite lattice and put a pentagon in every vertex of it. The obtained structure is, clearly, a $(5,8)$-plane, which is $8R_2$. In order to obtain $(5,q)$-planes with $q \geq 8$, one needs to modify the structure. The pentagons can be organized into pairs of adjacent ones. Every such pair, which is highlighted in the diagram below, can be changed into a $(5,3)$-polycycle E_{2n} with $n \geq 1$.



(16, 6, 2), $p2mg$ (24, 9, 3), $p31m$ (24, 9, 3), $p1$

Figure 42: Some (5, 9)-tori, which are $9R_2$



One obtains a $(5, 8+n)$-plane, which is $(8+n)R_2$. All above planes are periodic; hence, by taking the quotient (by a translation subgroup of their automorphism group), one obtains $(5, q)$-tori, which are qR_2.

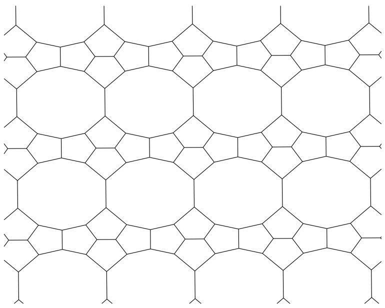
In order to prove the non-existence of $(5, q)$-torus with $q \leq 7$, it is sufficient to use Theorem 4.4(iii); it yields $f_q = 0$, which is impossible. □

Theorem 8.3 *For any $q \geq 8$, there exist an infinity of $(5, q)$-spheres, which are qR_2.*

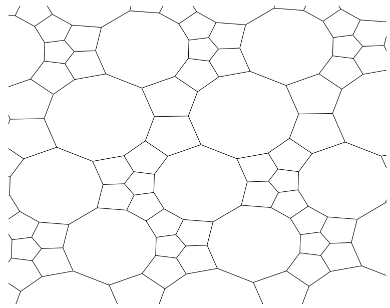
Proof. Take two Dodecahedra and remove one vertex from each of them. By merging the three pending edges, one obtains a $8R_2$ (5, 8)-sphere with three 8-gonal faces and 18 5-gonal faces partitioned into two (5, 3)-polycycles A_3. Conclusion follows from Theorem 1.5.

In order to prove the result for $q > 8$, one needs to find an initial graph, which is qR_2. For any $n \geq 0$, take three (5, 3)-polycycles E_{2n} (the polycycle "E_0" is the gluing of two (5, 3)-polycycles D) and glue them along their open edges. The resulting graph has two boundary sequences of the form $(23^{2+n})^3$; each could be filled by three $(6+n)$-gons by adding one vertex. Those two vertices can be truncated and replaced by (5, 3)-polycycle A_3. The resulting $(5, 9+n)$-sphere is $(9+n)R_2$ and one can apply Theorem 1.5, in order to get infiniteness. □

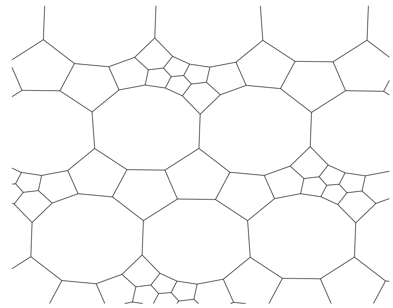
See on Figure 42 some (5, 9)-tori, which are $9R_2$. See on Figures 43 and 44 some (5, 10)-tori, which are $10R_2$.



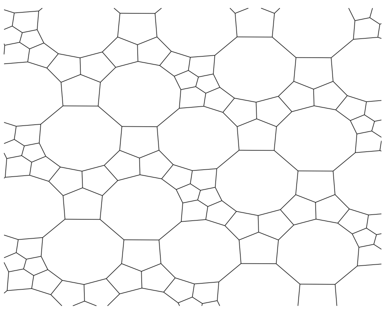
$(20, 8, 2)$, $p2mm$



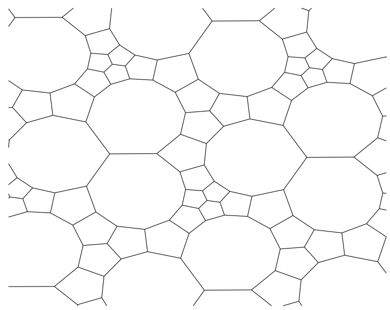
$(30, 12, 3)$, $p2$



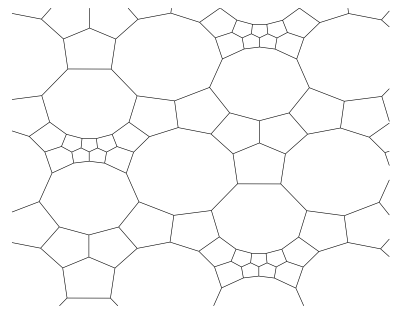
$(30, 12, 3)$, $p2$



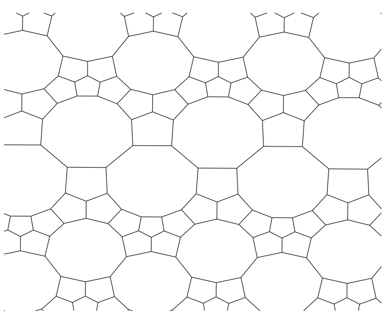
$(30, 12, 3)$, $p2$



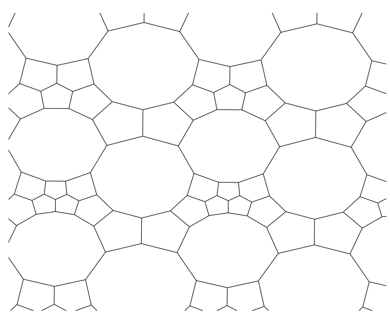
$(30, 12, 3)$, $p3$



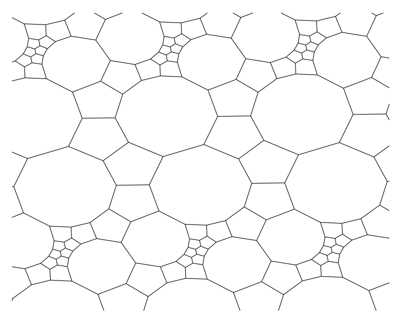
$(40, 16, 4)$, cm



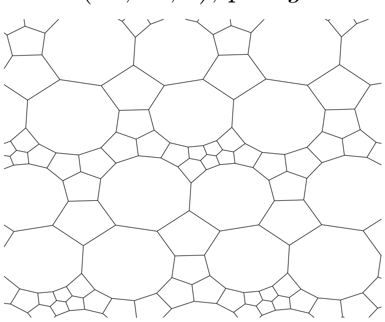
$(40, 16, 4)$, $p2mg$



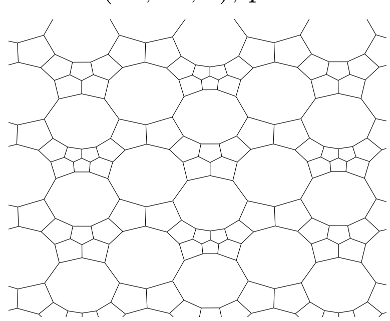
$(40, 16, 4)$, pm



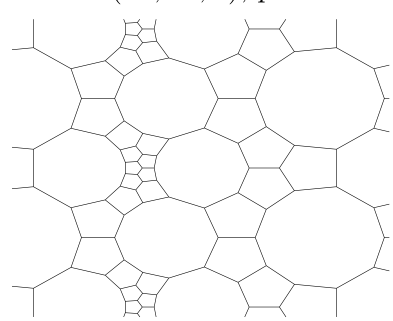
$(40, 16, 4)$, $p2$



$(40, 16, 4)$, $p2$

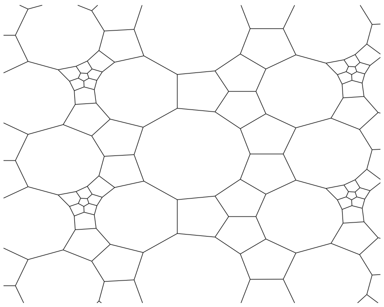


$(40, 16, 4)$, cm

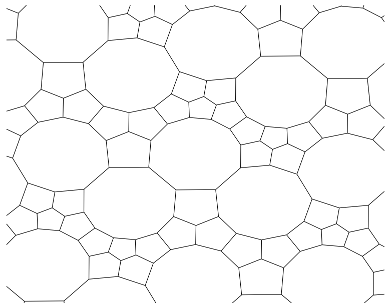


$(40, 16, 4)$, pm

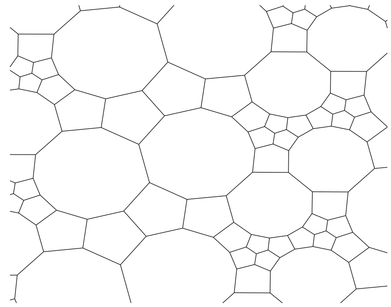
Figure 43: Some $(5, 10)$-tori, which are $10R_2$ (first part)



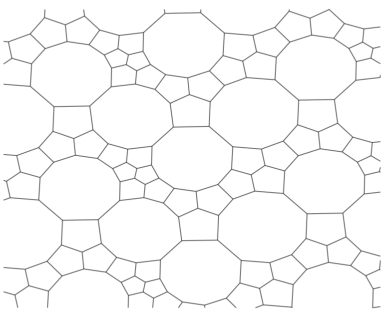
$(40, 16, 4)$, $p1$



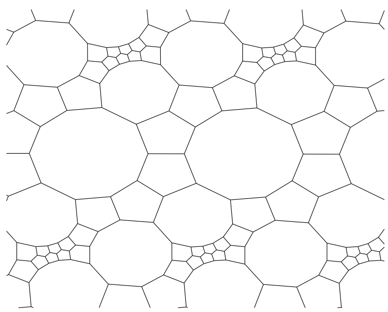
$(40, 16, 4)$, $p2$



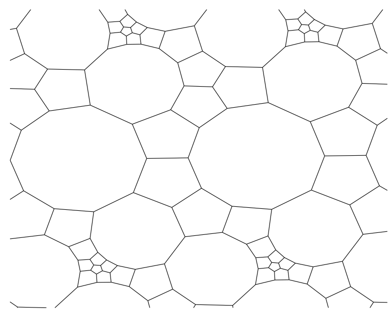
$(40, 16, 4)$, $p2$



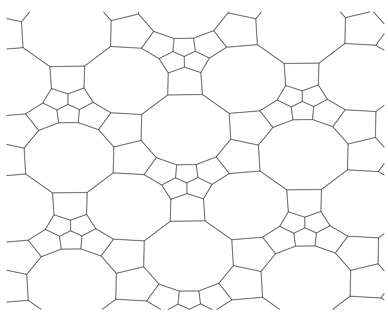
$(40, 16, 4)$, $p2$



$(40, 16, 4)$, $p2$



$(40, 16, 4)$, $p1$



$(40, 16, 4)$, $p2mg$

Figure 44: Some $(5, 10)$-tori, which are $10R_2$ (second part)

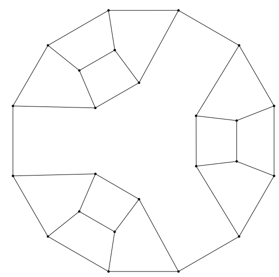
9 Maps qR_3

9.1 Classification of $(4,q)$-maps qR_3

Theorem 9.1 *(i) There is no $(4,q)$-torus, which is qR_3, for any q.*

(ii) The list of $(4,q)$-spheres, which are qR_3, consists of:

- *Strictly face-regular Nr. 35 and the following 2-connected $(4,12)$-sphere.*



24, D_{3h}

- *Unique sphere F_q, having two $(4,3)$-polycycles S_2 and two $(4,3)$-polycycles $P_{2(q-6)}$, if $q \geq 8$. It is of symmetry D_{2h} if $q=8$ and D_2 if $q>8$.*
- *Unique sphere G_q, having two $(4,3)$-polycycles S_1 and three $(4,3)$-polycycles $P_{2(q-5)}$, if $q \geq 7$. It is of symmetry D_{3h} if $q=7$ and D_3 if $q>7$.*
- *Unique sphere H_q, having two points incident to three q-gons and three $(4,3)$-polycycles $P_{2(q-3)}$, if $q \geq 5$. It is of symmetry D_{3h} if $q=5$ and D_3 if $q>5$.*
- *A family of spheres $K_{q,b}$, for $1 \leq b \leq q-5$, with two $(4,3)$-polycycles $P_{2(q-4)}$, two polycycles $P_{2(b+1)}$ and two polycycles $P_{2(q-3-b)}$. If $b=1$, then the symmetry is O_h if $q=6$ and D_4, otherwise. If $b>1$, then the symmetry is D_{2d} if $b=\frac{q-4}{2}$ and D_2, otherwise.*

Proof. (i) The set of 4-gonal faces of such a $(4,q)$-map qR_3 is splitted into $(4,3)$-polycycles, S_1, S_2, P_{2k} for $2 \leq k \leq 7$. Consider the graph $q(G)$. Since the original map is qR_3, this graph is 3-valent.

Every vertex, which is incident to three q-gonal faces, corresponds to a 3-gonal face of $q(G)$. Every $(4,3)$-polycycle S_1 also corresponds to a 3-gonal face. Every $(4,3)$-polycycle S_2 corresponds to a *digon*, (i.e., 2-gonal face). On the other hand, all P_{2k} correspond to 4-gonal faces. A 3-valent graph, whose faces have gonality at most 4, does not exist on torus and, clearly, has at most 8-vertices on the sphere.

(ii) Take a $(4,q)$-sphere G, consider its associated map $G'=q(G)$ and assume $q \geq 11$. G' is a 3-valent plane graph with faces of gonality at most 4.

There are exactly five such maps: Tetrahedron, *Bundle* (i.e., 3-valent 2-vertex graph with three 2-gonal faces), $Prism_2$, $Prism_3$ and Cube.

If G' is Tetrahedron, then its faces are all 3-gonal; hence, they all correspond to $(4,3)$-polycycles S_1 or to vertices. Clearly, in order for the face to be q-gonal, they should be all S_1 or all vertices. This correspond to strictly face-regular Nr. 35 or to Tetrahedron.

If G' is Bundle, then, clearly, the graph is the one, indicated above.

If G' is $Prism_2$, then its face-set consists of two digons and two 4-gons. Hence, the map G is formed of two $(4,3)$-polycycles S_2, one $(4,3)$-polycycle P_{2k} and one $(4,3)$-polycycle $P_{2k'}$. Given a polycycle P_{2k}, which is adjacent to the q-gonal faces F_1, F_2, F_3 and F_4, the sequence of $k-1$ 4-gons can be adjacent either to F_1 and F_3, or to F_2 and F_4. Hence, we need to fix the orientations of the $(4,3)$-polycycles P_{2k}.

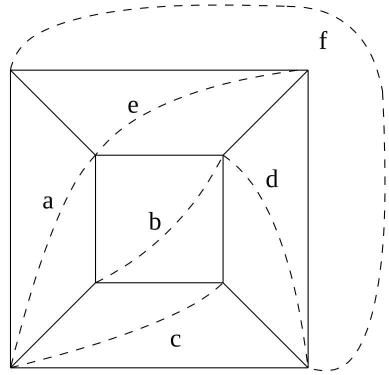
On every 4-gon, there are two possibilities for orienting the polycycles P_{2k}. So, one has a total of four possibilities and, after reducing by isomorphism, two possibilities. One possibility corresponds to a q-gonal faces having corona $4q4q444q$, which is impossible. So, the orientation of 4-gonal faces should be done in such a way, that the coronae are of the form $4^{q-7}q4q444q$; hence, $k=k'$ and we obtain the sphere F_q.

If G' is $Prism_3$, then two 3-gonal faces of G' correspond, in the sphere G, to S_1 or vertices, incident only to q-gonal faces. Three 4-gonal faces correspond to the $(4,3)$-polycycles P_{2k_i} with $1 \leq i \leq 3$. We must find the values of k_i and the orientations of those polycycles.

Denote by $(F_j)_{1\leq j\leq 3}$ and $(F'_j)_{1\leq i\leq 3}$, two cycles of faces of length 3. Let us consider the faces F_i. Their boundary sequences are of the form, either $q\alpha q4q4$, or $q\alpha q4^{k_i-1}q4$, or $q\alpha q4^{k_i-1}q4^{k_{i'}-1}$. Here α is void, if the faces F_i are incident to a common vertex, and α=44, if the faces F_i are adjacent to a common $(4,3)$-polycycle S_1. Clearly, the first pattern, i.e., $q\alpha q4q4$, is impossible, since it implies gonality 5 or 7. So, the pattern $q\alpha q4^{k_i-1}q4^{k_{i'}-1}$ is also impossible and the faces F_i have their boundary of the form $q\alpha q4^{k_i-1}q4$. This implies $k_1=k_2=k_3$.

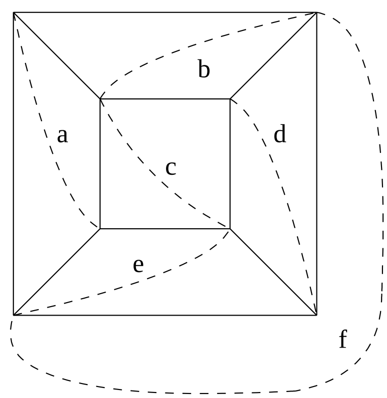
For faces F'_i, one obtains, by the same argument, that their boundary is of the form $q\alpha' q4^{k_i-1}q4$ with α' being void for a vertex and equal to 44 for a polycycle S_1. So, one has $\alpha=\alpha'$, i.e., either we have two vertices, or we have two $(4,3)$-polycycles S_1. We obtain the series G_q and H_q.

Let us assume now that G' is Cube. By the previous analysis, all 4-gonal faces are organized in $(4,3)$-polycycles P_{2k}. We need to fix the orientation on every 4-gonal face. Since there are two choices for every one of the six faces, this makes a total of 64 choices. Every q-gonal face should be incident to at least one sequence of 4-gons. This reduces to 22 cases. Under symmetry, this reduces to just 3 cases. The first case is depicted below.



The letters correspond to the length of the $(4,3)$-polycycles in the following way: a corresponds to $P_{2(a+1)}$. Every vertex is incident to three 4-gonal faces. So, three dotted paths appear. If two of those paths are incident to the vertex, i.e., if the corresponding $(4,3)$-polycycle is incident to an isolated 4-gon to the q-gonal face, then the length of the last dotted path is set. It gives $b=e=f=c=q-4$. Consider now the case of vertex being incident to just one dotted path. By previous assignment, one gets the length $a=d=1$. This map is $K_{q,1}$.

The second case is depicted below.

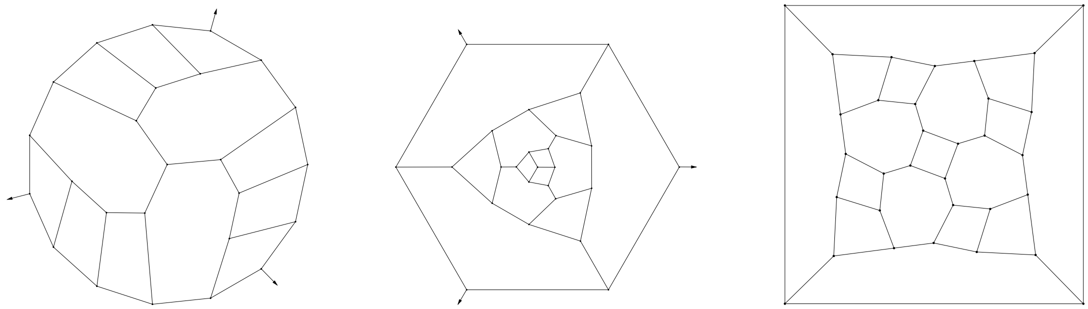


By the same argument, one gets $a=d=q-5$. This reasoning for other vertices yields the equations:

$$c+e=c+b=b+f=e+f=q-4.$$

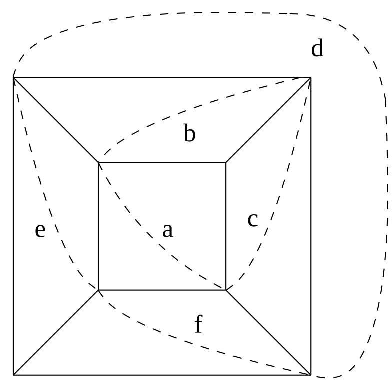
It yields $e=f$, $c=f$ and $c=q-4-b$. The corresponding map is $K_{q,b}$.

The third case is depicted below.



26, D_3, (H_7) 26, D_{3h}, (G_7) 32, D_4, $(K_{7,1})$

Figure 45: All weakly face-regular $(4, 7)$-spheres, which are $7R_3$



We obtain $a = b = c = d = e = f = q - 5$. Two remaining vertices, which are not incident to any dotted path, yield the equation $3(q - 5) = q - 3$, i.e., $q = 6$, which is already covered. □

See on Figures 45, 46, 47 and 48 the lists of weakly face-regular $(4, q)$-spheres, which are qR_3 for q=7, 8, 9 and 10, respectively.

9.2 $(5, q)$-maps qR_3

Theorem 9.2 *(i) A $(5, 7)$-torus is $7R_3$ if and only if it is $5R_1$.*

Moreover, corresponding $(5, 7)$-plane, which is $5R_1$, belongs to the Case 17 from Table 4.

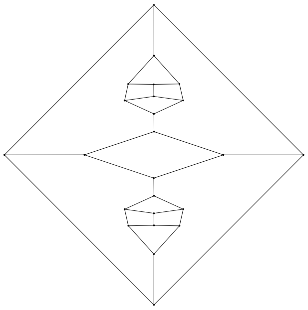
(ii) A $(5, 7)$-sphere, which is $7R_3$, satisfies to $x_0 + x_3 = 20$, where x_i denotes the number of vertices incident to i 5-gonal faces.

Proof. Assume first that the torus is $7R_3$. One has the standard relations:

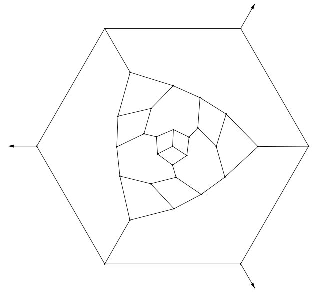
$$f_5 = f_7 \text{ and } e = 6f_7 .$$

Furthermore, the property $7R_3$ yields:

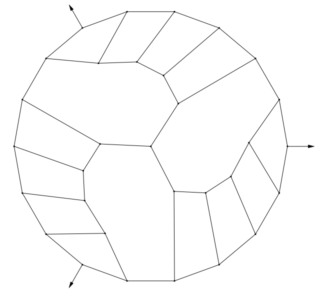
$$e_{7-7} = \frac{3}{2} f_7, \quad e_{5-7} = 4f_7 \text{ and } e_{5-5} = \frac{1}{2} f_7 .$$



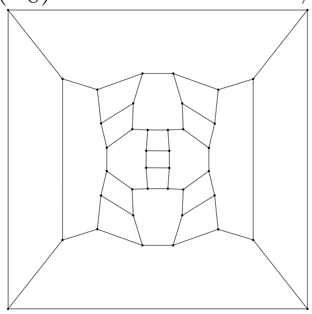
24, D_{2h}, (F_8)



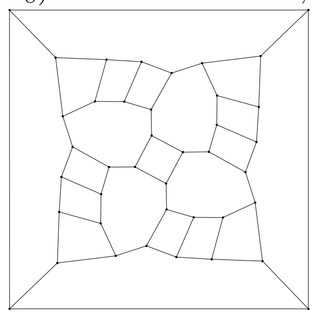
32, D_3, (G_8)



32, D_3, (H_8)

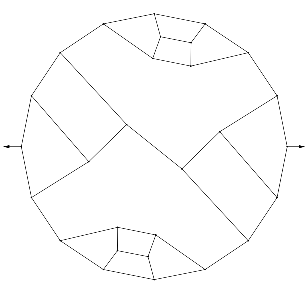


40, D_{2d}, $(K_{8,1})$

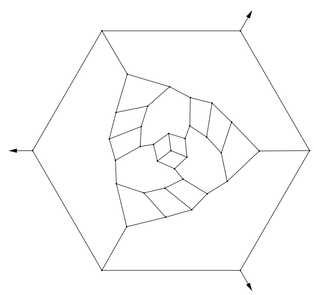


40, D_4, $(K_{8,2})$

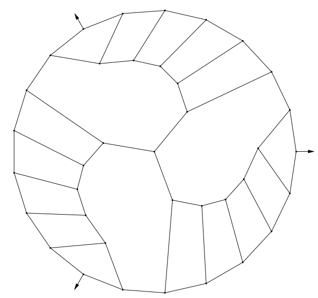
Figure 46: All weakly face-regular $(4, 8)$-spheres, which are $8R_3$



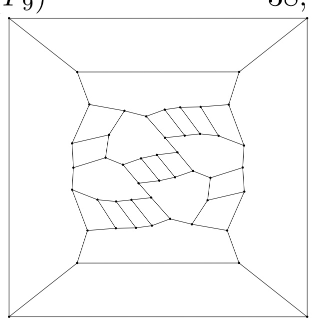
28, D_2, (F_9)



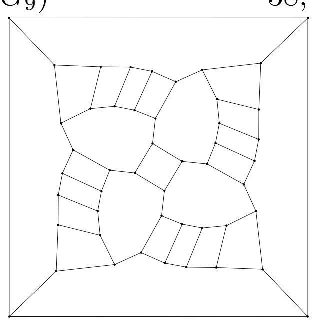
38, D_3, (G_9)



38, D_3, (H_9)

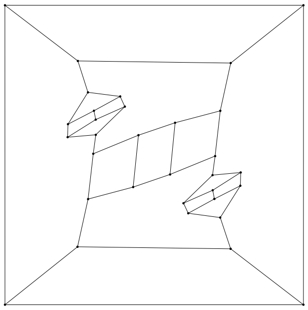


48, D_2, $(K_{9,2})$

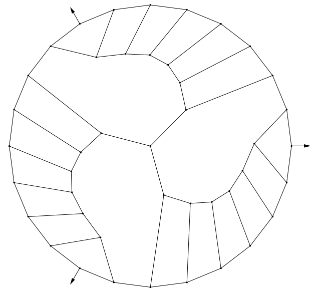


48, D_4, $(K_{9,1})$

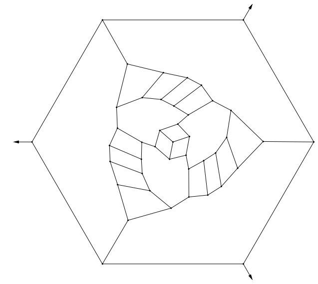
Figure 47: All weakly face-regular $(4, 9)$-spheres, which are $9R_3$



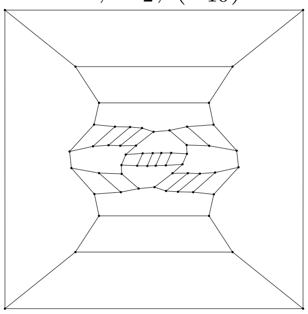
32, D_2, (F_{10})



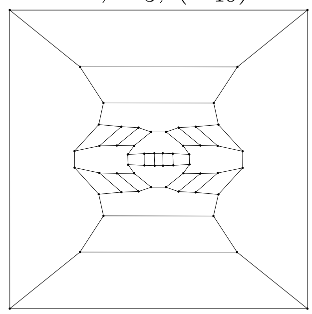
44, D_3, (H_{10})



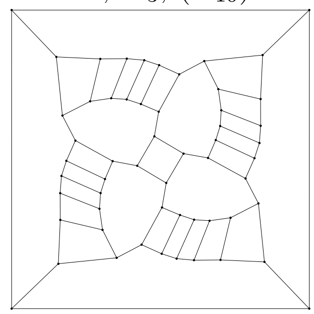
44, D_3, (G_{10})



56, D_2, $(K_{10,2})$



56, D_{2d}, $(K_{10,3})$



56, D_4, $(K_{10,1})$

Figure 48: All weakly face-regular $(4, 10)$-spheres, which are $10R_3$

Then, by expressing above numbers in terms of x_i, one obtains:

$$2e_{7-7} = 3x_0 + x_1, \quad e_{5-7} = x_1 + x_2 \text{ and } 2e_{5-5} = x_2 + 3x_3 \; .$$

By combining above equalities, we get the equality:

$$0 = 3f_7 + f_7 - 4f_7 = 2e_{7-7} - e_{5-7} + 2e_{5-5} = 3x_0 + 3x_3,$$

which yields $x_0 = x_3 = 0$ and so, $x_1 = 3f_7$, $x_2 = f_7$. Denote by $f_{5,k}$ the number of 5-gonal faces, which are adjacent to exactly k 5-gonal faces. Again, by easy counting, one obtains:

$$x_1 = 5f_{5,0} + 3f_{5,1} + f_{5,2} \text{ and } 2x_2 = 2f_{5,1} + 4f_{5,2} \; .$$

Since $x_2 = f_7 = f_5 = f_{5,1} + f_{5,2}$, we get $f_{5,2} = 0$, which imply $f_{5,0} = 0$ and so, our torus is $5R_1$.

On the other hand, if the torus is $5R_1$, then it holds, by the same computation, $x_0 = x_3 = 0$, $x_1 = 3f_7$ and $x_2 = f_7$.

Now, denoting by $f_{7,k}$ the number of 7-gonal faces, which are adjacent to exactly k 7-gonal faces, one obtains:

$$x_2 = 7f_{7,0} + 5f_{7,1} + 3f_{7,2} + f_{7,3} \text{ and } 2x_1 = 2f_{7,1} + 4f_{7,2} + 6f_{7,3} \; .$$

Hence, we get:

$$\begin{aligned} 0 &= 6f_7 - 2x_1 \\ &= 6(f_{7,0} + f_{7,1} + f_{7,2} + f_{7,3}) - (2f_{7,1} + 4f_{7,2} + 6f_{7,3}) \\ &= 6f_{7,0} + 4f_{7,1} + 2f_{7,2} \; . \end{aligned}$$

Therefore, $f_{7,0} = f_{7,1} = f_{7,2} = 0$ and the torus is $7R_3$.

In the case of spheres, the proof is very similar; we only indicate the different formulae:

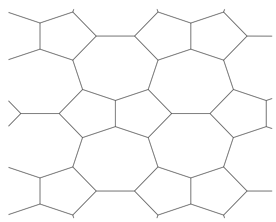
$$f_5 = 12 + f_7, \quad \text{and} \quad e = 30 + 6f_7 \ ,$$

which yields $3x_0 + 3x_3 = 2e_{7-7} - e_{5-7} + 2e_{5-5} = 60$. □

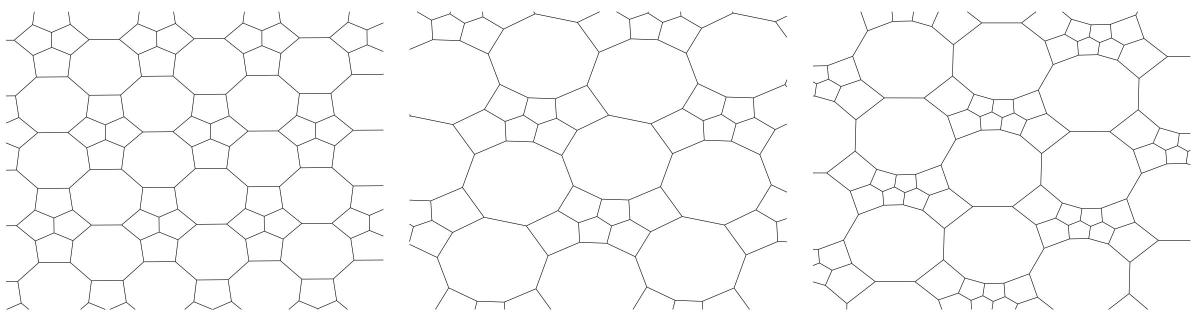
Moreover, the corona argument, as on Figure 2 in [DFSV00], excludes corona 5575577 and a variation of 5575757 (only three variations of last corona are possible). The remaining of the above theorem is proved in [DFSV00].

Theorem 9.3 *For any $q \geq 7$, there exist a $(5,q)$-torus, which is qR_3.*

Proof. Take the following picture of a $(5,7)$-torus, $5R_1$



and replace every one of the pair of adjacent pentagons by an elementary $(5,3)$-polycycle E_{2n}. The obtained $(5,7+n)$-torus is $(7+n)R_3$. See below the result for $q = 8$, 9 and 10.
□



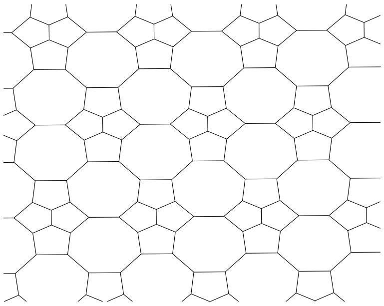
Conjecture 9.4 *For any $q \geq 7$, there exist an infinity of $(5,q)$-spheres, which are qR_3.*

Theorem 9.5 *There exist an infinity of $(5,q)$-spheres, which are qR_3, for $q = 9$, 10 and* 12.

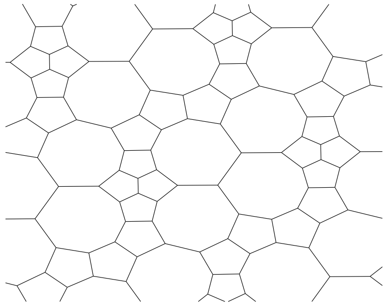
Proof. Theorem 1.5 can be applied to the first sphere on Figure 86. By truncating the two opposite vertices of the 3-fold axis of the second sphere of Figure 86 and filling those truncations by $(5,3)$-polycycles A_3, one obtains a $(5,12)$-sphere, which is $12R_3$ and for which Theorem 1.5 can be applied.

Theorem 1.5 can be applied also to the first sphere on Figure 89. □

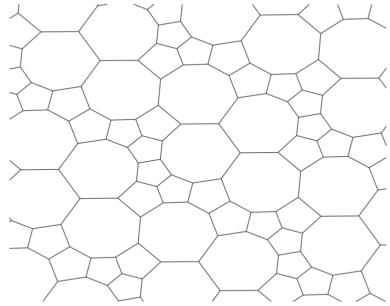
See on Figures 49, 50 and 51 some $(5,8)$-tori, which are $8R_3$. See on Figures 52, 53 and 54 some $(5,q)$-tori, which are qR_3 for $q = 9$ or 10.



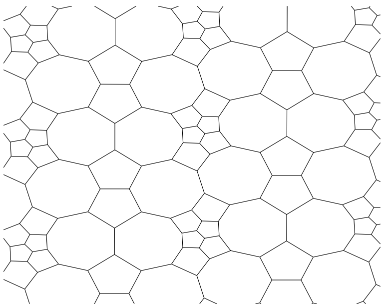
$(12,4,2)$, $c2mm$



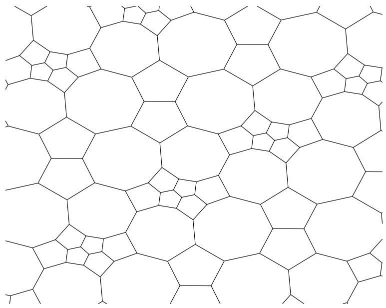
$(24,8,4)$, $p2$



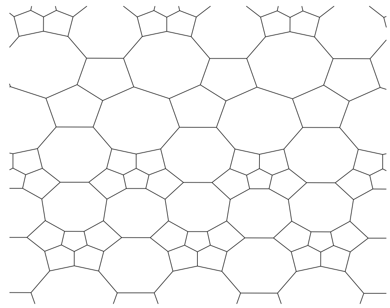
$(24,8,4)$, $p2$



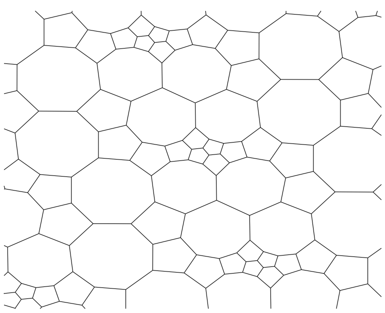
$(24,8,4)$, $p2$



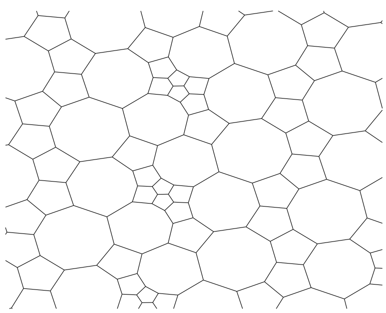
$(24,8,4)$, $p2$



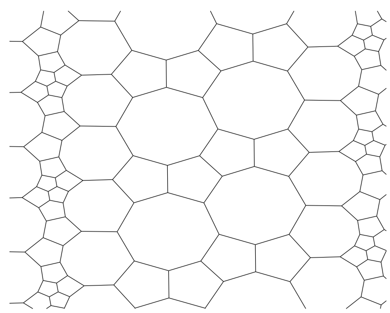
$(36,12,6)$, $p2$



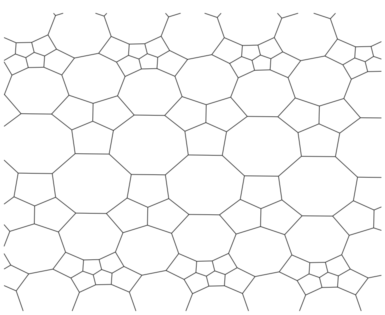
$(36,12,6)$, $p2$



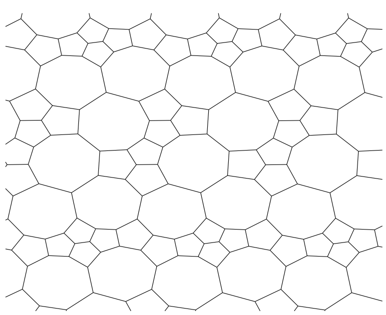
$(36,12,6)$, $p2$



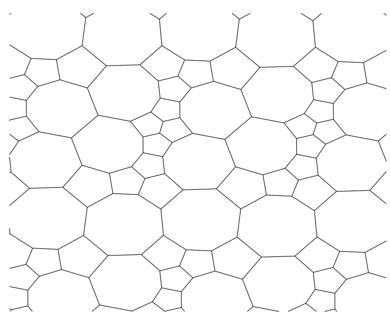
$(36,12,6)$, $p2$



$(36,12,6)$, $p2$

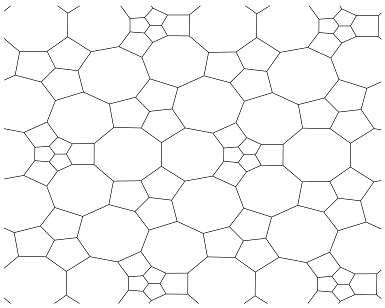


$(36,12,6)$, $p2$

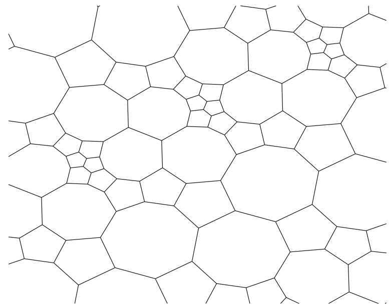


$(36,12,6)$, $p2$

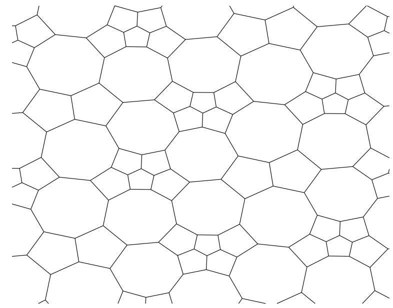
Figure 49: Some $(5,8)$-tori, which are $8R_3$ (first part)



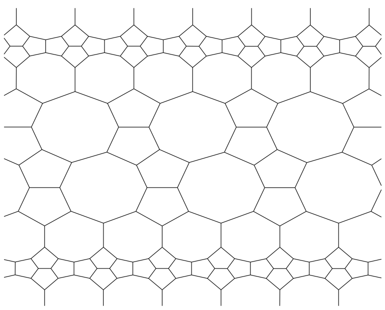
$(36, 12, 6)$, $p31m$



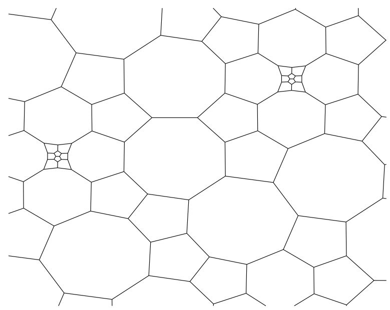
$(36, 12, 6)$, $p2$



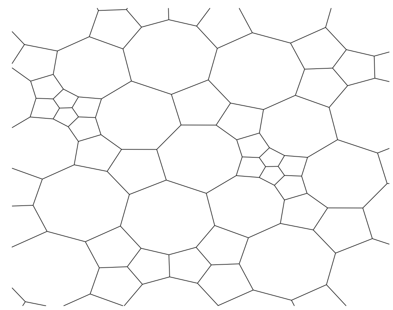
$(36, 12, 6)$, $p2$



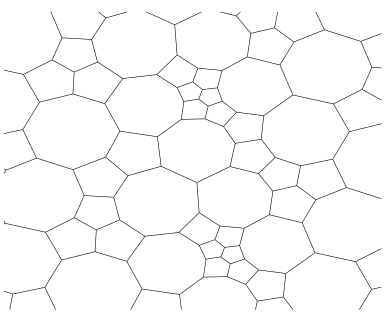
$(36, 12, 6)$, $p2$



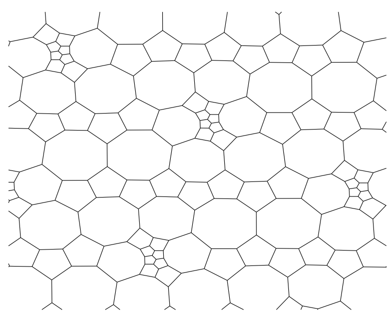
$(48, 16, 8)$, $p2$



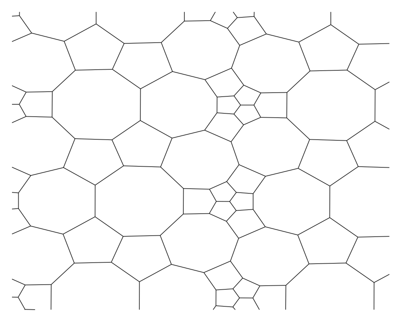
$(48, 16, 8)$, $p2$



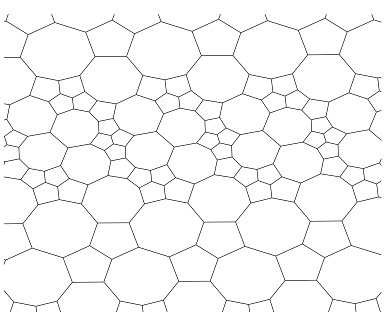
$(48, 16, 8)$, $p1$



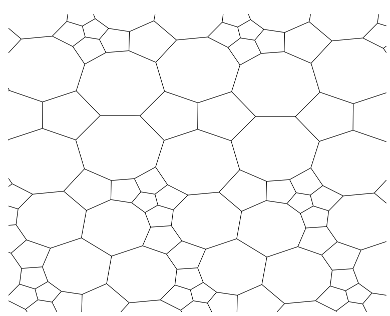
$(48, 16, 8)$, $p2$



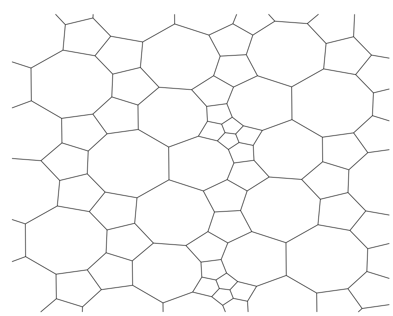
$(48, 16, 8)$, $p2mg$



$(48, 16, 8)$, $p2$

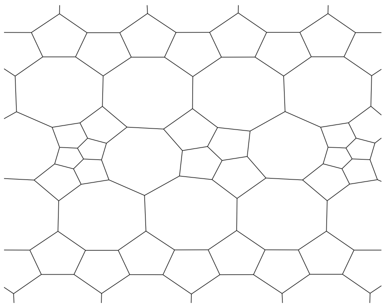


$(48, 16, 8)$, $p2$

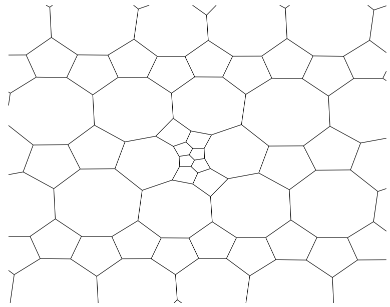


$(48, 16, 8)$, $p2$

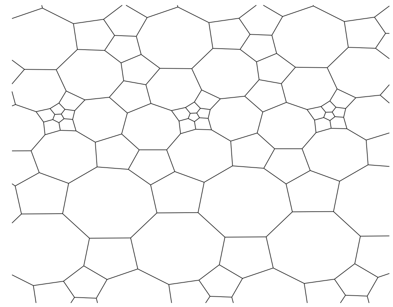
Figure 50: Some $(5, 8)$-tori, which are $8R_3$ (second part)



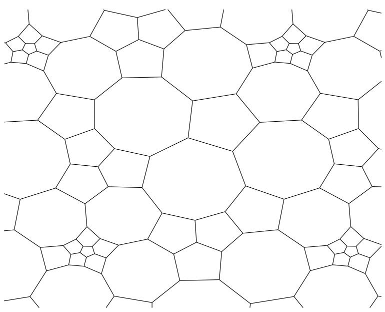
$(48, 16, 8)$, $p2$



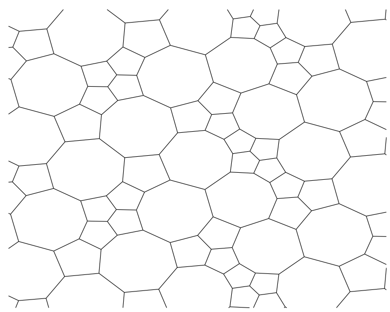
$(48, 16, 8)$, $p2$



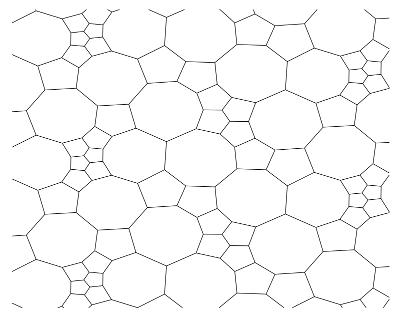
$(48, 16, 8)$, $p1$



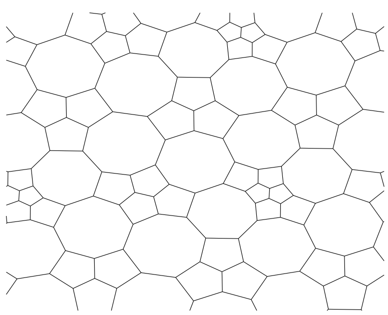
$(48, 16, 8)$, $p1$



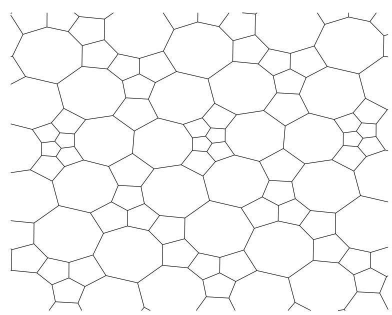
$(48, 16, 8)$, $p2$



$(48, 16, 8)$, $p2$

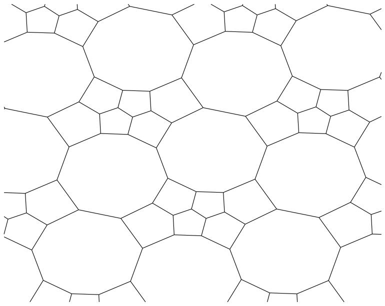


$(48, 16, 8)$, $p2$

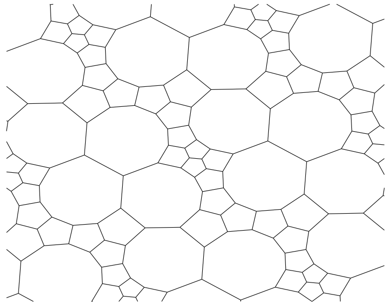


$(48, 16, 8)$, $p2$

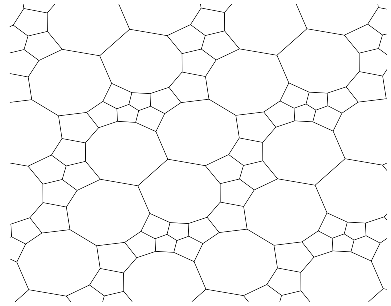
Figure 51: Some $(5, 8)$-tori, which are $8R_3$ (third part)



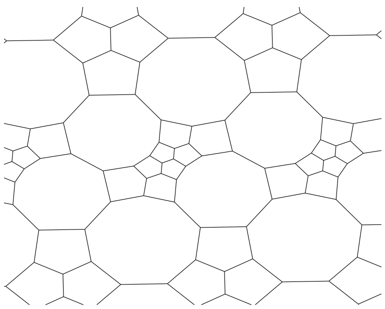
$(16, 6, 2)$, $p2$



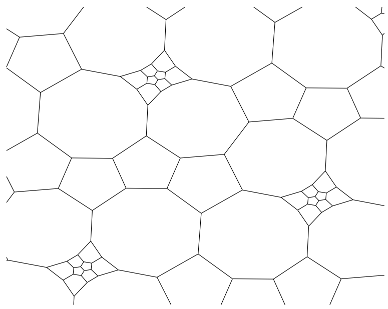
$(32, 12, 4)$, $p2$



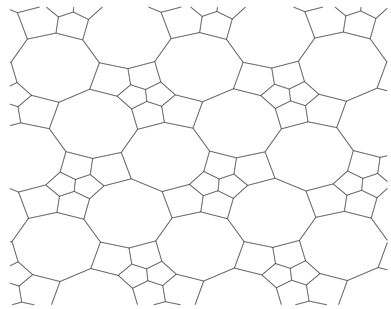
$(32, 12, 4)$, $p2$



$(32, 12, 4)$, $p2$



$(32, 12, 4)$, $p2$



$(32, 12, 4)$, $p2gg$

Figure 52: Some $(5, 9)$-tori, which are $9R_3$

10 Maps qR_4

Lemma 10.1 *Let G be a $(4, q)$-map, which is qR_4. Denote by x_i the number of vertices, which are contained into i 4-gonal faces. Then one has:*

$$\begin{cases} x_0 + x_3 = 8 & \text{on sphere,} \\ x_0 + x_3 = 0 & \text{on torus.} \end{cases}$$

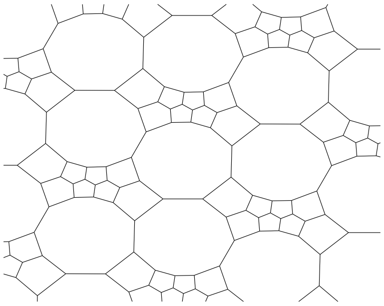
Proof. Denote by $G' = q(G)$ the map formed by q-gonal faces; it is a 4-valent map. The set of 4-gonal faces of G is partitioned into $(4, 3)$-polycycles S_1, S_2 and P_{2k} with $k \geq 2$. Denote by $f_{S_1}, \dots$ the corresponding number of such polycycles.

Those polycycles correspond, respectively, to 3-, 2- and 4-gonal faces of the map G'. The other faces of G' are 3-gonal ones, corresponding to vertices of G, which are incident to three q-gonal faces.

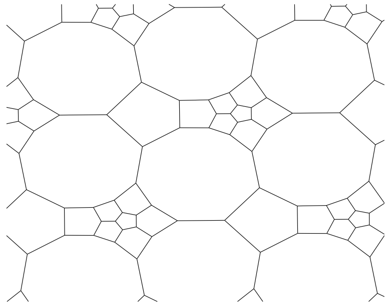
Hence, the map G' is a 4-valent one, whose faces are 2-, 3- or 4-gonal. The Euler formula for those maps is $4\chi = 2p_2 + p_3$ with p_i being the number of i-gonal faces and $\chi = 2$ for spheres and 0 for torus. One has $p_2 = f_{S_2}$ and $p_3 = f_{S_1} + x_0$. So, it holds:

$$\chi = 2f_{S_2} + f_{S_1} + x_0.$$

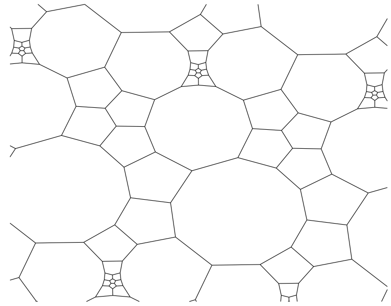
It turns out, that $x_3 = 2f_{S_2} + f_{S_1}$. $\square$



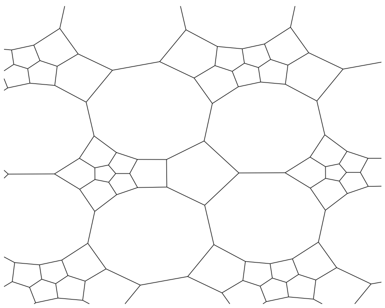
$(20, 8, 2)$, $p2$



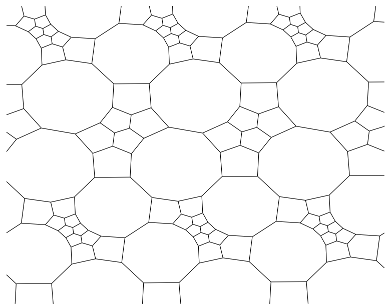
$(20, 8, 2)$, cm



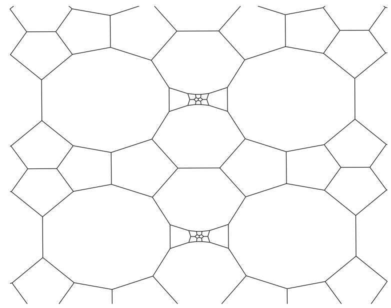
$(40, 16, 4)$, $p1$



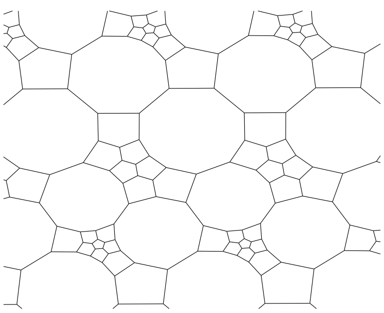
$(40, 16, 4)$, $p1$



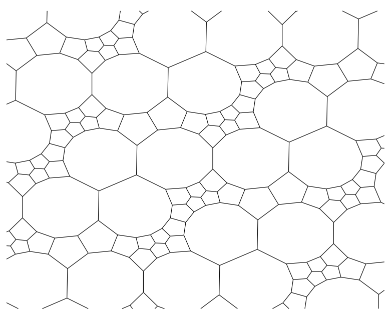
$(40, 16, 4)$, $p2$



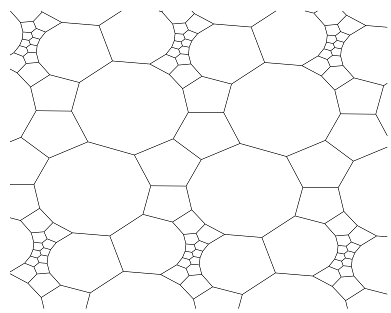
$(40, 16, 4)$, $p2mm$



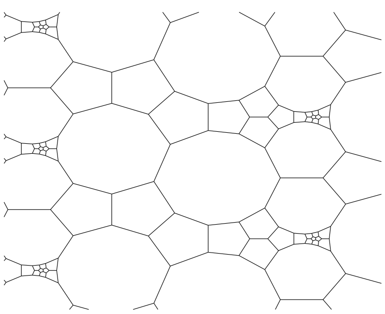
$(40, 16, 4)$, $p1$



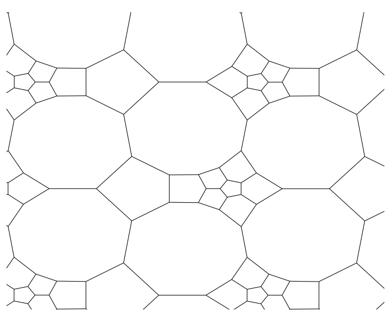
$(40, 16, 4)$, $p2$



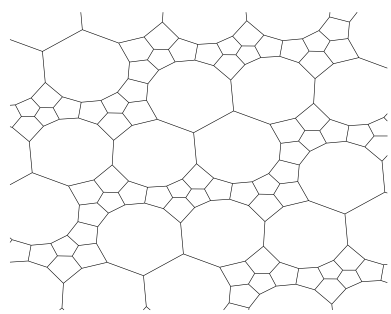
$(40, 16, 4)$, $p2$



$(40, 16, 4)$, $p1$

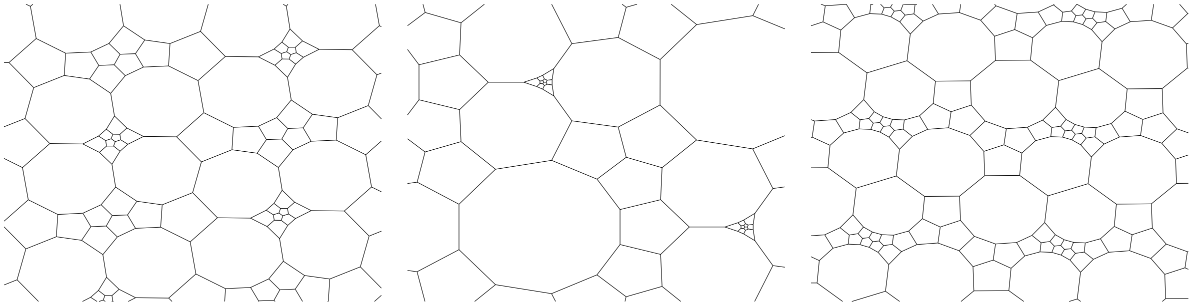


$(40, 16, 4)$, $p2mg$



$(40, 16, 4)$, $p2$

Figure 53: Some $(5, 10)$-tori, which are $10R_3$ (first part)



(40, 16, 4), $p2$ (40, 16, 4), cm (40, 16, 4), $p2$

Figure 54: Some (5, 10)-tori, which are $10R_3$ (second part)

The above theorem also admits a more standard proof, which does not make use of the classification of (4, 3)-polycycles. It is very useful for classifying the corresponding maps, in both, theoretical and computational, levels.

Theorem 10.2 *(i) $(4, q)$-tori, which are qR_4, exist if and only if $q \geq 8$.*

(ii) A $(4, 8)$-torus is $8R_4$ if and only if it is $4R_0$.

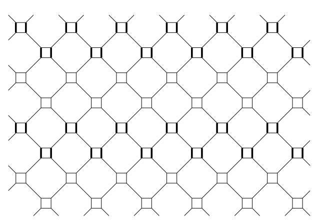
(iii) Any $(4, q)$-torus, which is qR_4, has no $(4, 3)$-polycycles S_i and has no vertices incident only to q-gons. Such a torus is described by orienting the $(4, 3)$-polycycles P_{2k} in a 4-valent tessellation of the torus.

(iv) Any $(4, 9)$-torus, which is $9R_4$, is described in terms of a perfect matching on a 4-valent tessellation of the torus by 4-gons.

Proof. (i) By standard argument, one obtains successively:

$$f_4 = \frac{q-6}{2} f_q \quad \text{and} \quad e_{4-4} = \frac{q-8}{2} f_q.$$

This implies that a $(4, q)$-torus, which is qR_4, exist if and only if $q \geq 8$. On the other hand, there exists an unique (4, 8)-plane, which is $8R_4$; we need to modify it, so as to obtain a $(4, q)$-plane. In order to do this, we use the following drawing



and transform every square with two bold edges into a (4, 3)-polycycle $P_{2(q-6)}$.

(ii) If $q = 8$, then $e_{4-4} = 0$, i.e., all 4-gons are isolated. On the other hand, if a (4, 8)-torus is $4R_0$, then each 8-gon is adjacent to at most four 4-gons and one concludes easily.

(iii) For any $(4, q)$-torus, which is qR_4, one obtains, by Lemma 10.1, the equality $x_0 + x_3 = 0$. Therefore, $x_0 = x_3 = 0$, then it holds $f_{S_i} = 0$. So, all $(4, 3)$-polycycles are of the form P_{2k}. On the other hand, the map $q(G)$ is, clearly, a 4-valent tessellation of the torus. Each vertex corresponds to a q-gonal face and each 4-gon corresponds to a $(4, 3)$-polycycle P_{2k}. We just need to fix the orientation of this polycycle and the value of k, in order to define the map.

(iv) If $q = 9$, then each $(4, 3)$-polycycle is of the form P_4 or P_6 and each q-gonal face is adjacent to exactly one polycycle P_6 on one of its two sequences of two 4-gons.

Every polycycle P_6 is adjacent to two q-gonal faces on their sequence of two 4-gons. This defines a matching in the graph $q(G)$. It is a perfect matching. □

Theorem 10.3 *All $(4, 8)$-spheres, which are $8R_4$, belong to the following list:*

1. *Two infinite series with $v = 32 + 8i$, which contain the $(4, 3)$-polycycle S_2. If i is odd, they are isomorphic of symmetry C_2, otherwise; if $i > 0$ is even, they are not isomorphic and one is of symmetry C_{2h}, the other of symmetry C_{2v}. If $i = 0$, the sphere of symmetry C_{2h} gains higher symmetry D_{2h} (see Figure 62).*

2. *Two infinite series with $v = 32 + 16i$, which contain four $(4, 3)$-polycycles S_1. One of the series is of symmetry D_{2d}, while the other is of symmetry D_{2h}. If $i = 0$, the sphere of symmetry D_{2d} gains the additional symmetry T_d (see Figure 63).*

3. *Four infinite series with $v = 80 + 24i$, which contain two $(4, 3)$-polycycles S_1 and six $(4, 3)$-polycycle P_6. Two series are of symmetry C_2, one of symmetry C_{2v}, one of symmetry C_{2h} (see Figure 64).*

4. *Two infinite series with $v = 144 + 16i$ of symmetry D_2, which contain twelve $(4, 3)$-polycycles P_6 (see Figure 65).*

5. *Three infinite series with $v = 128 + 32i$ of symmetry D_2, D_{2d} and D_{2h}, respectively, which contain twelve $(4, 3)$-polycycles P_6 (see Figure 66).*

6. *A list of sporadic examples, given on Figures 55—61.*

Proof. The enumeration was done by computer. It was done using two relations (see Theorems 4.7 and 10.1):

$$e_{4-4} = 12, \qquad x_0 + x_3 = 8,$$

i.e., if, in the enumeration, we found some partial graph with $e_{4-4} > 12$ or $x_0 + x_3 > 8$, then we can discard it.

The set of 4-gonal faces is partitioned into $(4, 3)$-polycycles. Clearly, the only possible polycycles are S_1, S_2, P_4, P_6, P_8 and P_{10}.

The enumeration consisted of a progressive set of steps. Very often, we run into infinite loops. But those infinite loops present a periodic structure for their boundary and one period just add another ring of faces. This reflects the fact that all infinite series are of "cylindrical" form.

Also, the property of being face-regular is local, i.e., only the neighbors are involved. This means that, if one period appear in the enumeration, then this period will reappear thereafter.

The first step takes, as initial group of faces, the $(4,3)$-polycycle P_{10}, circumscribed by 8-gons, and ends up in a cycle, with no graph appearing. This proves that P_{10} never appears in a $(4,8)$-sphere $8R_4$.

The second step takes, as initial group of faces, the $(4,3)$-polycycle S_2, circumscribed by 8-gons, and first creates a sporadic graph and then an infinite sequence of graphs. This proves that all $(4,8)$-spheres $8R_4$, having a polycycle S_2, are either the sporadic ones, or belong to one of the two such infinite series. In the following, we can assume that the graph has no S_2.

The third step takes, as initial group of faces, the $(4,3)$-polycycle S_1, circumscribed by 8-gons, and creates some sporadic spheres and six infinite series. In the following, we can assume $x_3 = 0$.

The fourth step takes, as initial group of faces, the $(4,3)$-polycycle P_8, circumscribed by 8-gons, creates some sporadic graphs and ends up in a cycle with no more graphs created.

The fifth step takes, as initial, the $(4,3)$-polycycle P_6, circumscribed by 8-gons. It creates some sporadic graphs and five infinite series. □

Theorem 10.4 *There exist an infinite series of* $(5,8)$*-spheres, which are* $8R_4$*.*

Proof. Take an edge of Dodecahedron and add a 4-gon in the middle of the edge. Inside of this 4-gon it is possible to put another Dodecahedron, by cutting an edge of it in the middle and gluing it inside the 4-gon. This construction can be generalized, by cutting opposite edge of Dodecahedra. See the first example on Figure 84. □

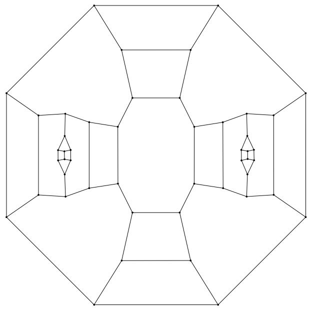
In above procedure, cutting opposite edges of Dodecahedra is just one of several possibilities. In fact, given one edge of Dodecahedron, there are five other edges, which can be cut, in order to obtain $(5,8)$-spheres $8R_4$.

Theorem 10.5 *A* $(5,7)$*-sphere, which is* $7R_4$ *and contains, in its set of* 5*-gonal faces, a* $(5,3)$*-polycycle with boundary sequence containing at most three* 2*, belongs to an infinite series of such spheres, having* $v = 20 + 24t$ *vertices with* $t \geq 1$ *and symmetry* D_{3d}*.*

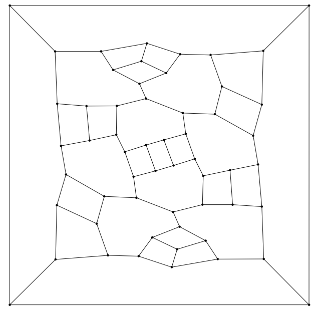
Proof. Denoting by v_3, v_2 the number of vertices of degree 3, 2 on the boundary, one obtains $v_3 \leq 2v_2$ and $v_2 \leq 3$.

The formula $f_5 = 6 - v_2 + v_3$, expressing the number of 5-gonal faces in a $(5,3)$-polycycle, yields $f_5 \leq 9$. An exhaustive enumeration amongst all $(5,3)$-polycycles with 9-pentagons yields the elementary $(5,3)$-polycycle A_3 as the only possibility.

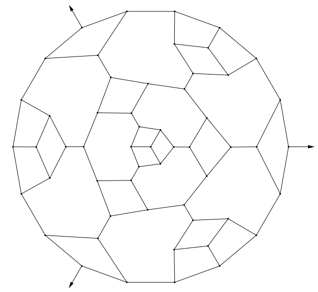
So, now we extend this polycycle by adding a ring of 7-gons around it. Since every 7-gon is adjacent to four 7-gons, we need to add another ring of 7-gons around them. Every 7-gonal face in those rings is adjacent to four 7-gonal faces. So, we are forced to add another ring of 5-gons. If we add another pentagon, then there is only one possibility for filling the structure and one adds two more pentagons, i.e., obtains a sphere.



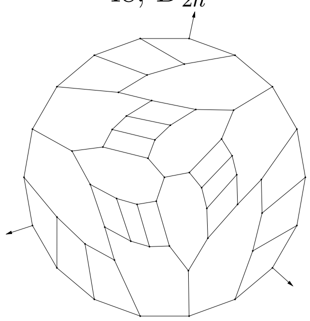
48, D_{2h}



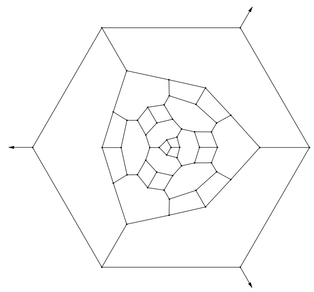
56, C_2



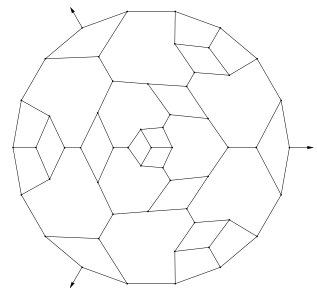
56, C_{3v}



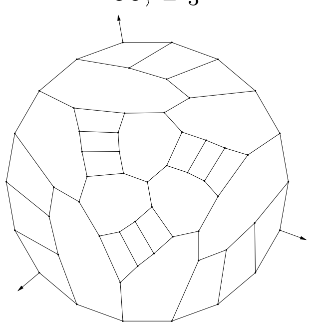
56, D_3



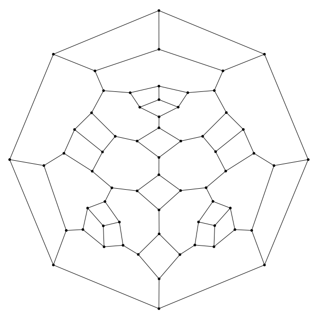
56, D_{3d}



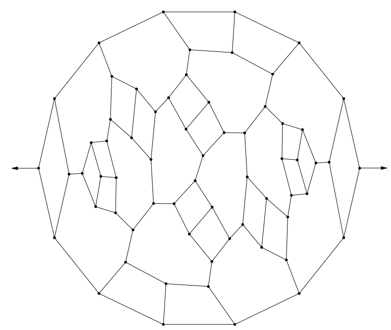
56, T_d



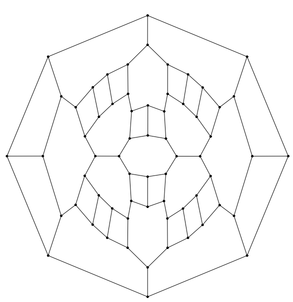
56, T_h



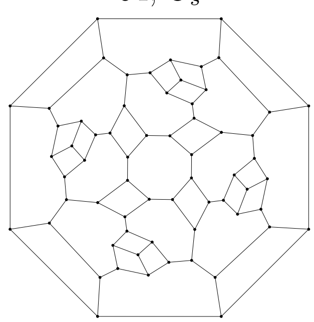
64, C_s



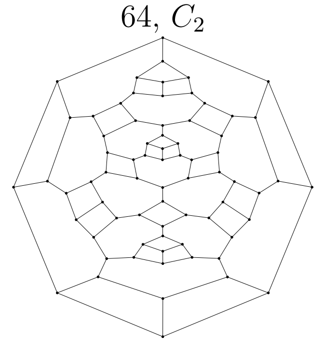
64, C_2



64, D_{2d}

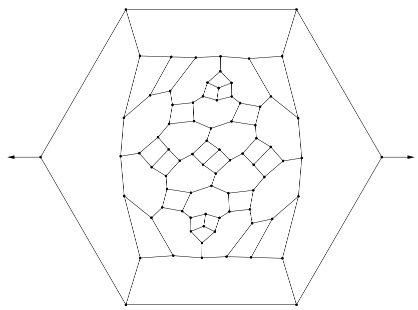


64, C_{4h}

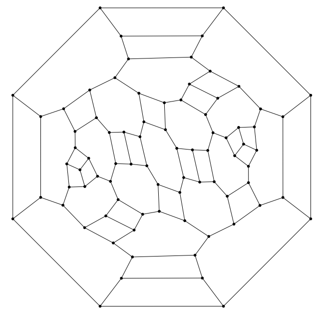


72, C_s

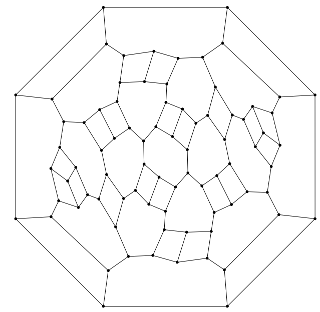
Figure 55: Sporadic $(4,8)$-spheres, which are $8R_4$ (first part)



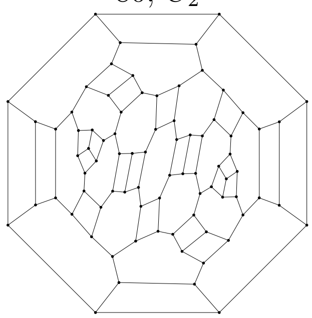
80, C_2



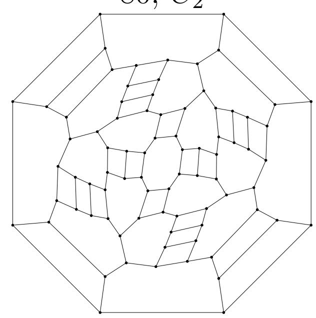
80, C_2



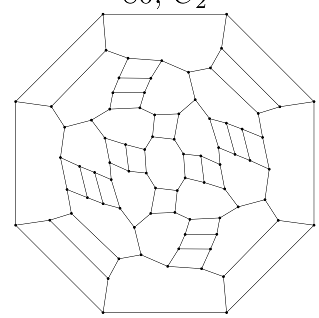
80, C_2



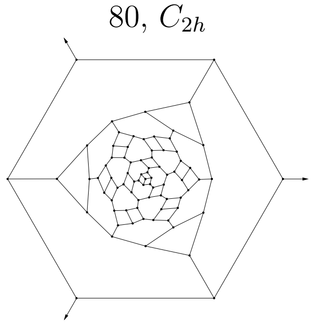
80, C_{2h}



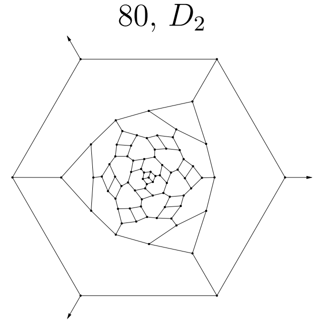
80, D_2



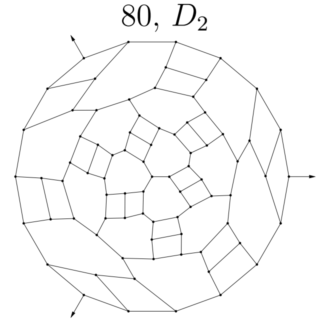
80, D_2



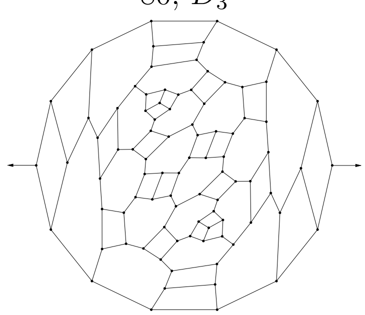
80, D_3



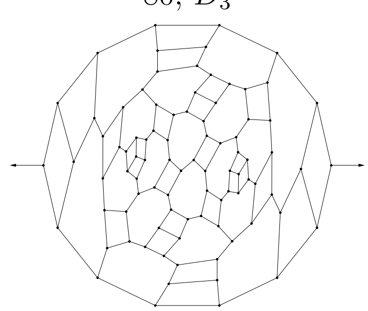
80, D_3



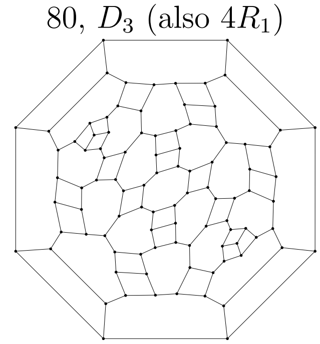
80, D_3 (also $4R_1$)



88, C_2



88, C_2



96, C_2

Figure 56: Sporadic $(4, 8)$-spheres, which are $8R_4$ (second part)

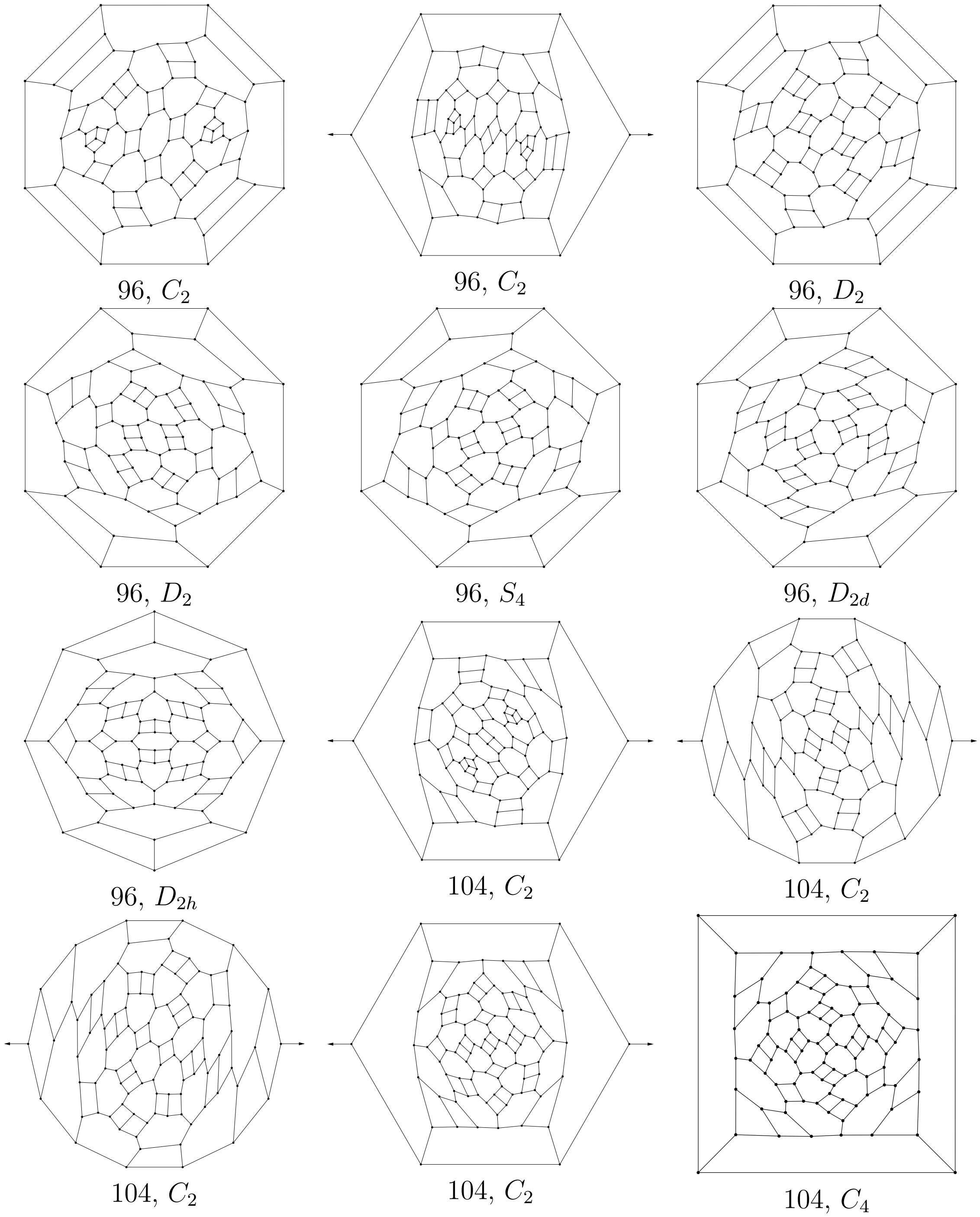
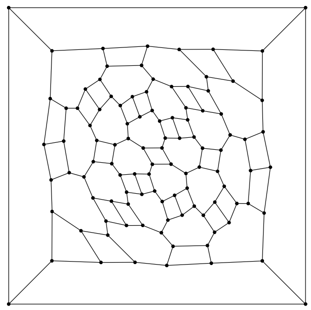
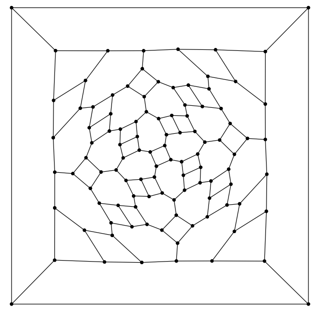


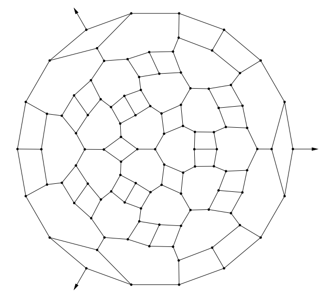
Figure 57: Sporadic $(4,8)$-spheres, which are $8R_4$ (third part)



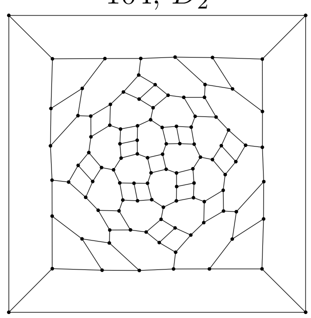
104, D_2



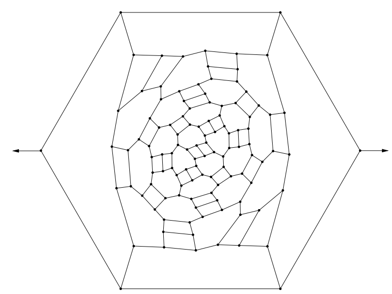
104, D_4



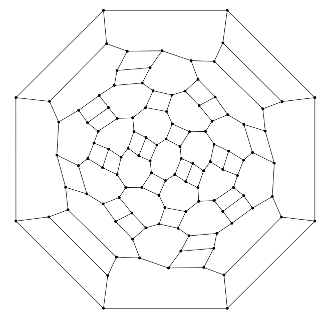
104, D_{3d}



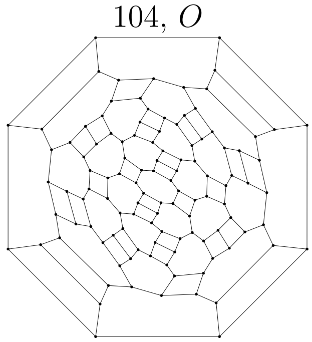
104, O



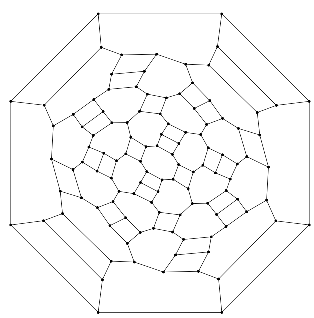
112, C_2



112, C_2



112, D_2



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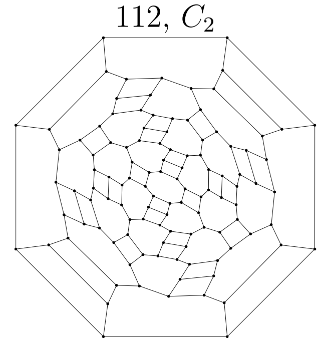
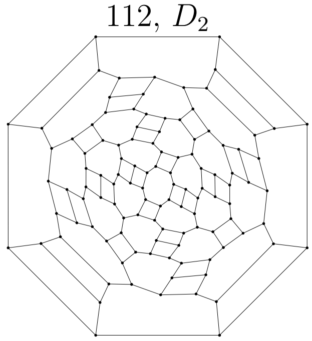
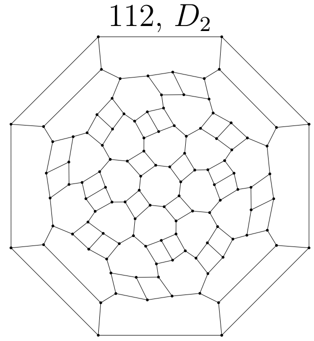


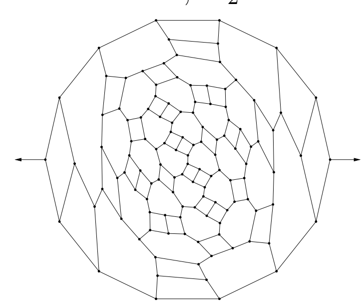
112, D_2



112, D_2

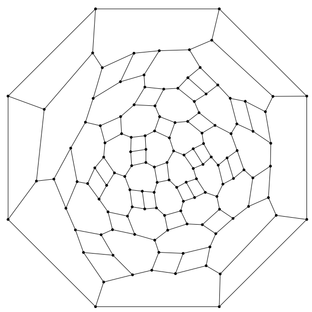


112, D_4

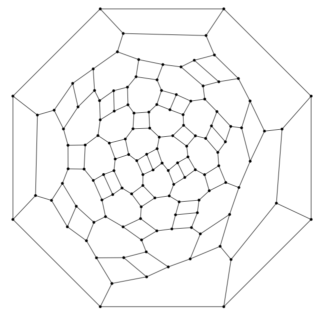


120, C_2

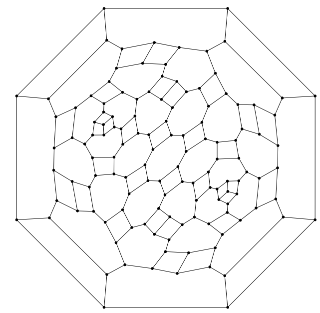
Figure 58: Sporadic $(4,8)$-spheres, which are $8R_4$ (fourth part)



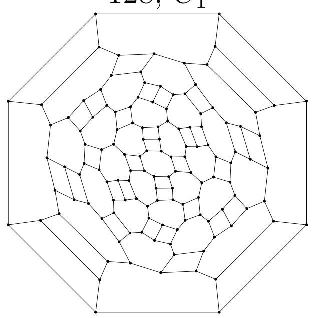
128, C_1



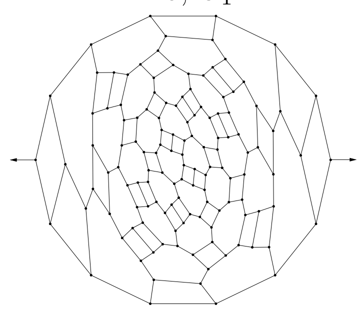
128, C_1



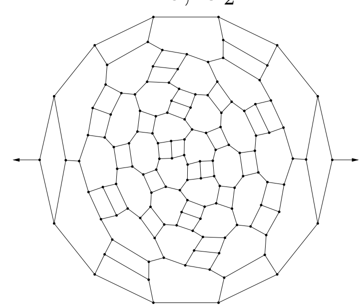
128, C_2



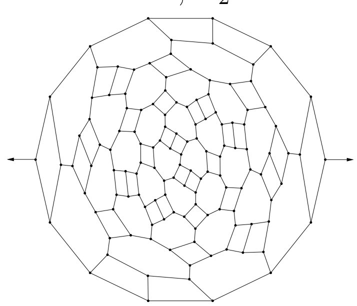
128, C_2



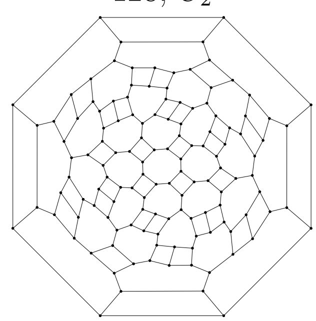
128, C_2



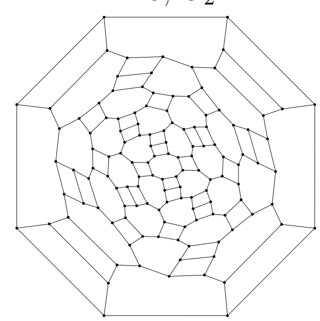
128, C_2



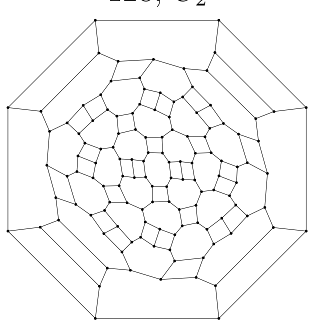
128, C_2



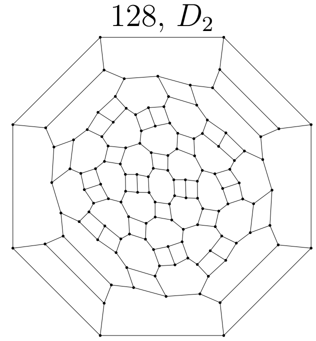
128, D_2



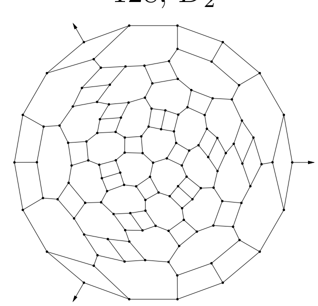
128, D_2



128, D_2

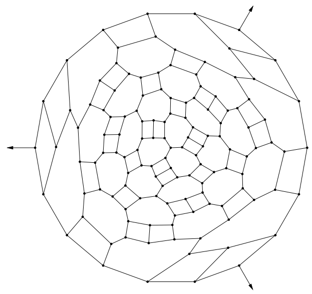


128, D_2

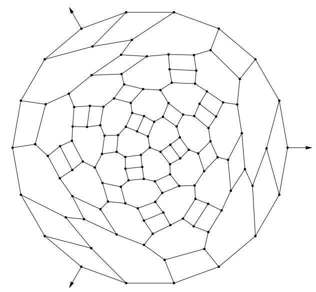


128, D_3

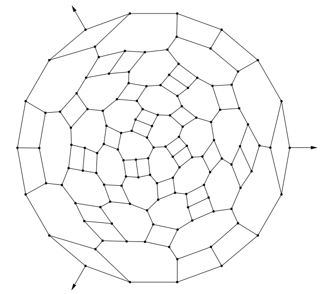
Figure 59: Sporadic $(4,8)$-spheres, which are $8R_4$ (fifth part)



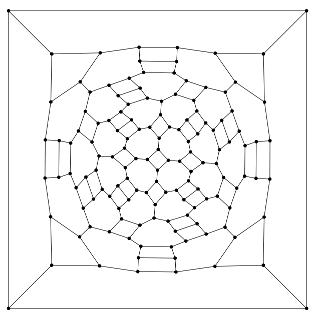
128, D_3



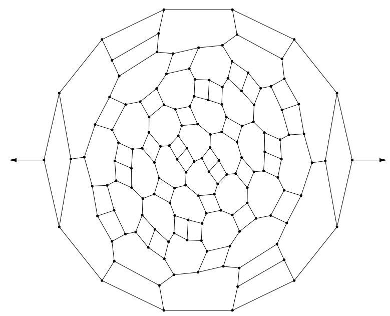
128, D_3



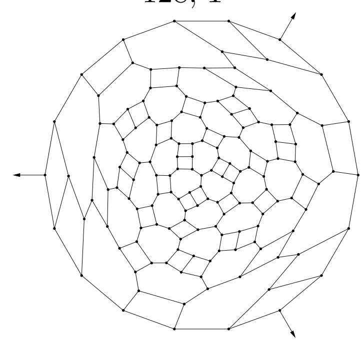
128, T



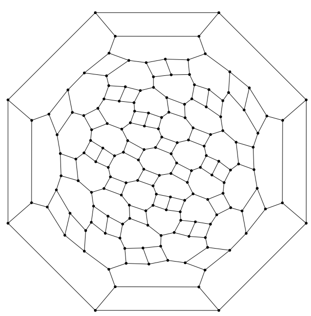
136, D_4



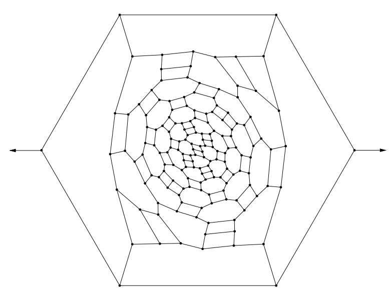
144, C_2



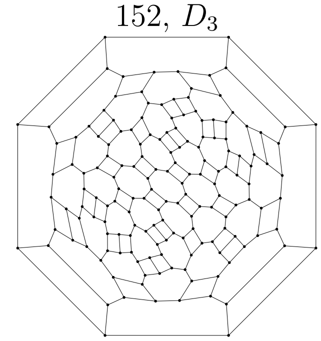
152, D_3



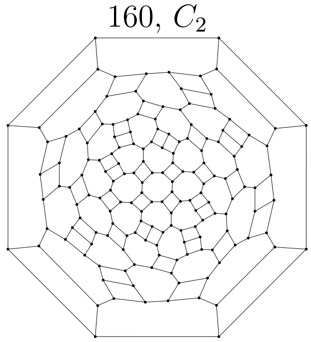
160, C_2



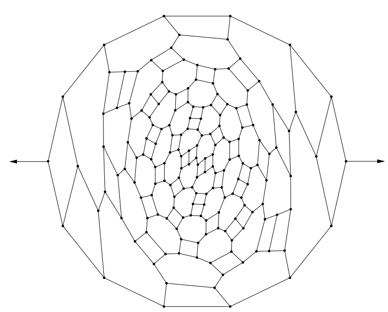
160, C_{2h}



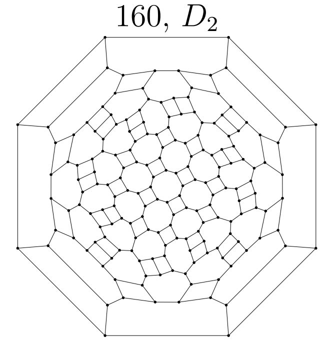
160, D_2



160, S_4

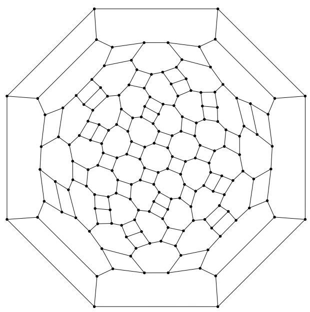


176, C_2

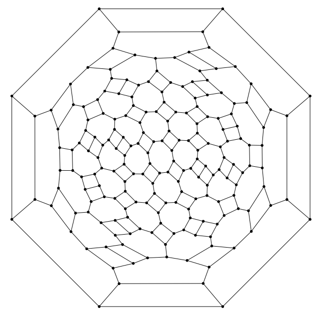


176, D_4

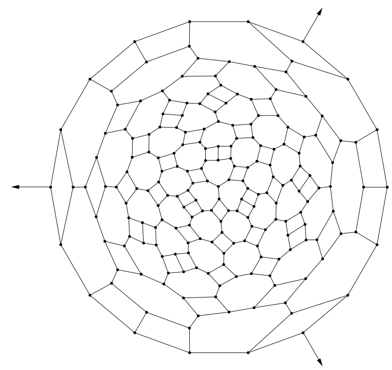
Figure 60: Sporadic $(4, 8)$-spheres, which are $8R_4$ (sixth part)



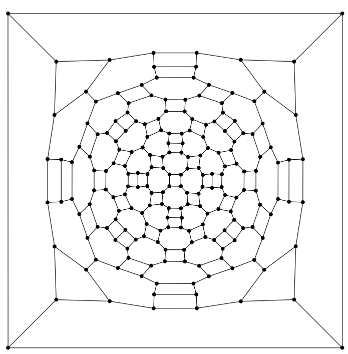
176, D_2



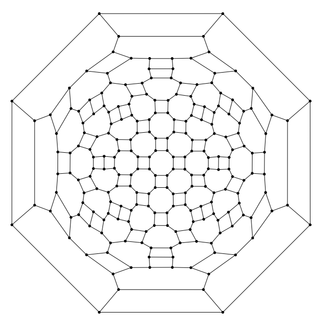
192, D_2



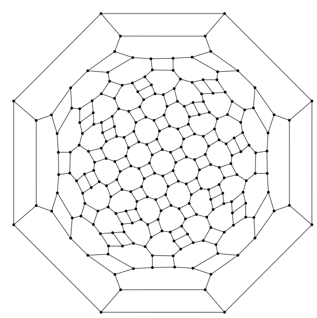
200, D_3



200, O_h

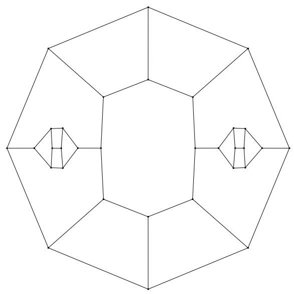


224, D_{2d}

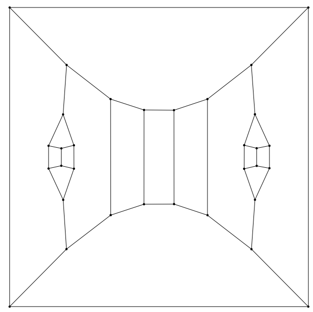


240, D_4

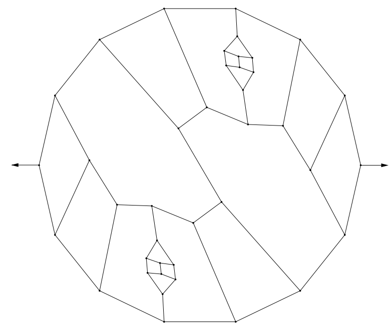
Figure 61: Sporadic $(4,8)$-spheres, which are $8R_4$ (seventh part)



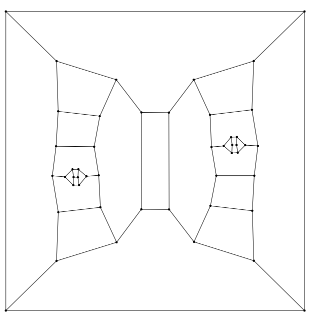
32, D_{2h}



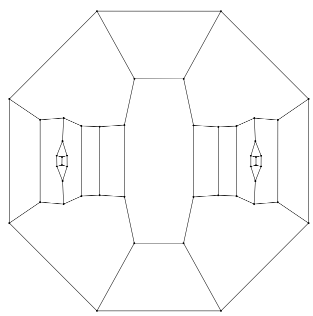
32, C_{2v}



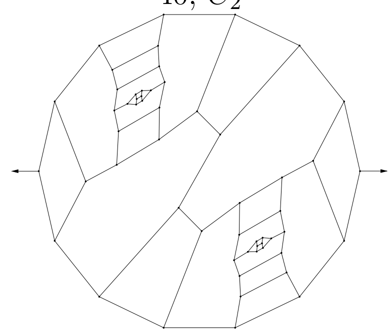
40, C_2



48, C_{2h}

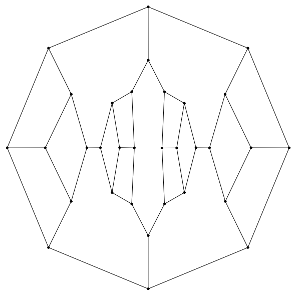


48, C_{2v}

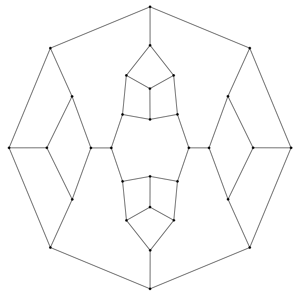


56, C_2

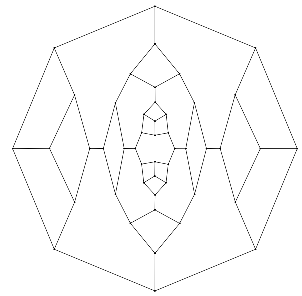
Figure 62: First members of infinite series of $(4, 8)$-spheres, which are $8R_4$ and contain two $(4, 3)$-polycycles S_2



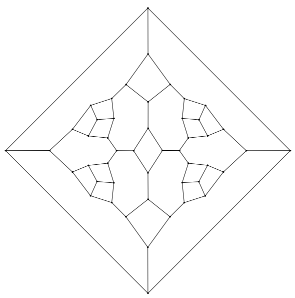
32, D_{2h} (also $4R_2$)



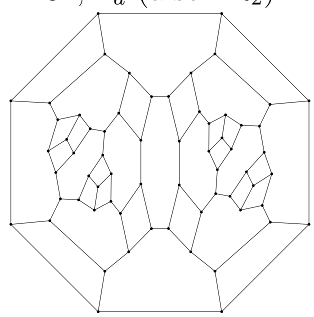
32, T_d (also $4R_2$)



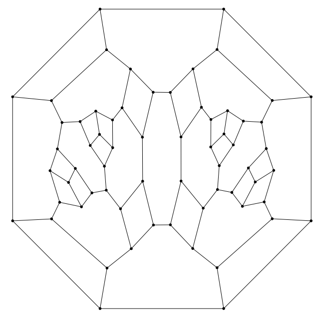
48, D_{2d}



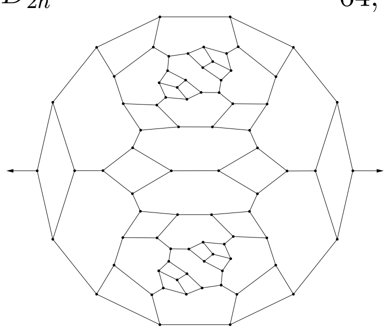
48, D_{2h}



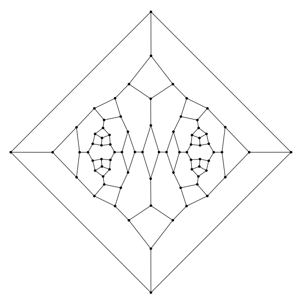
64, D_{2d}



64, D_{2h}



80, D_{2d}



80, D_{2h}

Figure 63: First members of infinite series of $(4,8)$-spheres, which are $8R_4$ and contain four $(4,3)$-polycycles S_1

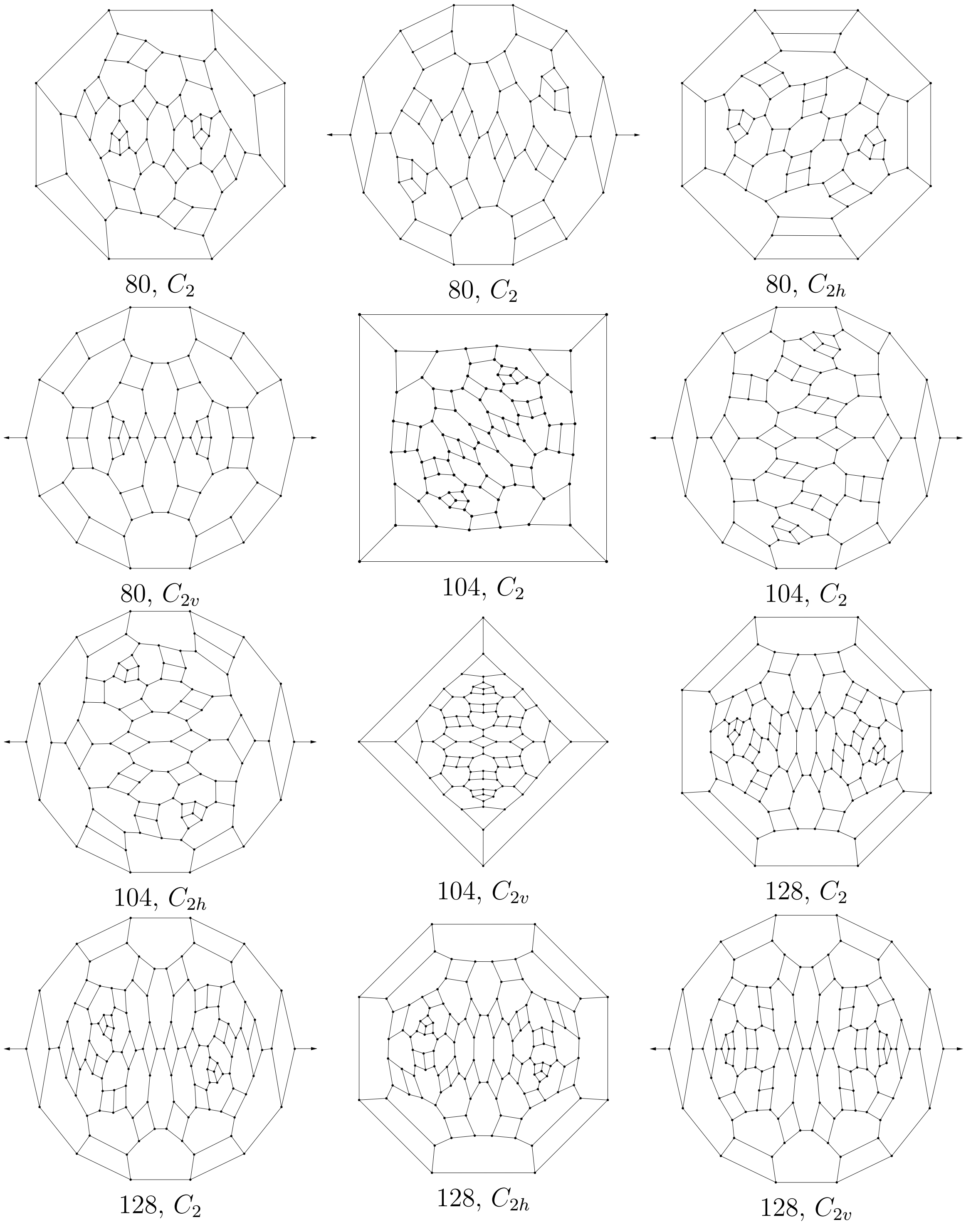


Figure 64: First members of infinite series of $(4, 8)$-spheres, which are $8R_4$ and contain two $(4, 3)$-polycycles S_1

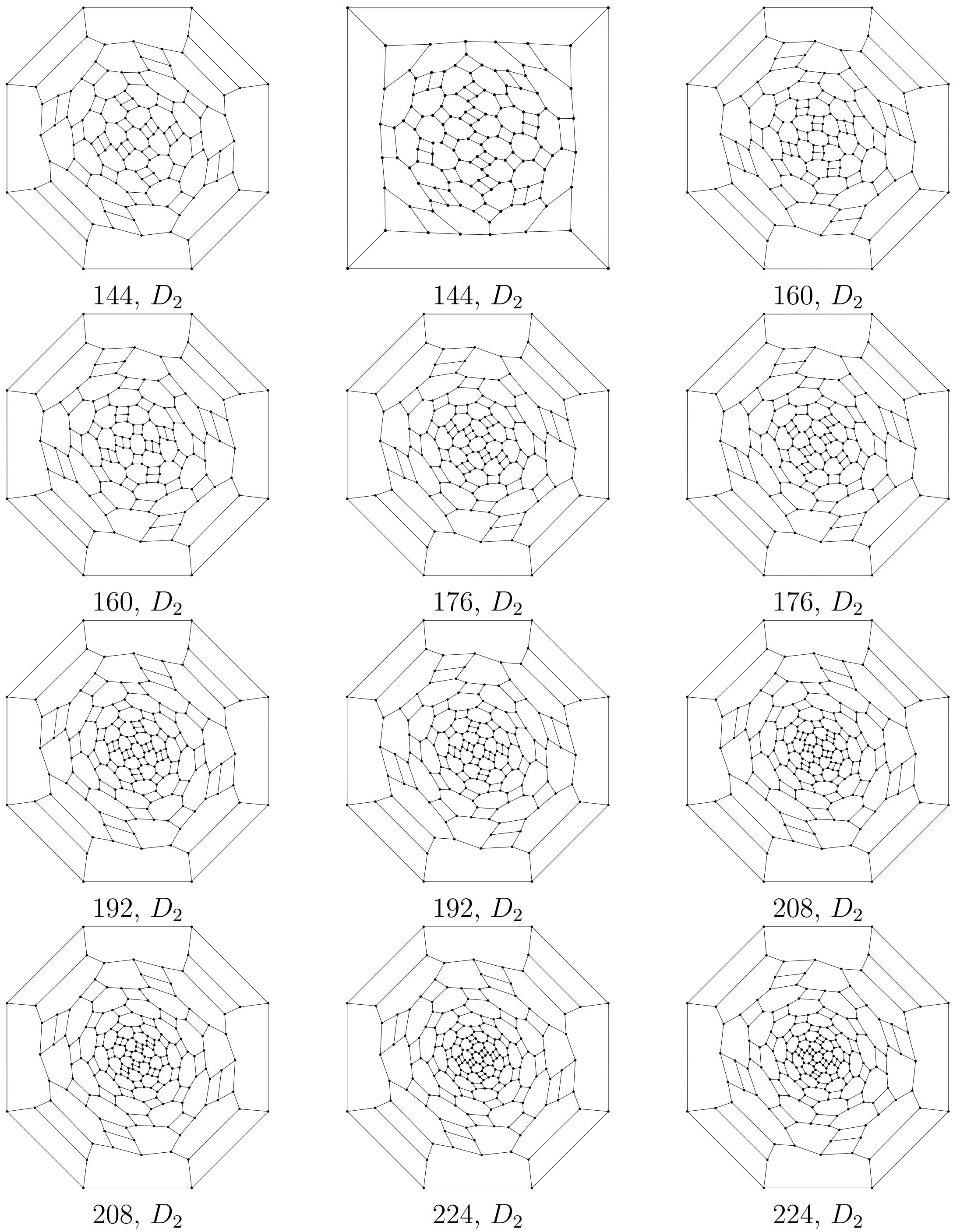
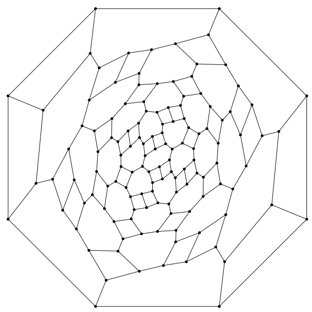
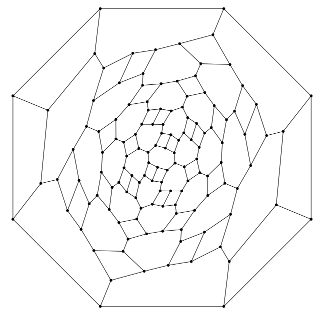


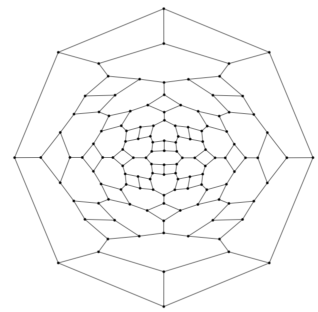
Figure 65: First members of infinite series of $(4, 8)$-spheres, which are $8R_4$, contain twelve $(4, 3)$-polycycles P_6 and have symmetry D_2



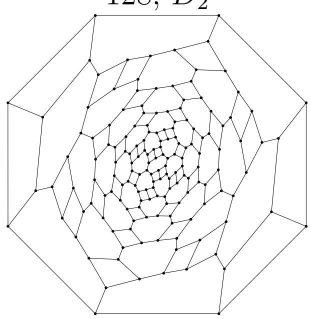
128, D_2



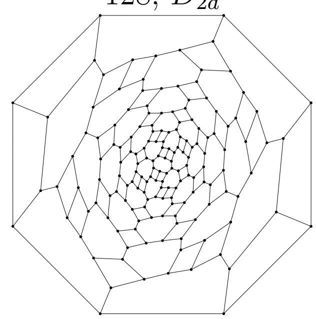
128, D_{2d}



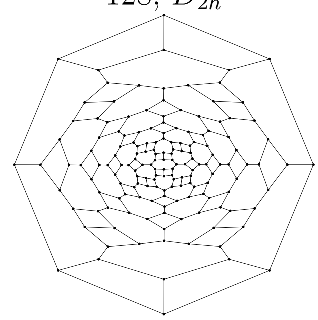
128, D_{2h}



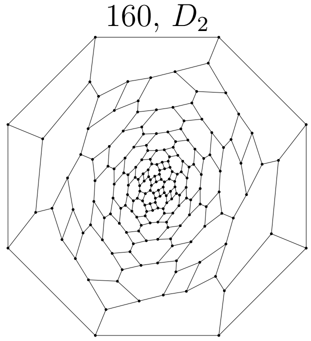
160, D_2



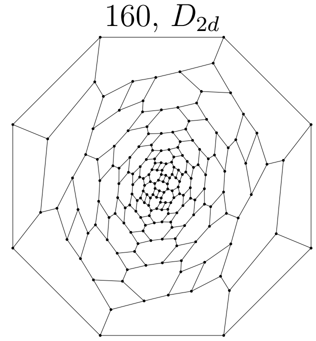
160, D_{2d}



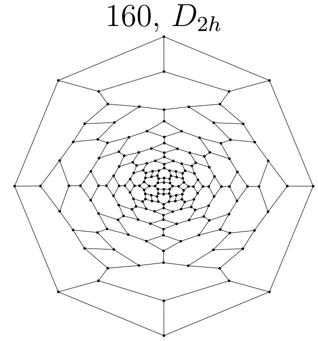
160, D_{2h}



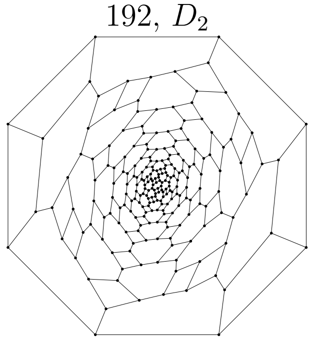
192, D_2



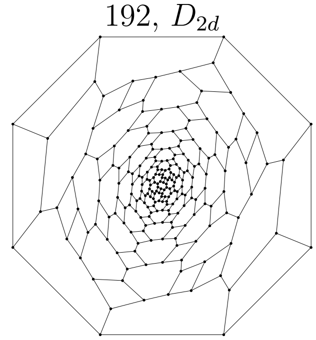
192, D_{2d}



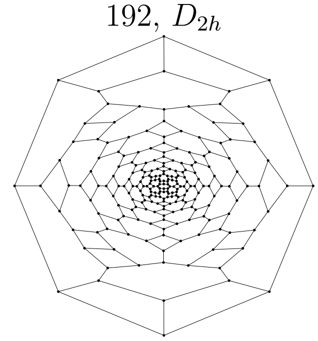
192, D_{2h}



224, D_2

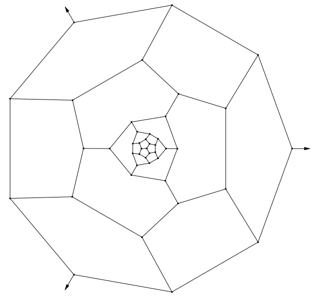


224, D_{2d}

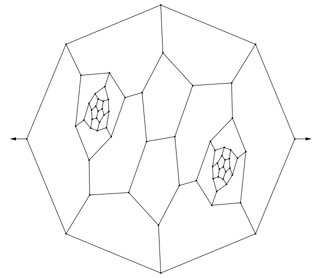


224, D_{2h}

Figure 66: First members of infinite series of $(4,8)$-spheres, which are $8R_4$, contain twelve $(4,3)$-polycycles P_6 and have symmetry D_2, D_{2d} or D_{2h}



44, D_{3d}



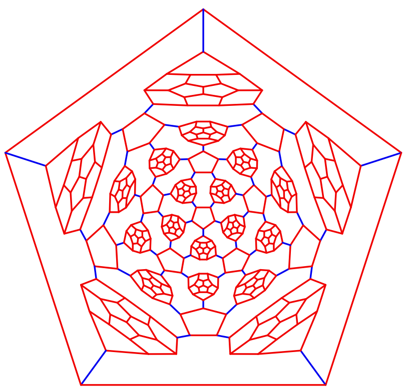
68, D_{3d}

Figure 67: All weakly face-regular $(5,7)$-spheres, which are $7R_4$ and have at most 76 vertices; they belong to an infinite series of those having $20+24t$ vertices (on Figure are given cases $t=1,2$; see Theorem 10.5)

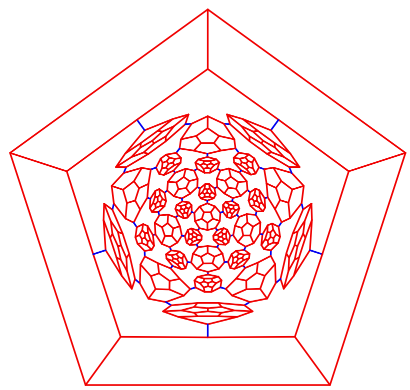
If not, then we add a ring of 7-gons and the argument can be repeated. Since the graph is finite, eventually, one will obtain a sphere belonging to the infinite series. □

$(5,7)$-spheres, which are $7R_4$ and which are not obtained by the above Theorem 10.5, probably, exist since there is a lot of $(5,7)$-tori, which are $7R_4$.

See below some examples of $(5,q)$-spheres with relatively high number of vertices.



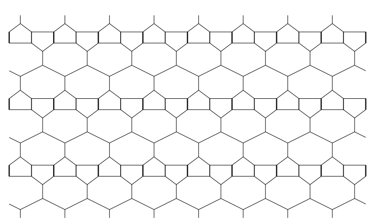
440, I_h
$(5,12)$-sphere $12R_4$



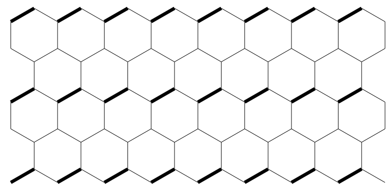
560, I_h
$(5,14)$-sphere $14R_4$

Theorem 10.6 *For any $q \geq 7$, there exists a $(5,q)$-torus, which is qR_4.*

Proof. The proof consists of taking the following initial $(5,7)$-plane $7R_4$, which is also $5R_2$:



In order to obtain a $(5,q)$-torus, which is qR_4, we need to modify the structure of the pentagons. The pentagons D can be paired to form the "polycycle E_0" in the following way:



Every one of those pairs of pentagons, can be replaced by an E_{2n} with $n \geq 1$. We obtain a $(5,7{+}n)$-plane, which is $(7{+}n)R_4$. Then needed torus is obtained as its quotient. □

Theorem 10.7 *There is an infinity of* $(5,q)$*-spheres, which are* qR_4*, for* $q = 10$*,* 13*,* 16*.*

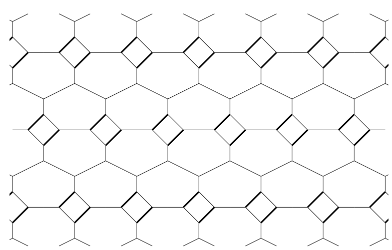
Proof. If one takes Cube, truncates some of its vertices and replace them by polycycles A_3, such that every 4-gon is incident to exactly t truncated vertices, then it is easy to see that the obtained graph is a $(5,4+3t)$-sphere, which is $(4+3t)R_4$. Clearly, for any $1 \leq t \leq 4$, such sets of vertices exist. The conclusion follows from Theorem 1.5. □

See on Figures 68, 69, 70, 71 some examples of $(5,q)$-tori, which are qR_4, for $q = 7$, 8, 9 and 10.

11 Maps qR_5

Theorem 11.1 *(i) A* $(4,q)$*-torus, which is* qR_5*, exists if and only if* $q \geq 7$*.*
(ii) Any $(4,7)$*-torus, which is* $7R_5$*, is also* $4R_0$*.*

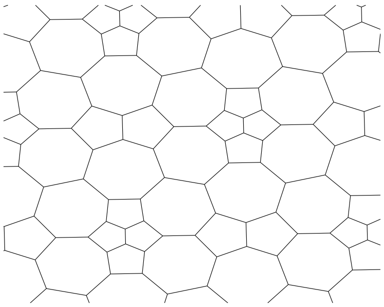
Proof. In order to prove (i), we take the following strictly face-regular $(4,7)$-plane $7R_5$.



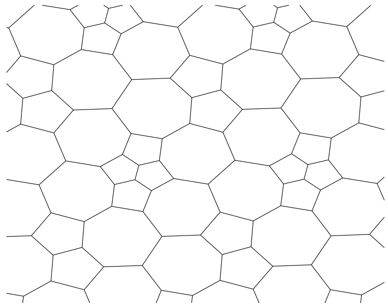
Replacing the isolated squares by $(4,3)$-polycycles $P_{2(2+n)}$, one obtains $(4,7{+}n)$-torus, which is $(7+n)R_5$. So, (i) is true.

For $(4,7)$-tori, which are $7R_5$, one gets, by direct counting, the equality $e_{4-4} = 0$; hence, the conclusion. □

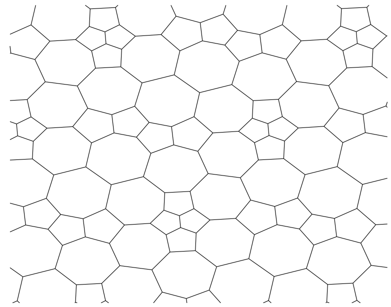
Note that there exist a $(4,7)$-torus, which is $4R_0$ but not $7R_5$.



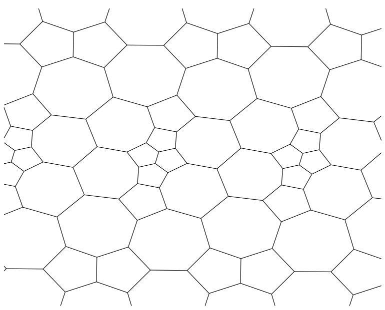
$(24, 6, 6)$, $c2mm$



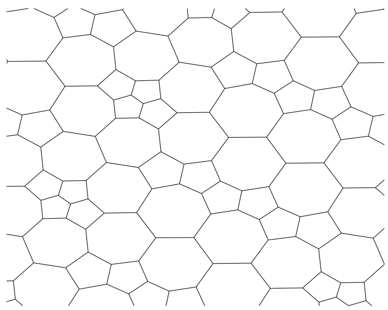
$(24, 6, 6)$, $p2$



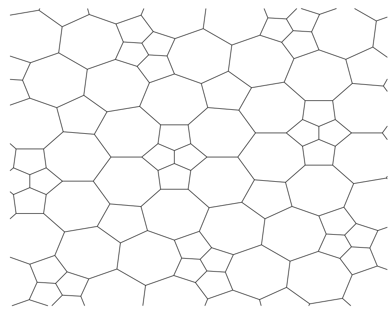
$(32, 8, 8)$, $p2$



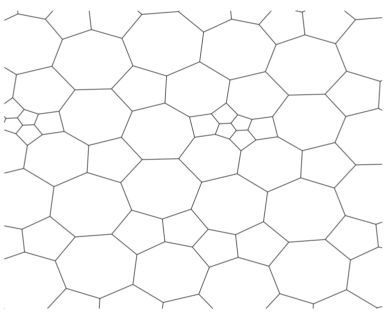
$(32, 8, 8)$, $p2$



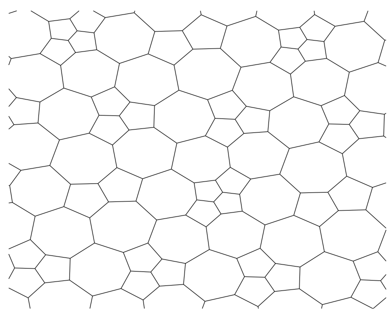
$(40, 10, 10)$, $p2$



$(40, 10, 10)$, $p2mg$

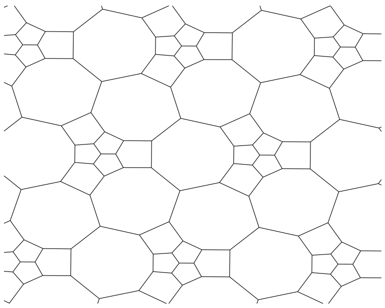


$(48, 12, 12)$, $p2$

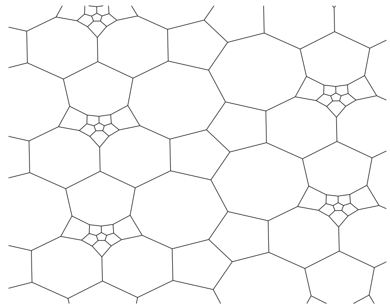


$(48, 12, 12)$, $p1$

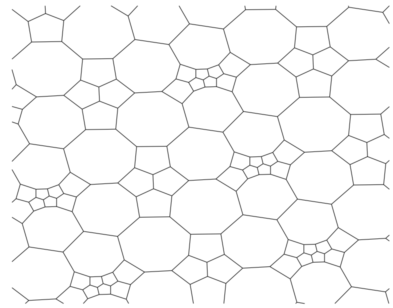
Figure 68: Some $(5, 7)$-tori, which are $7R_4$



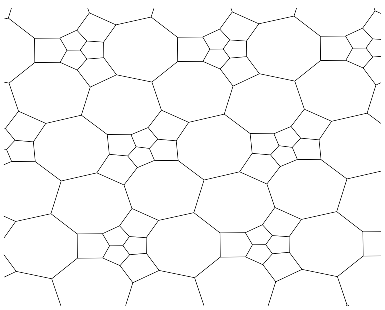
$(18, 6, 3)$, $p31m$



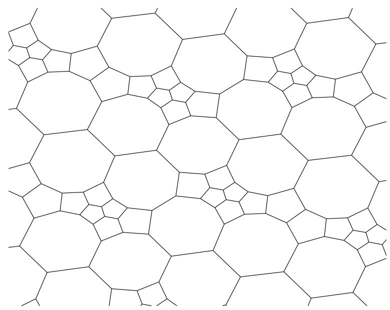
$(30, 10, 5)$, $p1$



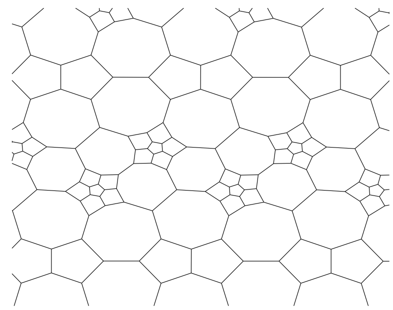
$(36, 12, 6)$, $p2$



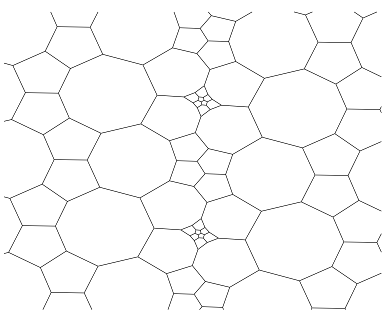
$(36, 12, 6)$, $p1$



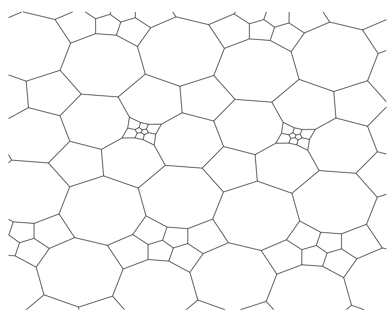
$(42, 14, 7)$, $p2$



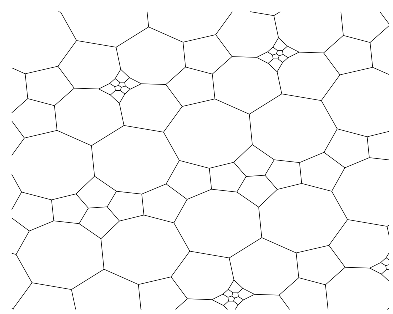
$(42, 14, 7)$, $p2$



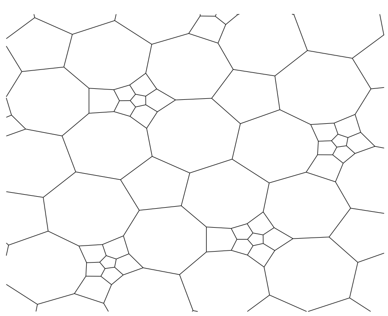
$(48, 16, 8)$, $p2$



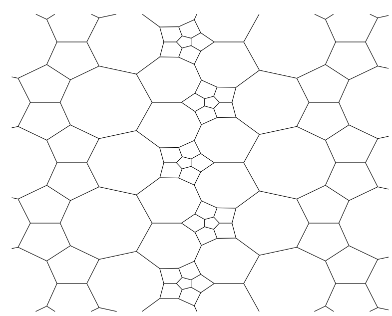
$(48, 16, 8)$, $p2$



$(48, 16, 8)$, $p2$

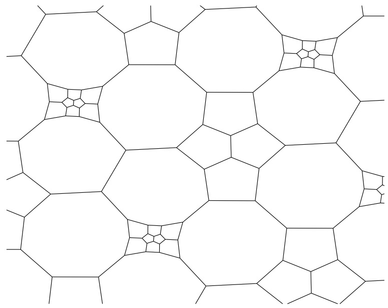


$(48, 16, 8)$, pg



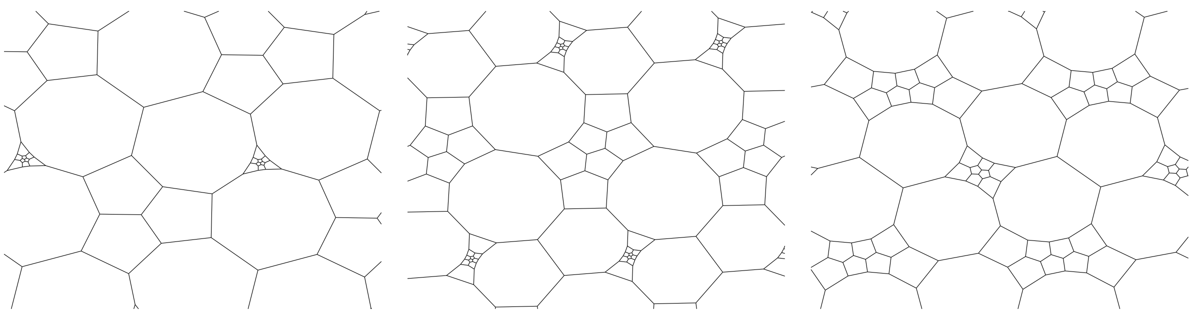
$(48, 16, 8)$, $p2mg$

Figure 69: Some $(5, 8)$-tori, which are $8R_4$

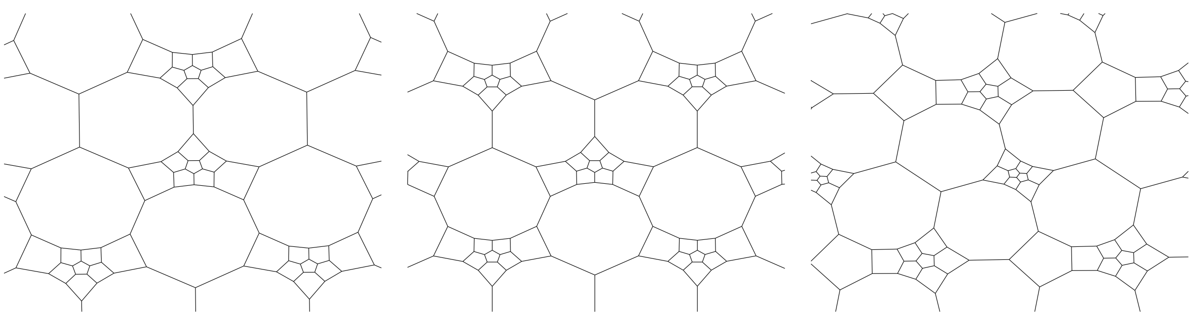


$(32, 12, 4)$, $p2$

Figure 70: A $(5, 9)$-torus, which is $9R_4$



$(30, 12, 3)$, $p3$ $(40, 16, 4)$, $p2$ $(40, 16, 4)$, $p2$



$(40, 16, 4)$, $c2mm$ $(40, 16, 4)$, $p2mg$ $(40, 16, 4)$, $p1$

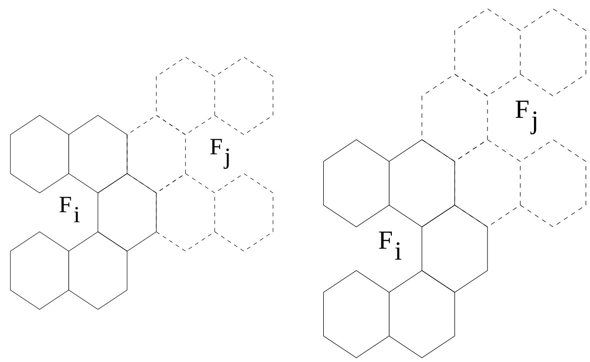
Figure 71: Some $(5, 10)$-tori, which are $10R_4$

Lemma 11.2 *Take a map G, such that its set $\mathcal{F}$ of faces is partitioned into two classes, $\mathcal{F}_1$ and $\mathcal{F}_2$, so that any face F in $\mathcal{F}_1$ is 6-gonal and adjacent to exactly five other faces of $\mathcal{F}_1$. Then it holds:*

(i) A face $F \in \mathcal{F}_2$ is adjacent only to faces in $\mathcal{F}_1$.

(ii) There exists a 3-valent map G', such that $G = GC_{2,1}(G')$ (cf. Goldberg-Coxeter construction in Section 1).

Proof. (i) follows from direct analysis of possible coronae of faces. Given a face $F \in \mathcal{F}_2$, denote by $N(F)$ the neighborhood of F in $\mathcal{F}_1$ (i.e., the set of all faces from $\mathcal{F}_1$, which are adjacent to F). Clearly, the set $\mathcal{F}_1$ is partitioned into $N(F_1), \dots, N(F_k)$. Suppose that two sets $N(F_i)$ and $N(F_j)$ have an adjacency. Then the following two cases are possible:



Both of those cases correspond to the local configuration arising in the Goldberg-Coxeter construction (see [Gold37], [Cox71], and [DuDe03]). Moreover, the choice of a local configuration determines the whole structure completely, i.e., there is only one choice globally.

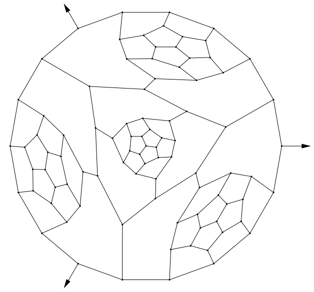
Now, define the map G' with faces corresponding to the set $\mathcal{F}_2$, edges corresponding to pairs $N(F_i)$, $N(F_j)$, having some adjacencies, and vertices corresponding to triples $N(F_i)$, $N(F_j)$, $N(F_k)$, having pairwise adjacencies. G' is a 3-valent plane graph and $GC_{2,1}(G')$ is isomorphic to G. □

Lemma 11.3 *The set of $(5,3)$-polycycles, having boundary sequence $(23^h)^g$, consists of:*

- *elementary $(5,3)$-polycycle A_2 with $h = 3$, $g = 2$,*
- *elementary $(5,3)$-polycycle A_3 with $h = 2$, $g = 3$,*
- *elementary $(5,3)$-polycycle A_5 with $h = 1$, $g = 5$,*
- *elementary $(5,3)$-polycycle D with $h = 0$, $g = 5$.*

Proof. Take a $(5,3)$-polycycle with boundary sequence $(23^h)^g$ and assume that it consists of several elementary $(5,3)$-polycycles put together.

Since it is a $(5,3)$-polycycle, the graph, formed by elementary $(5,3)$-polycycles with two of them being adjacent if they share a bridge, is a tree. This tree has at least one vertex of degree 1. Such a vertex correspond to an elementary $(5,3)$-polycycle, say, P.



92, T

Figure 72: The smallest $(5,8)$-sphere, which is $8R_5$

If P has at least twice the pattern 22 in its boundary sequence, then $h=0$ and we are done. Hence, one can assume that it has the pattern 22 only once in its boundary sequence; so, it is B_2 or B_3. But this is, clearly, impossible. Hence, the only possible cases are the elementary $(5,3)$-polycycles indicated above. $\square$

Theorem 11.4 *Take a $(5,q)$-map, which is qR_5, such that the corona of each q-gon is $q^5 5^{q-5}$ and whose graph $q(G)$ is connected.*

Then it is one of the following:

- *$GC_{2,1}$(Dodecahedron), i.e., the strictly face-regular Nr. 55,*
- *the unique $(5,7)$-sphere $7R_5$ with 260 vertices, depicted in Figure 73,*
- *the unique $(5,8)$-sphere $8R_5$ with 92 vertices, depicted in Figure 72,*
- *the unique $(5,9)$-sphere $9R_5$ with 68 vertices, depicted in Figure 78.*

Proof. The connectedness of $q(G)$ implies that the set of 5-gonal faces is partitioned into $(5,3)$-polycycles. The corona condition implies that the boundary sequence is of the form $(23^{q-6})^g$. Lemma 11.3 gives that the number g depends only on q.

After replacing those $(5,3)$-polycycles by g-gons, one gets a map G with 6- and g-gons only and with $g \leq 5$. Hence, this map is on the sphere; it satisfies the condition of Lemma 11.2 and one gets the result. $\square$

Theorem 11.5 *The smallest $(5,8)$-sphere, which is $8R_5$, is depicted on Figure 72.*

Proof. Euler formula for a $(5,8)$-sphere gives $12 = f_5 - 2f_8$; hence, $e = 30 + 9f_8$ and $v = 20 + 6f_8$. Since the example, given above, has 92 vertices, one can assume in the following, that $f_8 \leq 12$. Further enumeration yields:

$$e_{5-5} = 3f_8, \quad e_{8-8} = 5f_8 \quad \text{and} \quad e_{5-5} = 30 + (3 + \frac{1}{2})f_8 \,.$$

The set of 5-gonal faces is partitioned into elementary $(5,3)$-polycycles by bridges.

Since the sphere is $8R_5$, every 8-gon is adjacent to exactly three 5-gons; hence, the only elementary $(5,3)$-polycycles, which can appear in the decomposition, are A_3, A_4, A_5, D, B_3, C_3, E_1, E_2, E_3, E_4. Denote by f_{A_3}, $\dots$, their numbers and by t the number of adjacencies of elementary $(5,3)$-polycycles along their open edges.

By direct counting, one gets the equalities:

$$\left\{\begin{array}{lcl} e_{5-8} & = & 9f_{A_3}+10f_{A_4}+10f_{A_5}+5f_D+11f_{B_3}+12f_{C_3} \\ & & +9f_{E_1}+10f_{E_2}+11f_{E_3}+12f_{E_4}-2t \\ e_{5-5} & = & 18f_{A_3}+15f_{A_4}+10f_{A_5}+12f_{B_3}+6f_{C_3}+3f_{E_1}+5f_{E_2}+7f_{E_3}+9f_{E_4}+t\ . \end{array}\right.$$

Furthermore, the number t of adjacencies between $(5,3)$-polycycles satisfies the inequality:

$$2t \le 2f_D + 3f_{E_1} + 2f_{E_2}\ .$$

By combining above equalities and inequalities one gets:

$$\left\{\begin{array}{lcl} 2e_{5-8}-e_{5-5} & = & \frac{5}{2}f_8-30=\frac{5}{2}(f_8-12)\le 0 \\ 2e_{5-8}-e_{5-5} & = & 5f_{A_4}+10f_{A_5}+10f_D+10f_{B_3}+18f_{C_3} \\ & & +15f_{E_1}+15f_{E_2}+15f_{E_3}+15f_{E_4}-5t \\ & \ge & 5f_{A_4}+10f_{A_5}+5f_D+10f_{B_3}+18f_{C_3}+\frac{15}{2}f_{E_1}+10f_{E_2}+15f_{E_3}+15f_{E_4} \\ & \ge & 0\ . \end{array}\right.$$

This implies $2e_{5-8}-e_{5-5}=0$; hence, $f_8=12$ and $f_{A_4}=f_{A_5}=f_D=f_{B_3}=f_{C_3}=f_{E_i}=0$. So, the only polycycle, appearing in the decomposition, is A_3. So, by Theorem 11.4, the sphere is obtained by $GC_{2,1}(Tetrahedron)$ and replacing all triangles by the polycycles A_3. □

Theorem 11.6 *There is an infinity of $(5,q)$-spheres, which are qR_5, for $q=8$, 11, 14, 17, 20.*

Proof. The example for $q=8$, which is given in Theorem 11.5, has four $(5,3)$-polycycles A_3. The conclusion follows by Theorem 1.5.

In order to get a proof for other values of q, we need an initial example. The graph $GC_{2,1}(Tetrahedron)$ can be interpreted in the following way. The triangular faces can be shrunken to just one point. The obtained graph is Dodecahedron with a set $\mathcal{S}$ of four special vertices, corresponding to those faces. For any $1\le t\le 5$, there exist a set $\mathcal{S}_t$ of $4t$ vertices of Dodecahedron, such that every face is incident to t vertices of this set:

1. For $t=0$ or 5 (Dodecahedron or strictly face-regular Nr. 14), there is one possible set and it has symmetry I_h,
2. For $t=1$ or 4 (strictly face-regular Nrs. 6 or 13), there is one possible set and it has symmetry T,
3. For $t=2$ or 3 (strictly face-regular Nrs. 8,9 or 11,12), there are two possibilities, one of symmetry D_3, the other of symmetry T_h.

By truncating the vertices in one of those sets $\mathcal{S}_t$ and replacing the obtained triangles by elementary $(5,3)$-polycycles A_3, one gets a $(5, 5+3t)$-sphere, which is $(5+3t)R_5$. The proof of infiniteness is then identical to the case $q = 8$. □

In the above construction, we got a $(5,8)$-sphere by putting the spheres together in a path. If one create cycle, then one obtains higher genus surfaces, which are $8R_5$. So, for every $g \geq 0$, there is an infinity of $(5,8)$-surfaces of genus g, which are $8R_5$.

Also, above operation of removing a vertex leave us with six 5-gons in a circuit; hence, $5R_2$ holds. Clearly, if one manage to eliminate all $(5,3)$-polycycles A_3, so as to obtain a cycle, then one gets a $(5,8)$-surface of genus g, which is $8R_5$ and $5R_2$. Such a structure can be obtained for any g, $g \geq 2$. But for $g = 1$ it does not exist.

Theorem 11.7 *There is an infinity of* $(5,q)$*-spheres, which are* qR_5*, for* $q = 9$*, 12, 15, 18, 21.*

Proof. The graph, shown on Figure 78, is such a sphere. It can be obtained by taking $GC_{2,1}(Bundle)$ (see Subsection 3.1) and replacing every 2-gon by a $(5,3)$-polycycle A_2. The conclusion for $q = 9$ follows from Proposition 1.5.

On the other hand, if one takes the graph $GC_{2,1}(Bundle)$ and remove all 2-gons, then the obtained graph is Cube. For every $0 \leq t \leq 4$, there exist sets $\mathcal{S}_t$ of $2t$ vertices, such that every face of Cube is incident to exactly t vertices of $\mathcal{S}_t$. More precisely:

1. For $t = 0$ or 4 (Cube and strictly face-regular Nr. 10), there is one such set and it has the symmetry O_h.

2. For $t = 1$ or 3 (strictly face-regular Nrs. 2, 7), there is one such set and it is of symmetry D_{3d}.

3. For $t = 2$ (strictly face-regular Nrs. 4, 5), there are two such sets, one of symmetry T_d and the other of symmetry D_{2h}.

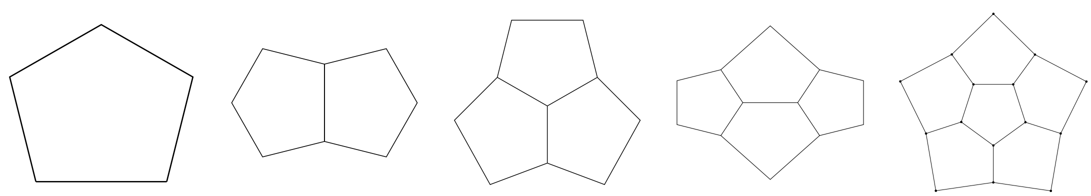
So, take Cube, truncate vertices belonging to the set $\mathcal{S}_t$ and fill them by $(5,3)$-polycycles A_3. Also, insert three 2-gons on the edges corresponding to the 2-gons of the original $(5,9)$-sphere, which is $9R_5$. One gets infiniteness by Theorem 1.5. □

By creating cycles in the chain of $(5,9)$-spheres put together, we get surfaces of genus $g \geq 1$. By changing all elementary $(5,3)$-polycycles A_3 into pairs of adjacent $(5,3)$-polycycles E_2, we get a $(5,9)$-map $5R_3$. Easy to see, that such construction is possible for any genus $g \geq 2$.

Theorem 11.8 *If* G *is a* $(5,q)$*-sphere, which is* qR_{q-2}*, then:*

(i) The graph $q(G)$ *is connected.*

(ii) The set of 5*-gonal faces belongs to the following set of* $(5,3)$*-polycycles:*



(iii) $q \le 7$.

(iv) If $q = 6$*, then such spheres are enumerated in [DeGr01] on pp 187-188 and presented on Figure 13.*

(v) If $q = 7$*, then such sphere has at least* 260 *vertices. If it has* 260 *vertices, then it is unique and has symmetry I (see Figure 73); otherwise, it has at least* 280 *vertices.*

Proof. Suppose that the graph $q(G)$ is not connected. This means that there exists a set of 5-gonal faces, on which at least two (say, t) connected components of $q(G)$ meet.

Since every q-gon is adjacent to exactly two pentagons, in the $(5,3)$-polycycle, the runs of 3 (i.e., sequence of 3 bounded by 2) of the boundary sequence have length at most one, i.e., every 3-valent vertex is bounded by two 2-valent vertices. This implies $v_3 \le v_2$ and so, $f_5 \le 0$, by Proposition 1.4. Hence, (i) is true.

Now, the connectedness of $q(G)$ implies that the set of 5-gonal faces form $(5,3)$-polycycles, i.e., they have $t = 1$ and $f_5 = 6 + v_3 - v_2 \le 6$. The set of $(5,3)$-polycycles, satisfying the condition that every run of 3 is of length at most one, is the announced one.

The set of faces of $q(G)$ comes from vertices of G, which are incident to three q-gonal faces (so, they are of gonality 3) and the $(5,3)$-polycycles (so, they are of gonality 5 or 6). Hence, by Lemma 4.2, $q(G)$ has a vertex of degree at most 5; so, $q-2 \le 5$ and (iii) is true.

The following formulae are easy:

$$f_5 = 12 + f_7, \quad e = 30 + 6f_7,$$
$$e_{7-7} = \tfrac{5}{2}f_7, \quad e_{5-7} = 2f_7 \text{ and } e_{5-5} = 30 + \tfrac{3}{2}f_7\ .$$

Denote by n_1, n_2, n_3, n_4, n_5 the respective number of such $(5,3)$-polycycles, depicted in (ii).

By direct counting, one obtains:

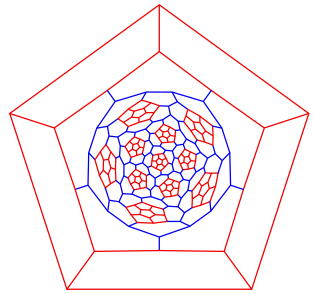
$$\begin{cases} 12 + f_7 = f_5 &= n_1 + 2n_2 + 3n_3 + 4n_4 + 6n_5, \\ 30 + \frac{3}{2}f_7 = e_{5-5} &= n_2 + 3n_3 + 5n_4 + 10n_5\ . \end{cases}$$

By eliminating n_5, one obtains also:

$$60 - f_7 = -10n_1 - 14n_2 - 12n_3 - 10n_4,$$

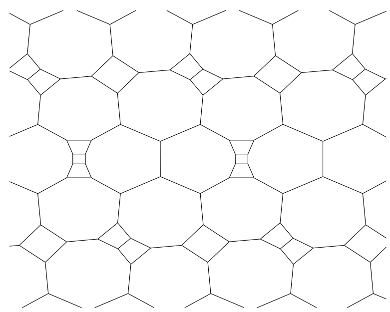
which implies $f_7 \ge 60$, i.e., that a $(5,7)$-sphere, which is $7R_5$, has at least 260 vertices. Furthermore, such a sphere with exactly 260 vertices has only the elementary $(5,3)$-polycycle A_5 and we conclude by Theorem 11.4. If it has more than 260 vertices, then $n_i > 0$ for some $1 \le i \le 4$, which implies $60 - f_7 \le -10$ and hence, the announced result. □

See below the only known $(4,8)$-torus, which is $8R_5$ and not $4R_1$:



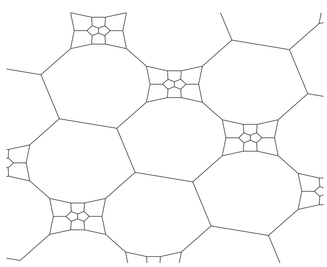
260, I

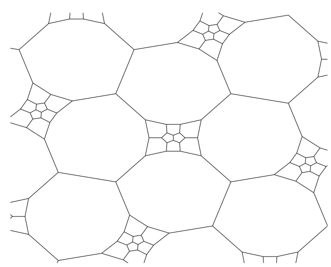
Figure 73: The only $(5,7)$-sphere, which is $7R_5$ and has less than 280 vertices



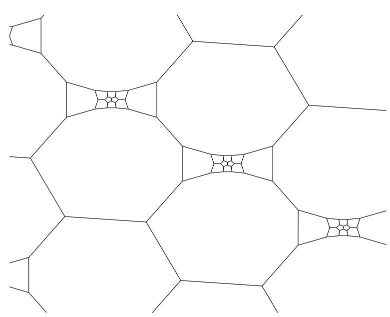
$(24,6,6)$, $p2$

See below two examples of $(5,10)$-tori, which are $10R_5$.





See below an example of a $(5,11)$-torus, which is $11R_5$.



$(24,10,2)$, $p2$

12 Maps qR_6

Theorem 12.1 *(i) Any $(4,q)$-torus, which is qR_6 and 3-connected, is also $4R_2$.*
(ii) All $(4,8)$-tori, which are $8R_6$, are also $4R_2$.

Proof. (i) The hypothesis of 3-connectedness of such a map G excludes the existence of $(4,3)$-polycycle S_2 in the set of 4-gons of this torus.

This means that, if one considers the corresponding 6-valent map $q(G)$, then all its faces are 3-gonal (vertices or $(4,3)$-polycycles S_1) or 4-gonal ($(4,3)$-polycycles P_{2k}). Clearly, by Euler formula, 4-gons are excluded, i.e., only S_1 exist and the map is $4R_2$.

(ii) In a $(4,8)$-torus, no 4-gon can be adjacent only to 4-gons, since it would imply the structure of Cube. If a $(4,8)$-torus contains a 4-gon, adjacent to three 4-gons, then it contains a 8-gon, adjacent to at least three 4-gons and hence, to at most five 8-gons. So, in a $(4,8)$-torus, the 4-gons are adjacent to at most two other 4-gons. The result then follows by usual counting and positivity. □

Theorem 12.2 *There are no $(4,q)$-spheres, which are qR_{q-2}, for $q \geq 8$.*

Proof. Every q-gonal face of such a sphere G would be adjacent to exactly two 4-gons. This means that the $(4,3)$-polycycle S_2 cannot appear in the decomposition of the set of 4-gonal faces. Hence, all faces of the sphere $q(G)$ are 3- or 4-gonal. So, we conclude by Lemma 4.2. □

Theorem 12.3 *There is no $(5,8)$-torus, which is $8R_6$.*

Proof. By re-doing computations of Theorem 11.8 for $q=8$, one would get that a $(5,8)$-torus, which is $8R_6$, has connected $8(G)$ and the set of its 5-gonal faces is partitioned into the $(5,3)$-polycycles, depicted in Theorem 11.8 (ii).

Hence, the torus $8(G)$ is 6-valent and its faces are triangles (intersection of three 8-gons), pentagons and hexagons.

Euler formula for 6-valent toric map is $\sum_i (3-i)p_i = 0$. Hence, there are no pentagonal and hexagonal faces in $8(G)$. So, G has no pentagonal faces, which is impossible. □

Theorem 12.4 *If a $(5,q)$-sphere is qR_j with $j \geq 6$ and $q(G)$ is connected, then $j \leq q-4$.*

Proof. By Lemma 4.2, the map $q(G)$ of such a sphere G contains at least one 2-gon. It is easy to see that the only $(5,3)$-polycycle with two vertices of degree 2 on the boundary is A_2. So, the q-gonal faces, which are adjacent to this $(5,3)$-polycycle A_2, are adjacent to at least four 5-gonal faces. □

Note that one can construct some $(5,10)$-spheres, which are $10R_6$. Take Dodecahedron and select a set S of its edges, such that every 5-gon is incident to exactly one edge of this set. Replacing those edges by the $(5,3)$-polycycles A_2, one obtains such spheres. Up to isomorphism, there exist five such sets in Dodecahedron and they yield five $(5,10)$-spheres $10R_6$ with 140 vertices and symmetry groups D_{3d}, C_2, D_2, D_3, T_h.

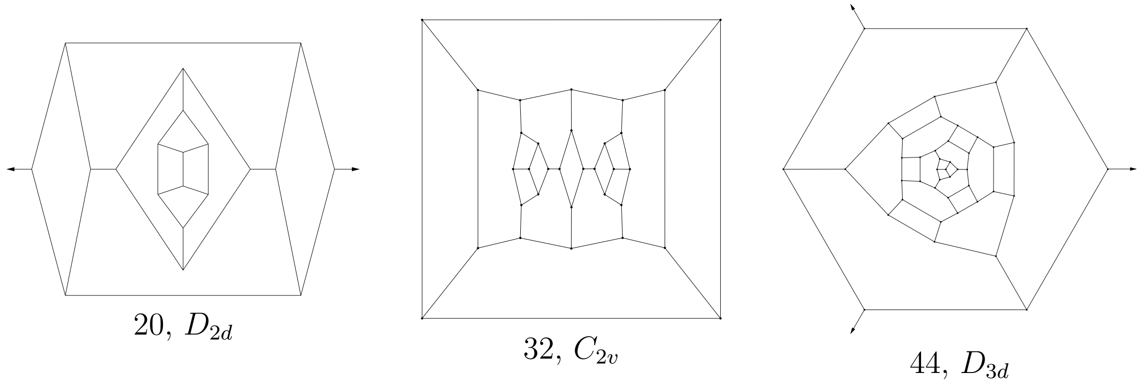


Figure 74: All weakly face-regular $(4,7)$-spheres, which are $7R_4$

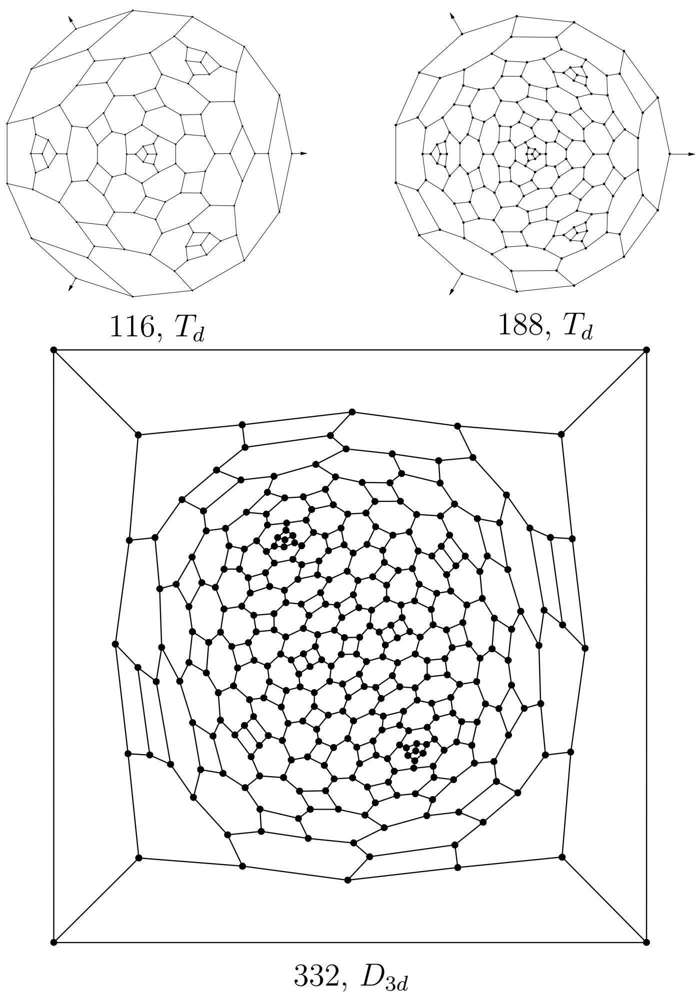
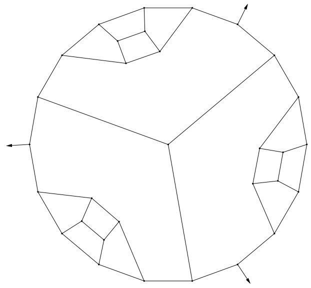
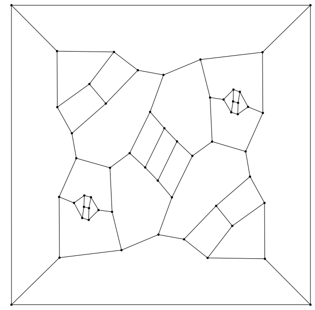


Figure 75: All known $(4,7)$-spheres, which are $7R_5$ (no one exists with at most 62 vertices)



32, D_3



56, C_2

Figure 76: All weakly face-regular $(4, 8)$-spheres, which are $8R_5$ and have at most 56 vertices

13 Other enumeration results: $(4, q)$-maps

In four remarks below (see also Table 1) we consider only $(4, q)$-spheres, which are different from Cube.

Remark 13.1 *Any $(4, 7)$-sphere $7R_j$ is one of the following:*

(i) Six strictly face-regular ones (Prism$_7$ and Nrs. 27–31).

(ii) Seven weakly face-regular ones: unique $7R_2$ (see Theorem 4.10), three $7R_3$ from Figure 45 and three $7R_4$ from Figure 74.

(iii) Undecided case $j = 5$ (Theorem 4.7(ii2) implies that $e_{4-4} = 12$ in this case). Two examples are known, Nr. 29 and the one on Figure 75.

Any $(4, 7)$-sphere $4R_i$ is one of the following:

(i) For $i = 0$, there exist an infinity of such spheres by Theorem 5.2.

(ii) For $i = 1$, besides 6 spheres on Figure 16 and Nrs. 27, 28, there exists an infinity of such spheres by Theorem 5.3.

(iii) For $i = 2$, all known ones are Prism$_7$, Nr. 29, an infinite series in Theorem 5.8, the list of known ones up to 110 vertices on Figure 20.

Remark 13.2 *For the $(4, 8)$-spheres, which are $8R_j$, one has $j \leq 6$ and:*

(i) All with $j \leq 2$ are given in Theorem 4.10.

(ii) All $8R_3$ are five spheres on Figure 46.

(iii) All known $8R_4$ are strictly face-regular ones Nrs. 31–34 and 15 spheres on Theorem 10.3.

(iv) All known $8R_5$ are two spheres on Figure 76.

(v) There is no $8R_6$ with at most 56 vertices. If there is such a sphere, it cannot be 3-connected.

No $4R_0$, $4R_3$ are possible by Theorem 5.1.

There is an infinity of $(4, 8)$-spheres, which are $4R_1$, by Theorem 5.4. See Figure 17 for all such spheres with at most 56 vertices.

By Theorem 5.9(i), there exist an infinite series of $(4, 8)$-spheres $4R_2$.

Note that there exist a $(4,8)$-torus, which is $4R_2$ but not $8R_6$, while the property $8R_5$ is not related to property $4R_1$.

Remark 13.3 *For the* $(4,9)$*-spheres, which are* $9R_j$*, one has* $j \leq 6$ *and:*

(i) All with $j \leq 2$ *are given in Theorem 4.10.*

(ii) All $9R_3$ *are strictly face-regular Nr. 35 and five spheres on Figure 47.*

(iii) All known $9R_4$ *are four spheres on Figure 77.*

(iv) All known $9R_5$ *are three spheres on Figure 78.*

(v) There is no $9R_6$ *with at most* 58 *vertices.*

For a $(4,9)$*-sphere, which is* $4R_j$*, one has:*

(i) No $4R_0$*,* $4R_3$ *are possible and no* $4R_1$ *with at most* 58 *vertices is known.*

(ii) Any $4R_1$ *has at least* 108 *vertices by Theorem 5.5.*

(iii) All known $4R_2$ *are* $Prism_9$*, strictly face-regular Nr. 35, two spheres shown on Figure 23 and an infinite series build in Proposition 5.9.*

Remark 13.4 *For the* $(4,10)$*-spheres, which are* $10R_j$*, one has* $j \leq 7$ *and:*

(i) All with $j \leq 2$ *are given in Theorem 4.10.*

(ii) All $10R_3$ *are six spheres on Figure 48.*

(iii) All known $10R_4$ *are strictly face-regular Nr. 38 and eight spheres on Figure 79.*

(iv) All known $10R_5$ *are two spheres on Figure 80.*

(v) No $10R_6$*,* $10R_7$ *with at most* 56 *vertices are known.*

For a $(4,10)$*-sphere, which is* $4R_i$*, one has:*

(i) $4R_0$*,* $4R_1$ *and* $4R_3$ *are impossible by Theorem 5.1.*

(ii) The only $4R_2$ *with at most* 56 *vertices are* $Prism_{10}$*, strictly face-regular Nr. 38 and an infinite series build in Theorem 5.11.*

14 Other enumeration results: $(5,q)$-maps

In four remarks below (see also Table 2) we consider only $(5,q)$-spheres, which are different from Dodecahedron.

Remark 14.1 *Any* $(5,7)$*-sphere* $7R_j$ *is one of the following:*

(i) Three strictly face-regular ones ($Barrel_7$ *and Nrs. 56, 57).*

(ii) Three weakly face-regular ones with $7R_1$ *given on Figure 37.*

(iii) 25 *weakly face-regular ones for* $j = 2$ *(see [DDS04]).*

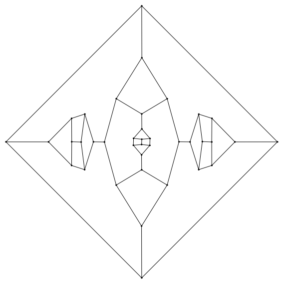
(iv) See on Figures 81, the only known $(5,7)$*-spheres, which are* $7R_3$*.*

(v) There is an infinity of $(5,7)$*-sphere, which are* $7R_4$*. Conjecturally, they all belong to an infinite series, whose first two members are presented on Figure 67.*

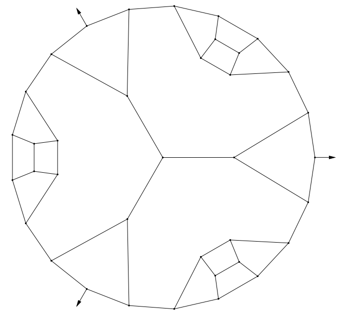
(vi) The only known $(5,7)$*-sphere, which is* $7R_5$*, is presented on Figure 73; an infinite series of such spheres may exist.*

Any $(5,7)$*-sphere* $5R_i$ *is one of the following:*

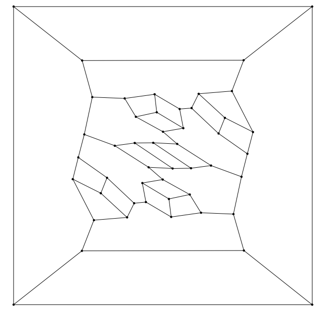
(i) Three strictly face-regular ones from (i) above.



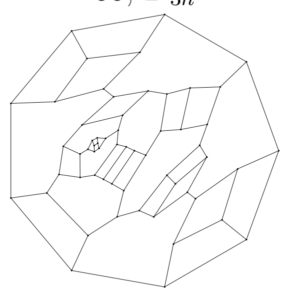
38, C_{2v}



38, D_{3h}

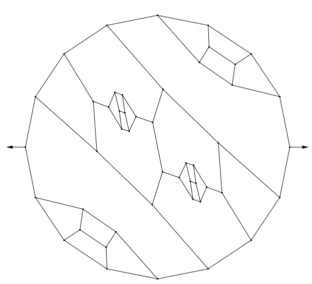


48, D_2

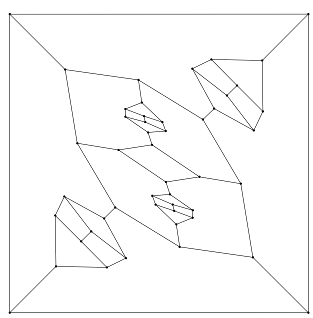


58, C_1

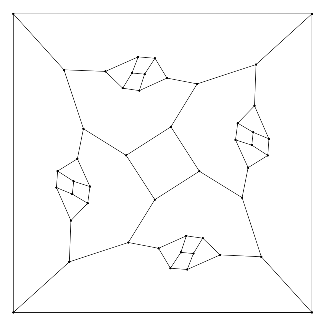
Figure 77: All weakly face-regular $(4,9)$-spheres, which are $9R_4$ and have at most 58 vertices



48, C_2



48, D_2



48, D_4

Figure 78: All weakly face-regular $(4,9)$-spheres, which are $9R_5$ and have at most 58 vertices

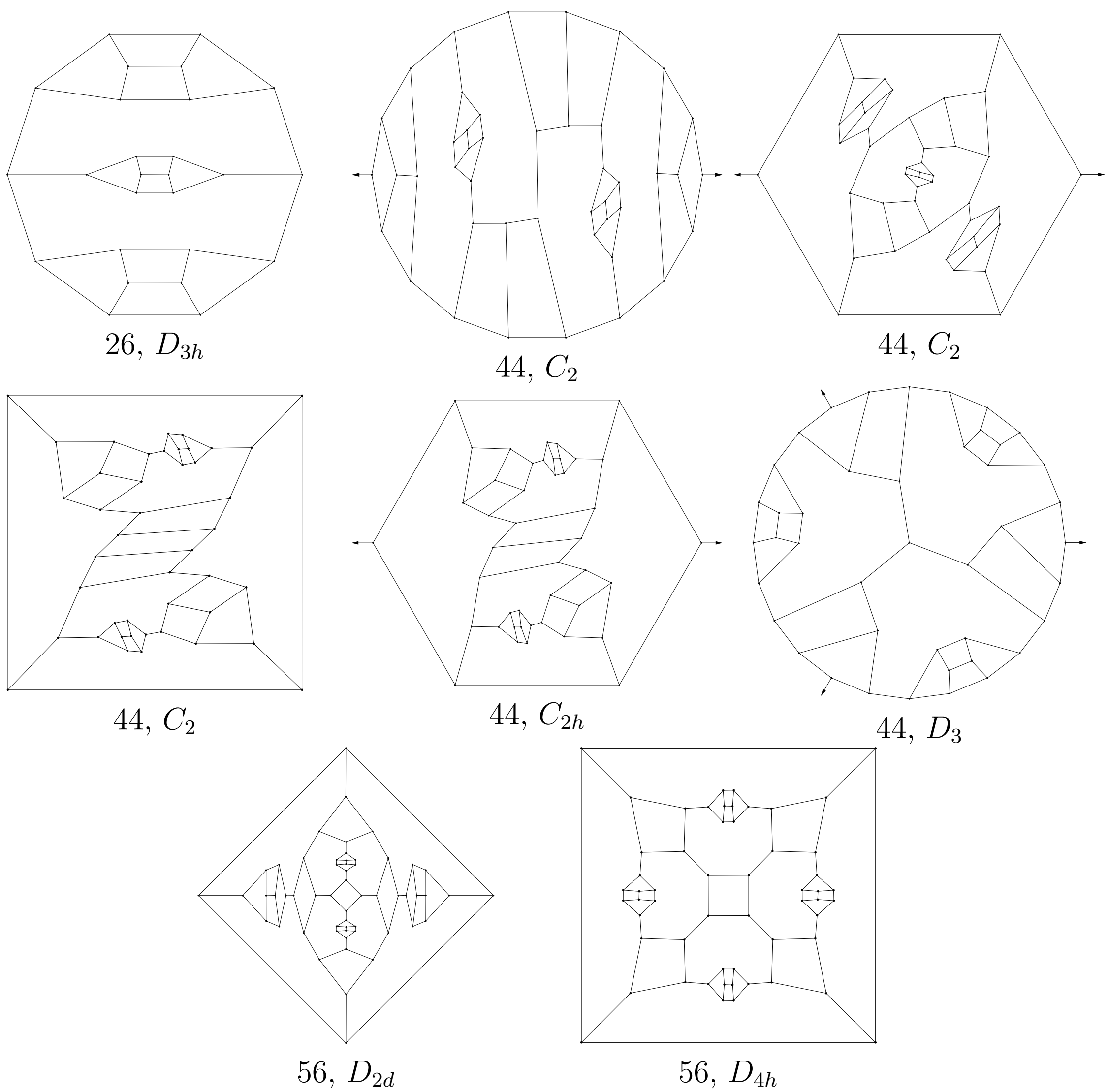
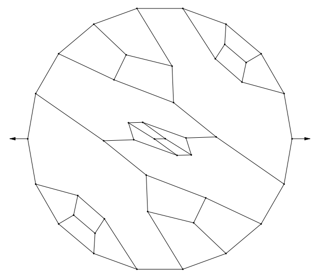
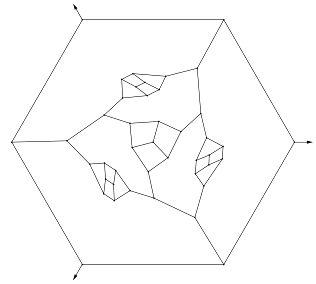


Figure 79: All weakly face-regular $(4,10)$-spheres, which are $10R_4$ and have at most 56 vertices



44, C_2



44, D_3

Figure 80: All weakly face-regular $(4,10)$-spheres, which are $10R_5$ and have at most 56 vertices

(ii) The spheres $5R_2$ *(they have at least* 84 *vertices; an infinite series of such spheres with one cycle is constructed in [HaSo04]).*

(iii) Two spheres $5R_3$*, which are given on Figure 26.*

Remark 14.2 *For a* $(5,8)$*-sphere, which is* $8R_j$*, one has* $j \leq 6$ *and:*

(i) All $8R_0$ *are strictly face-regular ones* $Barrel_8$ *and Nr. 58.*

(ii) All known $8R_1$ *are four spheres on Figure 38, there is a finite number of such spheres by Theorem 4.4(ii).*

(iii) All known $8R_2$ *are five spheres on Figure 82, the infinite series of* $(5,8)$*-spheres with* 8*-gons organized in exactly one ring (see Figure 12 in [DDS04]) and strictly-face regular one Nr. 59 with* 92 *vertices.*

(iv) All known $8R_3$ *are four spheres on Figure 83.*

(v) All known $8R_4$ *are four spheres on Figure 84.*

(vi) there is an infinite series of $8R_5$ *by Theorem 11.6 and the smallest one has* 92 *vertices (see Theorem 11.5).*

Remark 14.3 *For a* $(5,9)$*-sphere, which is* $9R_j$*, one has* $j \leq 7$ *and:*

(i) All $9R_0$ *are strictly face-regular* $Barrel_9$ *and two spheres on Figure 29.*

(ii) All known $9R_1$ *are five spheres on Figure 39, there is a finite number of such spheres by Theorem 7.2.*

(iii) All known $9R_2$ *are four spheres on Figure 85.*

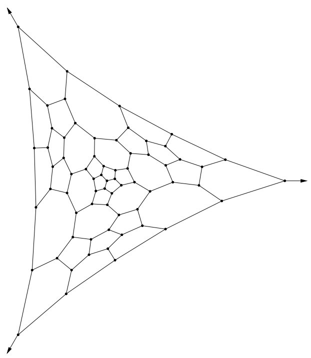
(iv) All known $9R_3$ *are three spheres on Figure 86.*

(v) The smallest $9R_5$ *is presented on Figure 87 and an infinite series of such spheres is constructed in Theorem 11.7.*

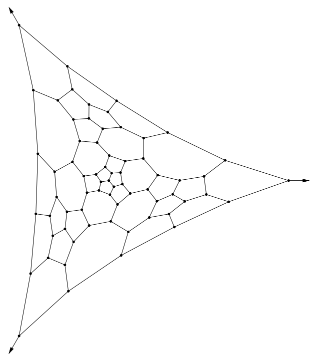
(vi) There are no $9R_4$ *with at most* 68 *vertices.*

Remark 14.4 *For a* $(5,10)$*-sphere, which is* $10R_j$*, one has* $j \leq 8$ *and:*

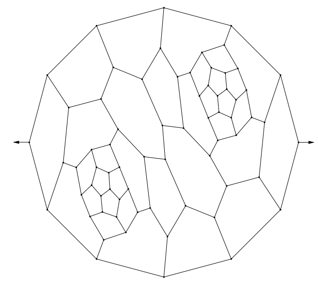
(i) All $10R_0$ *are strictly face-regular ones* $Barrel_{10}$*, Nr. 60 and three spheres on Figure 30.*



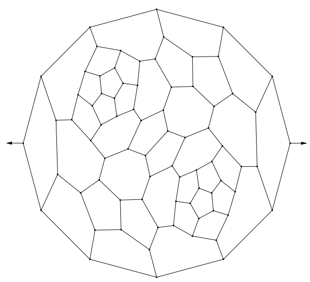
68, C_3



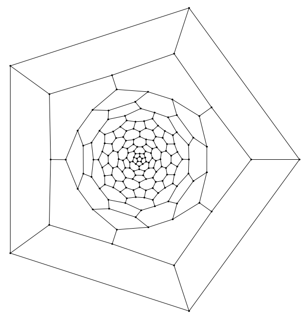
68, T



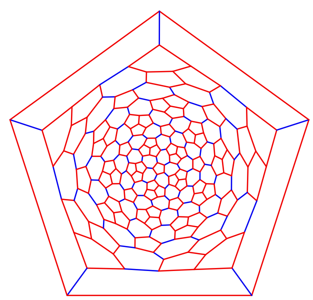
68, D_2



76, C_2



180, D_5



260, I

Figure 81: All weakly face-regular $(5,7)$-spheres, which are $7R_3$ and have at most 76 vertices, an example with 180 vertices and the first one with symmetry I

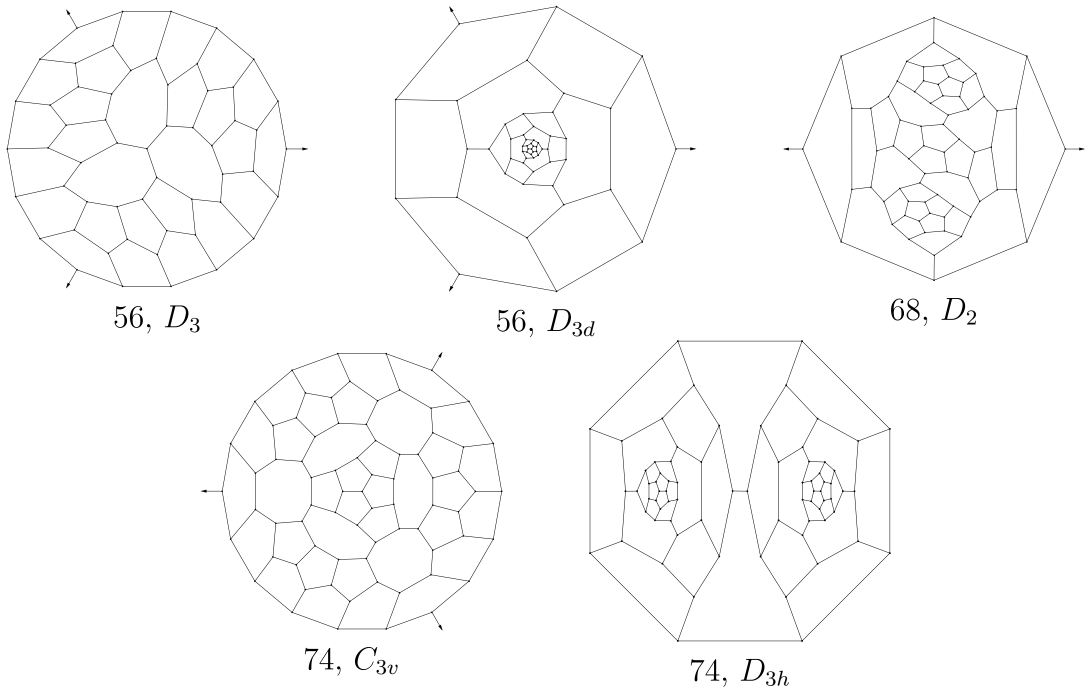


Figure 82: All weakly face-regular $(5, 8)$-spheres, which are $8R_2$ and have at most 74 vertices and more than one circuit.

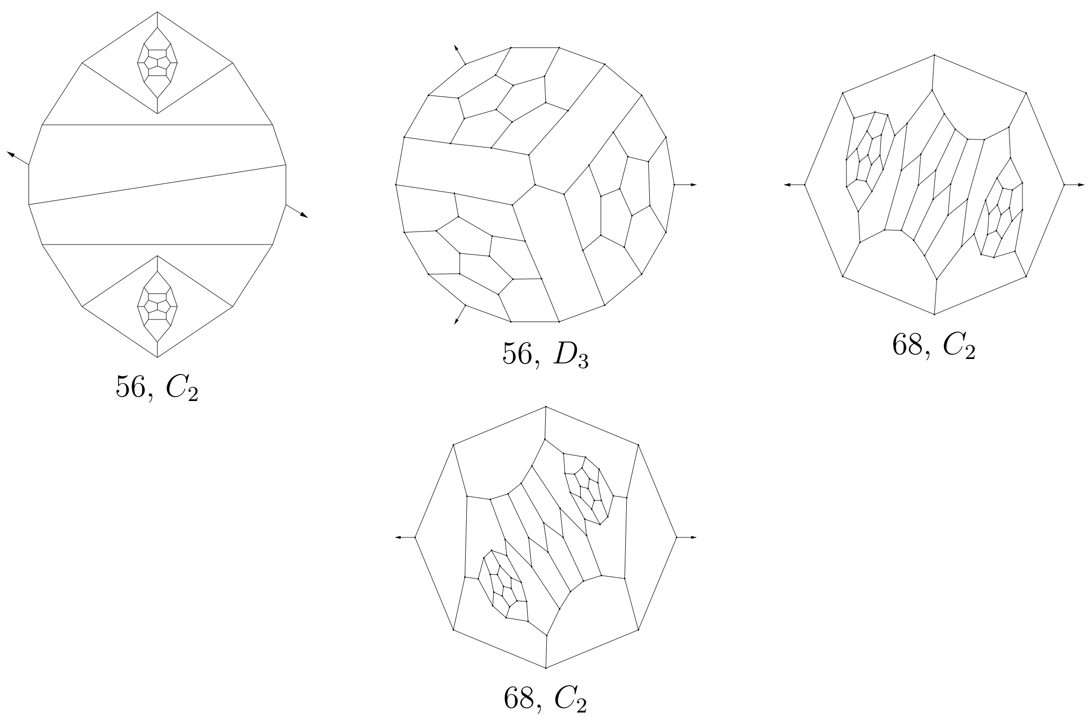
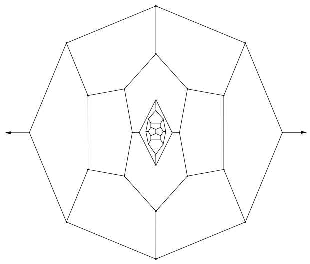
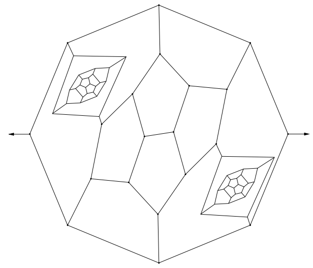


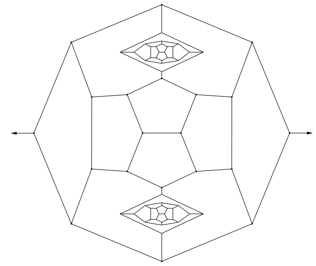
Figure 83: All weakly face-regular $(5, 8)$-spheres, which are $8R_3$ and have at most 74 vertices



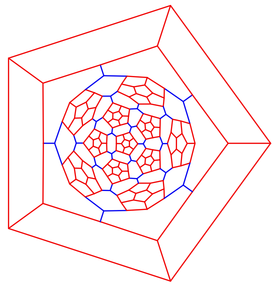
44, D_{2d}



68, C_2

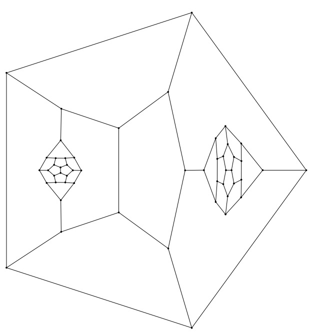


68, D_{2h}

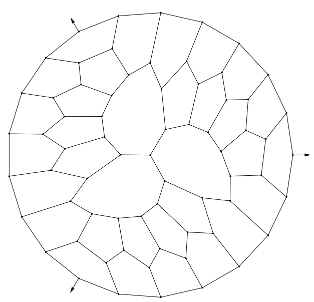


200, I_h

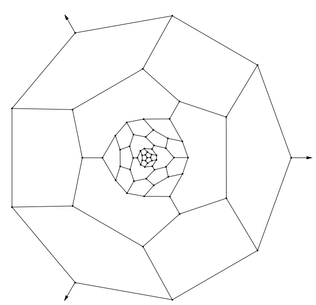
Figure 84: All weakly face-regular $(5,8)$-spheres, which are $8R_4$ and have at most 74 vertices, and the one with 200 vertices and symmetry I_h



52, D_{2d}

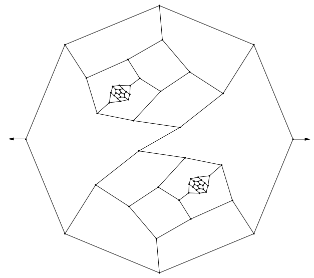


68, D_3

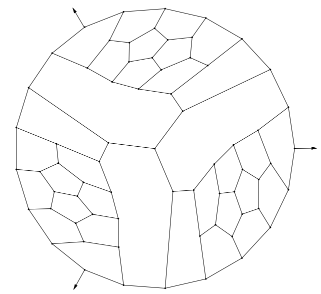


68, D_{3h}

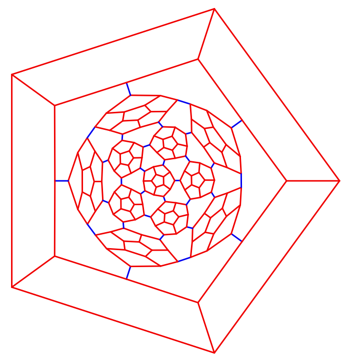
Figure 85: All weakly face-regular $(5,9)$-spheres, which are $9R_2$ and have at most 68 vertices and more than one circuit



68, C_2

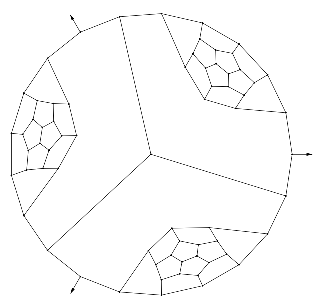


68, D_3



180, I_h

Figure 86: All weakly face-regular $(5,9)$-spheres, which are $9R_3$ and have at most 68 vertices, and smallest $(5,9)$-sphere $9R_3$ of symmetry I or I_h



68, D_3

Figure 87: The only weakly face-regular $(5,9)$-sphere, which is $9R_5$ and have at most 68 vertices

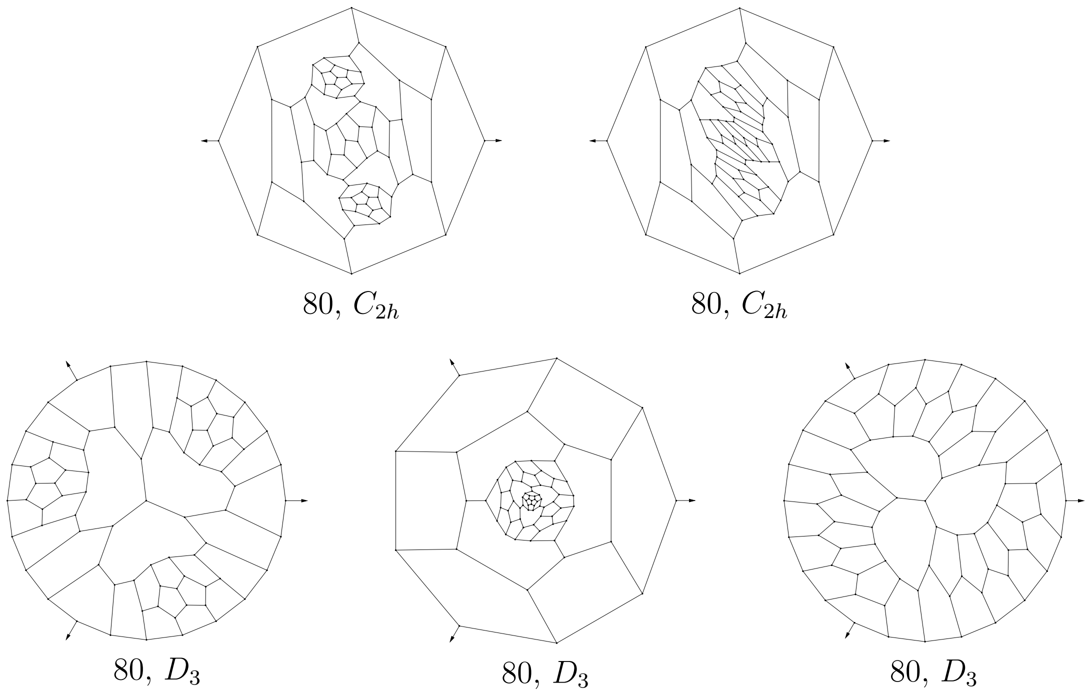


Figure 88: All $(5, 10)$-spheres, which are $10R_2$, have more than one ring of 10-gons (with all rings of 10-gons of length greater than 2) and have at most 80 vertices

(ii) All known $10R_1$ *are ten spheres on Figure 40.*

(iii) All known $10R_2$ *are* 12 *spheres on [DDS04], an infinity of non* 3*-connected such spheres obtained by inserting Dodecahedron (on an edge separating two pentagons, which are not adjacent) to a* $10R_2$ *sphere and five spheres on Figure 88. See [DDS04] for an iteration of this construction from the Dodecahedron.*

(iv) All known $10R_3$ *are three spheres on Figure 89.*

(v) All known $10R_4$ *are three spheres on Figure 90.*

(vi) There is no $10R_j$ *with* $j \geq 5$ *and at most* 80 *vertices.*

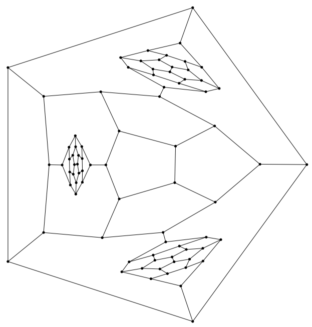
15 2-connectivity: non-polyhedral (p, q)-spheres

If a graph is not 3-connected, then this means that deleting of some two edges disconnect it, i.e., that two faces share more than one edge. For all examples in this paper (except for Theorem 6.5), those two faces are of the same gonality q and share exactly two edges.

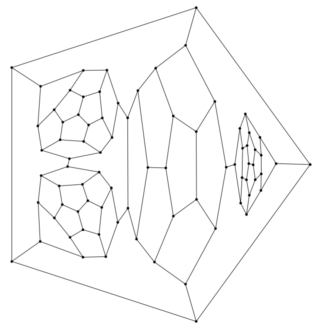
Also it turns out, that in all known examples those two faces are of the same gonality q.

Moreover, in all those examples, between two common edges of q-gons lies one of four following graphs:

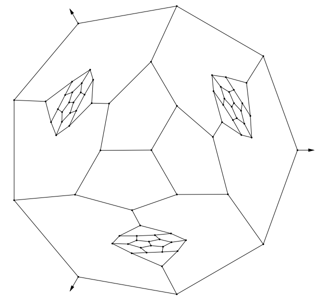
- I_0 = Tetrahedron-e (a shortened notation for Tetrahedron without one edge),



80, C_s

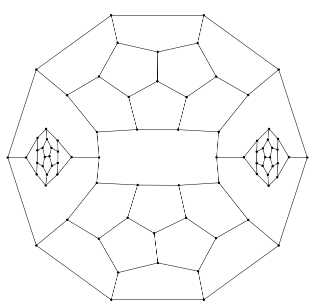


80, C_s

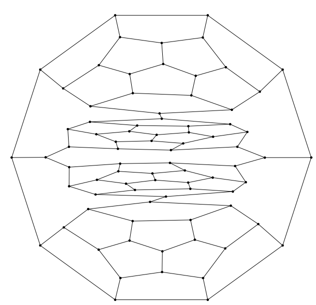


80, D_3

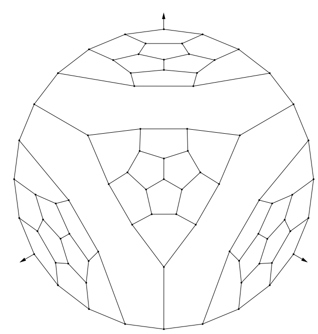
Figure 89: All weakly face-regular $(5, 10)$-spheres, which are $10R_3$ and have at most 80 vertices



80, D_{2h}



80, D_{2h}



80, T_d

Figure 90: All weakly face-regular $(5, 10)$-spheres, which are $10R_4$ and have at most 80 vertices

- I_1 = Cube-e,
- I_2 = Dodecahedron-e (see Remark 14.4),
- I'_1 = a pair I_0 of adjacent triangles with I_1 put on their common edge,
- I'_2 = a pair I_0 of adjacent triangles with I_2 put on their common edge.

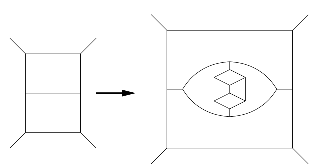
The appearance of those graphs is as follows:

- I_1 appears for $p = 4$ and (q, j)= $(8, 3)$ 1st on Figure 46, $(8, 4)$ on Figure 62, $(8, 5)$ all on Figure 76; $(9, 3)$ 1st on Figure 47, $(9, 4)$ all but one on Figure 77, $(9, 5)$ all on Figure 78; $(10, 3)$ 1st on Figure 48, $(10, 4)$ all but one on Figure 79, $(10, 5)$ all on Figure 80.
- I'_1 appears for $p = 4$ and $(q, j) = (7, 4)$ 1st on Figure 74.
- I_2 appears for $p = 5$ and (q, j)= $(9, 2)$ 1st on Figure 85, $(9, 3)$ all but last on Figure 86; $(10, 2)$ in exceptional $M_2(4, 10)$, $(10, 3)$ all on Figure 89, $(10, 4)$ 1st on Figure 90.
- I'_2 appears for $p = 5$ and $(q, j) = (8, 3)$ second on Figure 83, $(q, j) = (8, 4)$ all but last on Figure 84.

We have seen in Section 2.1 that the enumeration of strictly face-regular (p, q)-map $3R_1$ or $2R_0$ was more complicate than for other cases. The full description of only 2-connected weakly face-regular (p, q)-maps $2R_0$ or $3R_1$ is an open problem. One case that admits a complete solution is the $(3, 7)$-spheres, which are $3R_1$. The *PT-insertion* on an edge, consist of inserting a pair of triangle along that edge. Given a $(4, 7)$-sphere a *PT-insertion set* $\mathcal{S}$ consists of a set of edges, incident to squares only and such that every square is incident to exactly one edge of $\mathcal{S}$. A $(4, 7)$-sphere is said to be of class I if its only $(4, 3)$-polycycles are P_6, P_{10} and S_2 (see Section 1.2).

Theorem 15.1 *(i) Take a $(3, 7)$-sphere, which is $3R_1$. Then it arises from a PT-insertion on a PT-insertion set of a $(4, 7)$-sphere, which is Cube or of class I.*

(ii) Any $(4, 7)$-sphere G of class I, having a $(4, 3)$-polycycle S_2 arise from another $(4, 7)$-sphere G' of class I, by replacing an edge e of G', which belongs to a PT-insertion set by the following structure:

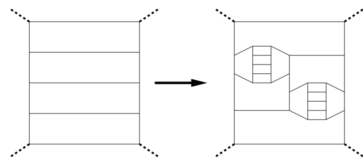


All $(4, 7)$-spheres of class I arise from successive application of the above operation to $(4, 7)$-sphere, having only P_6 and P_{10} as $(4, 3)$-polycycles.

Proof. (i) Take a $(3,7)$-sphere G which is $3R_1$ and remove all pairs of adjacent triangles. The obtained $(4,7)$-sphere G', clearly, has a PT-insertion set from which one can reconstruct the pairs of adjacent triangles. If the sphere is the Cube, then there is nothing to prove. Otherwise, the 4-gons of G' are grouped in $(4,3)$-polycycles, which should have an even number of 4-gons. Clearly, only P_6, P_{10} and S_2 are possible.

(ii) is obvious. □

The above class of $(4,7)$-spheres, whose 4-gons are in $(4,3)$-polycycles P_6 and P_{10} is very large. Up to 62 vertices and except of $(4,7)$-spheres $4R_1$, one has 11 possibilities. Also, this class is infinite, as proved by the transformation below:



16 Remaining questions

We list here remaining problems, which we considerer to be most interesting:

1. Decide finiteness or not for $(4,7)$-spheres $7R_5$ and $(5,7)$-spheres $7R_5$.

2. Decide finiteness or not for $(5,7)$-spheres $7R_3$ and $(5,10)$-spheres $5R_3$.

3. Decide existence of $(5,q)$-tori qR_5

One of the most interesting questions, that arise in this research is to check our conjecture that an infinity of (p,q)-spheres qR_j exists if and only if a (p,q)-torus qR_j exists. Some testing ground for it is to decide finiteness or not for $(4,9)$-spheres $9R_4$ and $(5,13)$-spheres $13R_0$.

For every pair pR_i, qR_j and fixed genus $g \geq 2$, the number of strictly face-regular possibilities is, clearly, finite. But it is, certainly, extremely large. An interesting question would be to decide, what type of strict face-regularity can appear on surfaces of genus $g \geq 2$.

Another direction is to study all weakly face-regular (p,q)-maps of valency s, $s \geq 4$; all such strictly face-regular maps are found in [De02]. One can also permit $p = q$ in the main problem.

Complete classification looks impossible and many finite classes would be extremely large.

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