Zigzags in plane graphs and generalizations

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I. Simple two-faced polyhedra

Polyhedra and planar graphs

A graph is called k-connected if after removing any set of $k-1$ vertices it remains connected.

The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face of P.

Theorem *(Steinitz)*
(i) A graph G is the skeleton of a 3-polytope if and only if it is planar and 3-connected.
(ii) P and P' are in the same combinatorial type if and only if $G(P)$ is isomorphic to $G(P')$.

The dual graph G^* of a plane graph G is the plane graph formed by the faces of G, with two faces adjacent if they share an edge.

Simple two-faced polyhedra

A polyhedron is called simple if all its vertices are 3-valent. If one denote p_i the number of faces of gonality i, then Euler's relation take the form:

$$12 = \sum_i (6-i)p_i \ .$$

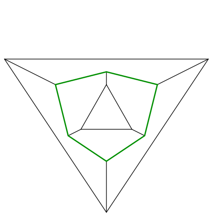
A simple planar graph is called two-faced if the gonality of its faces has only two possible values:

a and b, where $3 \le a < b \le 6$.

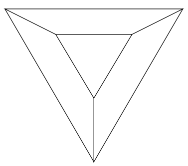
We consider mainly classes q_n, i.e. simple planar graphs with n vertices and $(a,b) = (q,6)$;

there are 3 cases: 3_n, 4_n, 5_n.

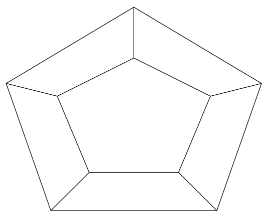
(a, b)	Polyhedra	Exist if and only if	p_a	n
$(5, 6)$	5_n (fullerenes)	$p_6 \in N - \{1\}$	$p_5 = 12$	$n = 20 + 2p_6$
$(4, 6)$	4_n	$p_6 \in N - \{1\}$	$p_4 = 6$	$n = 8 + 2p_6$
$(3, 6)$	3_n	$p_6/2 \in N - \{1\}$	$p_3 = 4$	$4 + 2p_6$
$(4, 5)$	4 dual deltahedra	$p_5 = 2, 3, 4, 5$	$p_4 = 5, 4, 3, 2$	$n = 10, 12, 14, 16$
$(3, 5)$	Dürer's Octahedron	$p_5 = 6$	$p_3 = 2$	$n = 12$
$(3, 4)$	Prism_3	$p_4 = 3$	$p_3 = 2$	$n = 6$


z=6; $30_{6,6}$

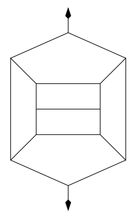
D_{3d}


z=18 $_{3,6}$

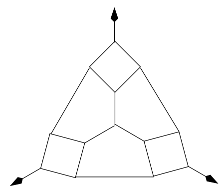
D_{3h}


z=30 $_{5,10}$

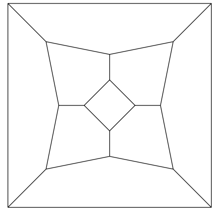
D_{5h}


z=8^2; $20_{0,4}$

D_{2d}


z=8^3; $18_{0,3}$

D_{3h}


z=8; $40_{8,8}$

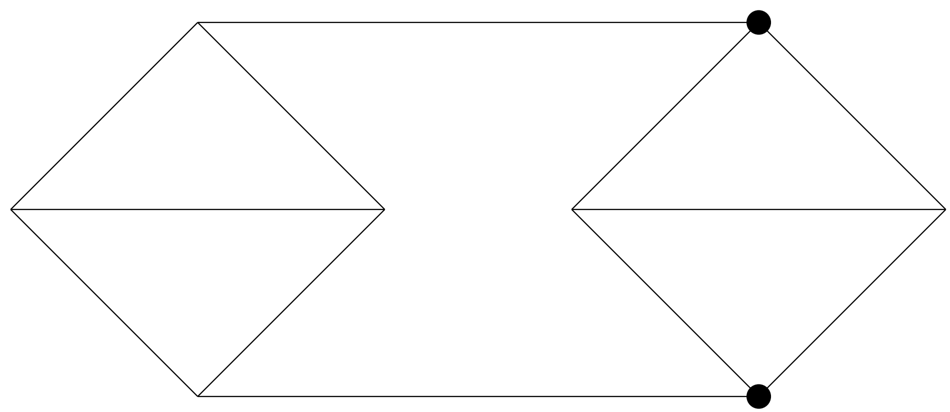
D_{4d}

k-connectedness

Theorem

(i) Any 3-valent plane graph without (>6)-gonal faces is 2-connected.

(ii) Moreover, any 3-valent plane graph without (>6)-gonal faces is 3-connected except of the following serie G_n:



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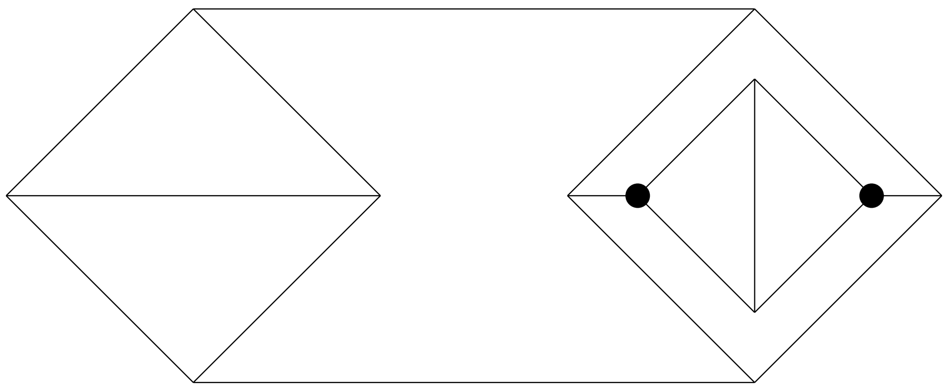

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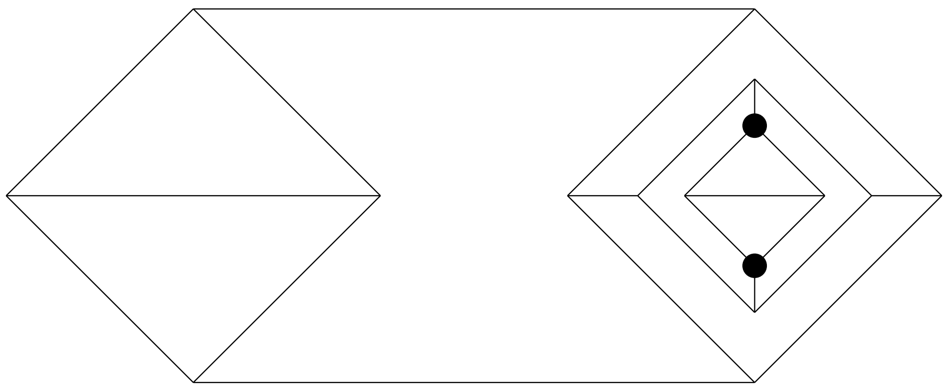


Point groups

(point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group)

Theorem*(Mani, 1971)*
Given a 3*-connected planar graph* G*, there exist a* 3*-polytope* P*, whose group of isometries is isomorphic to* $Aut(G)$ *and* $G(P) = G$*.*

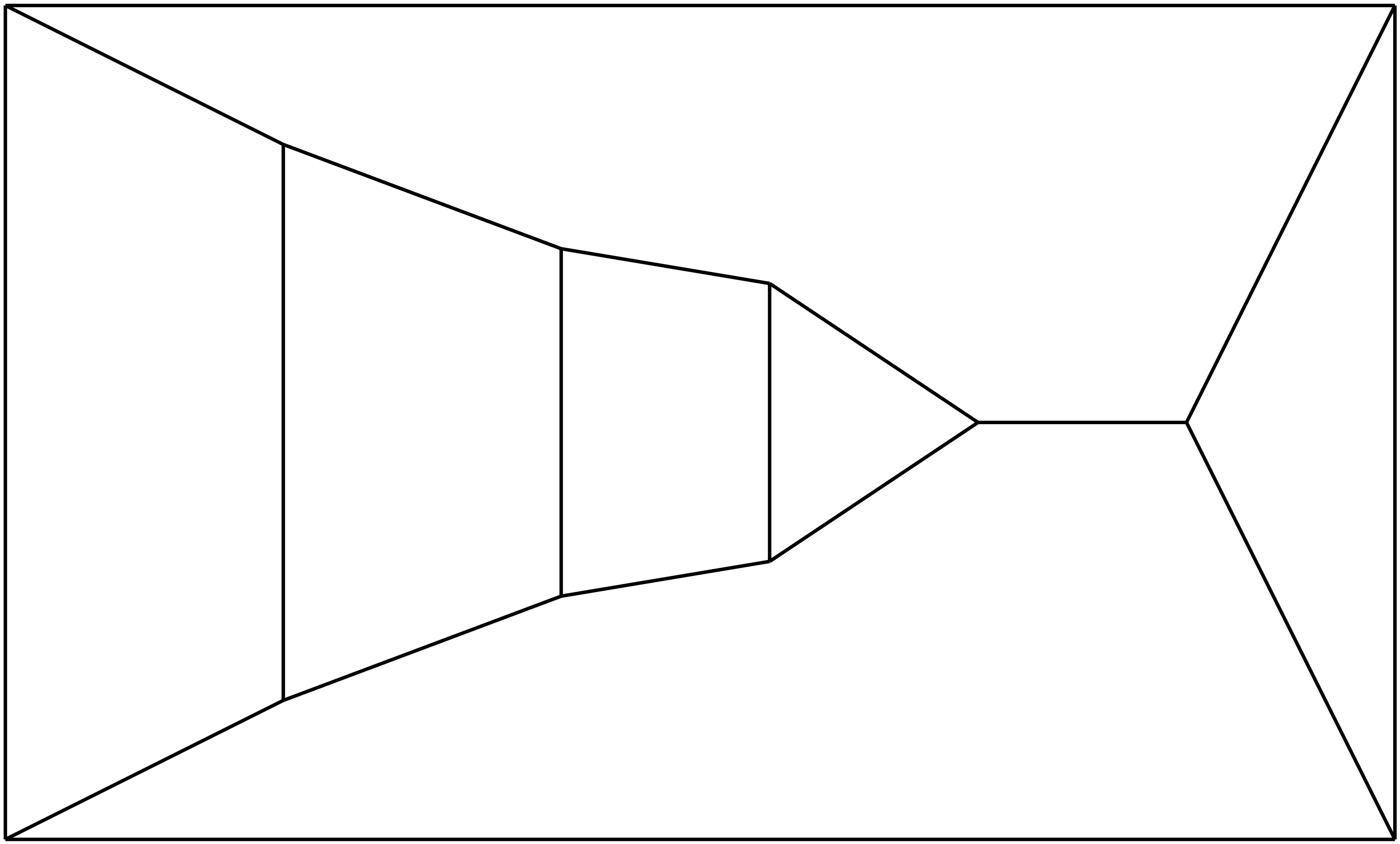
So, $Aut(G)$ of plane graphs G are finite subgroups of $O(3)$.
The symmetry groups of graphs q_n are known:

- For 3_n: D_2, D_{2h}, D_{2d}, T, T_d (Fowler and al.)
- For 4_n: C_1, C_s, C_2, C_i, C_{2v}, C_{2h}, D_2, D_3, D_{2d}, D_{2h}, D_{3d}, D_{3h}, D_6, D_{6h}, O, O_h (Dutour and Deza)
- For 5_n: C_1, C_2, C_i, C_s, C_3, D_2, S_4, C_{2v}, C_{2h}, D_3, S_6, C_{3v}, C_{3h}, D_{2h}, D_{2d}, D_5, D_6, D_{3h}, D_{3d}, T, D_{5h}, D_{5d}, D_{6h}, D_{6d}, T_d, T_h, I, I_h (Fowler and al.)

II. Zigzags

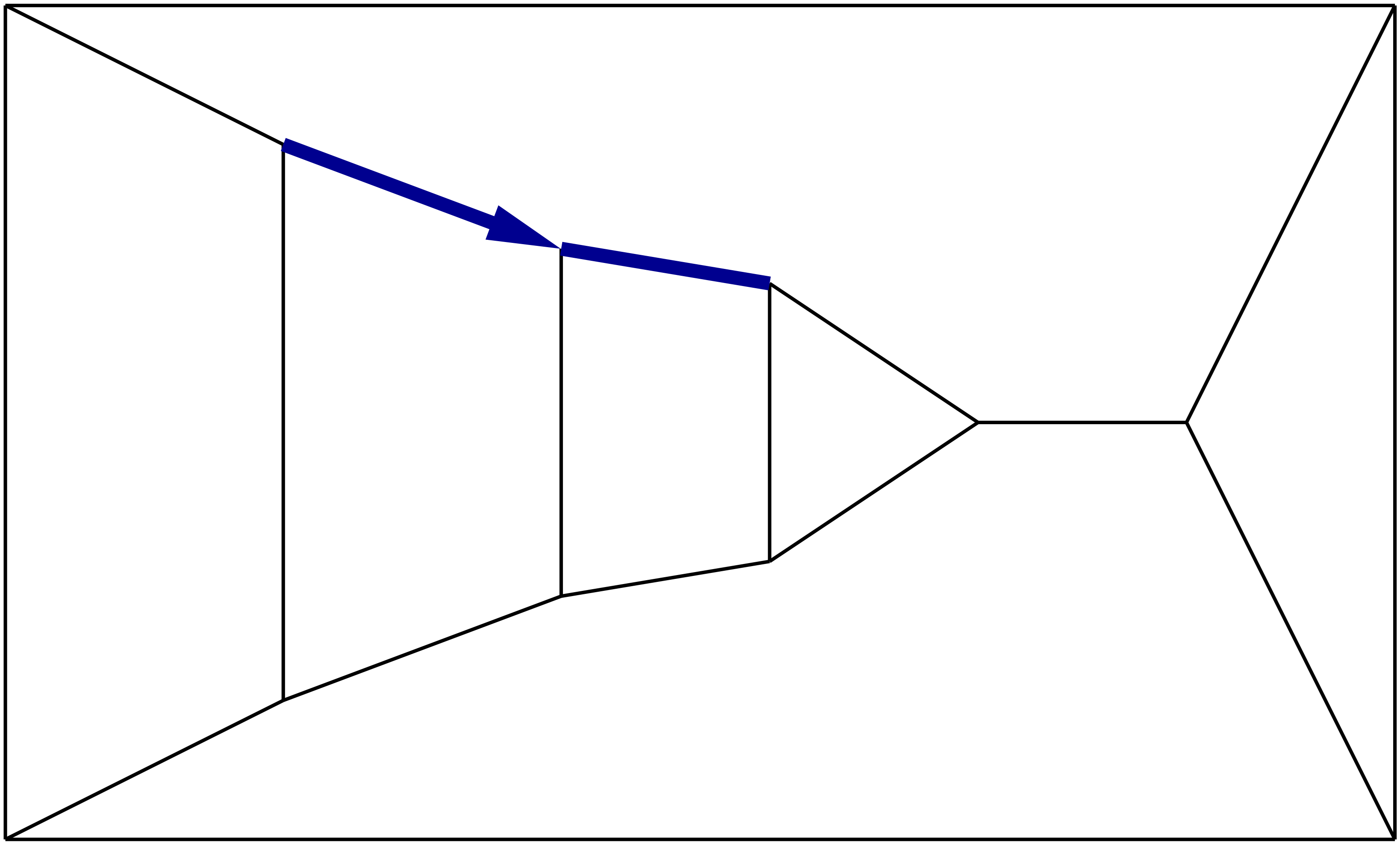
Zigzags

A plane graph G



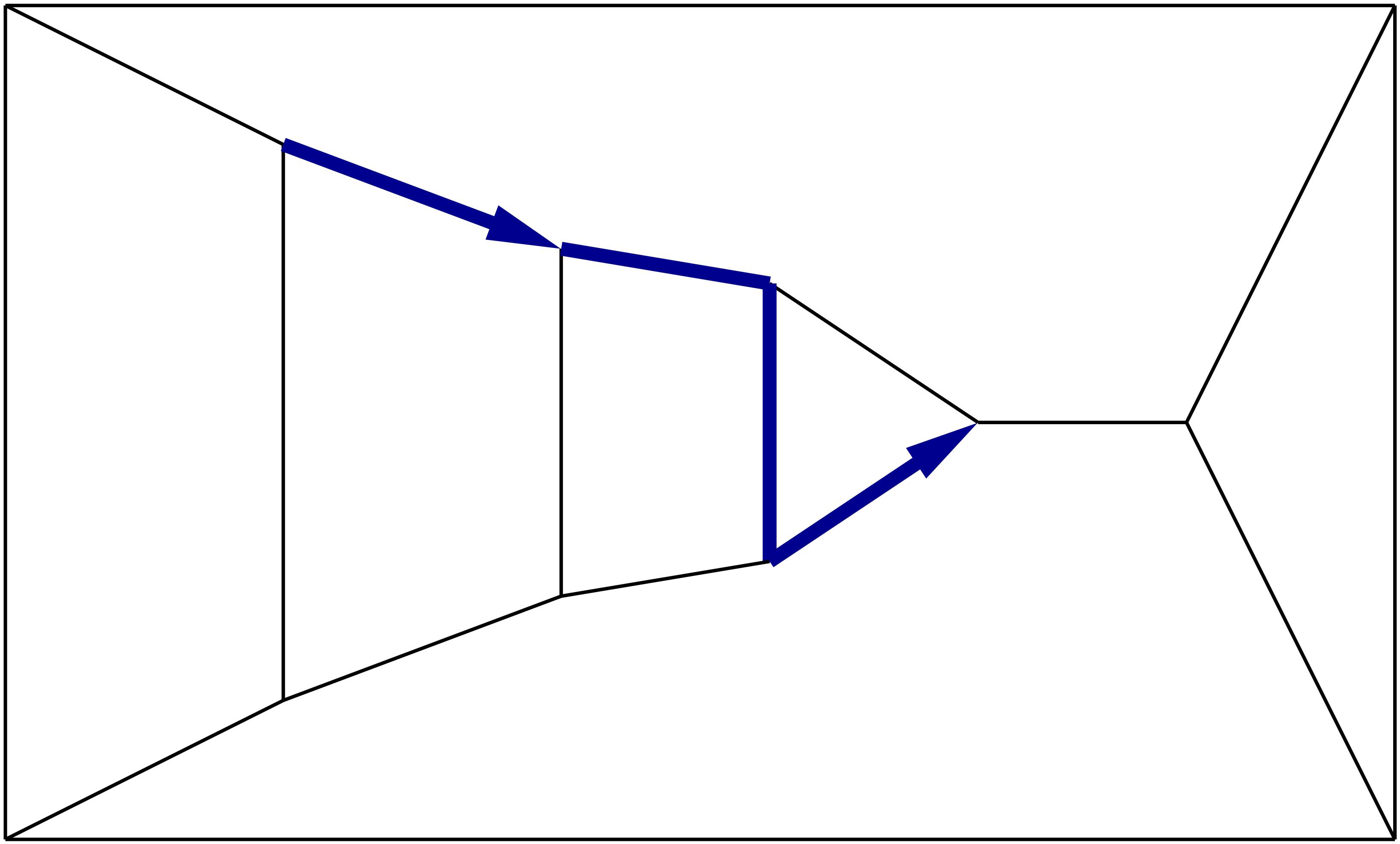
Zigzags

Take two edges



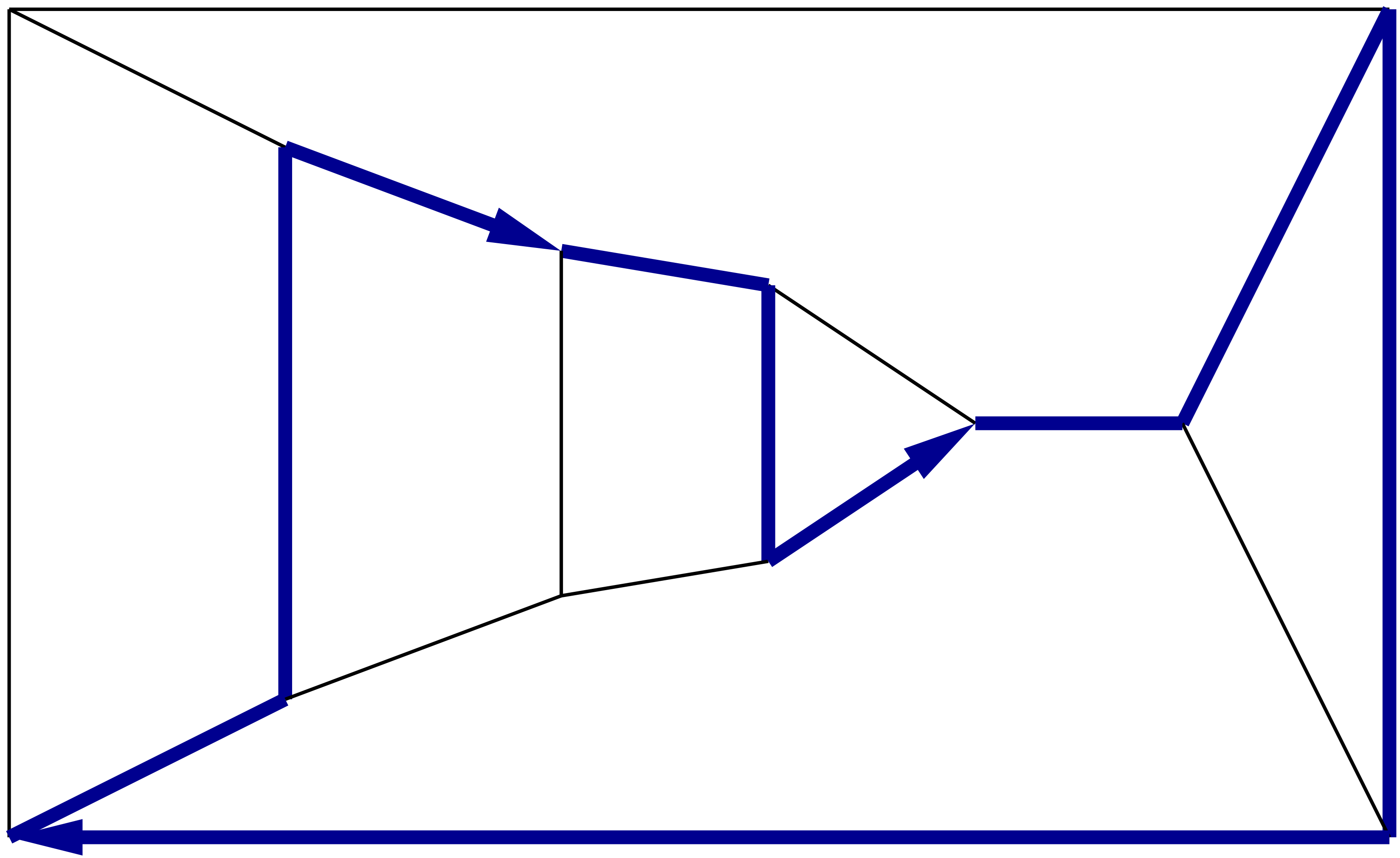
Zigzags

Continue it left−right alternatively ....



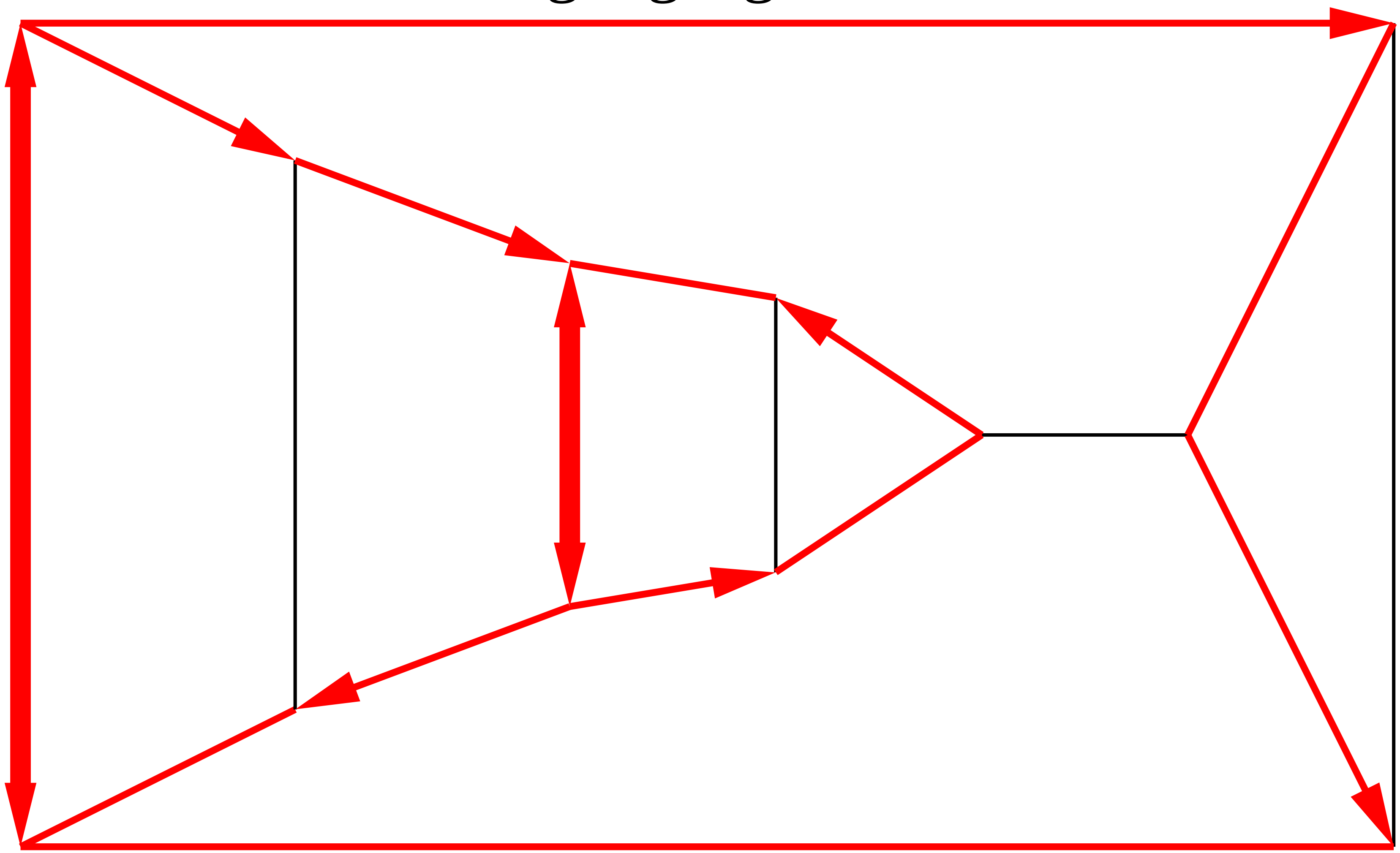
Zigzags

... until we come back



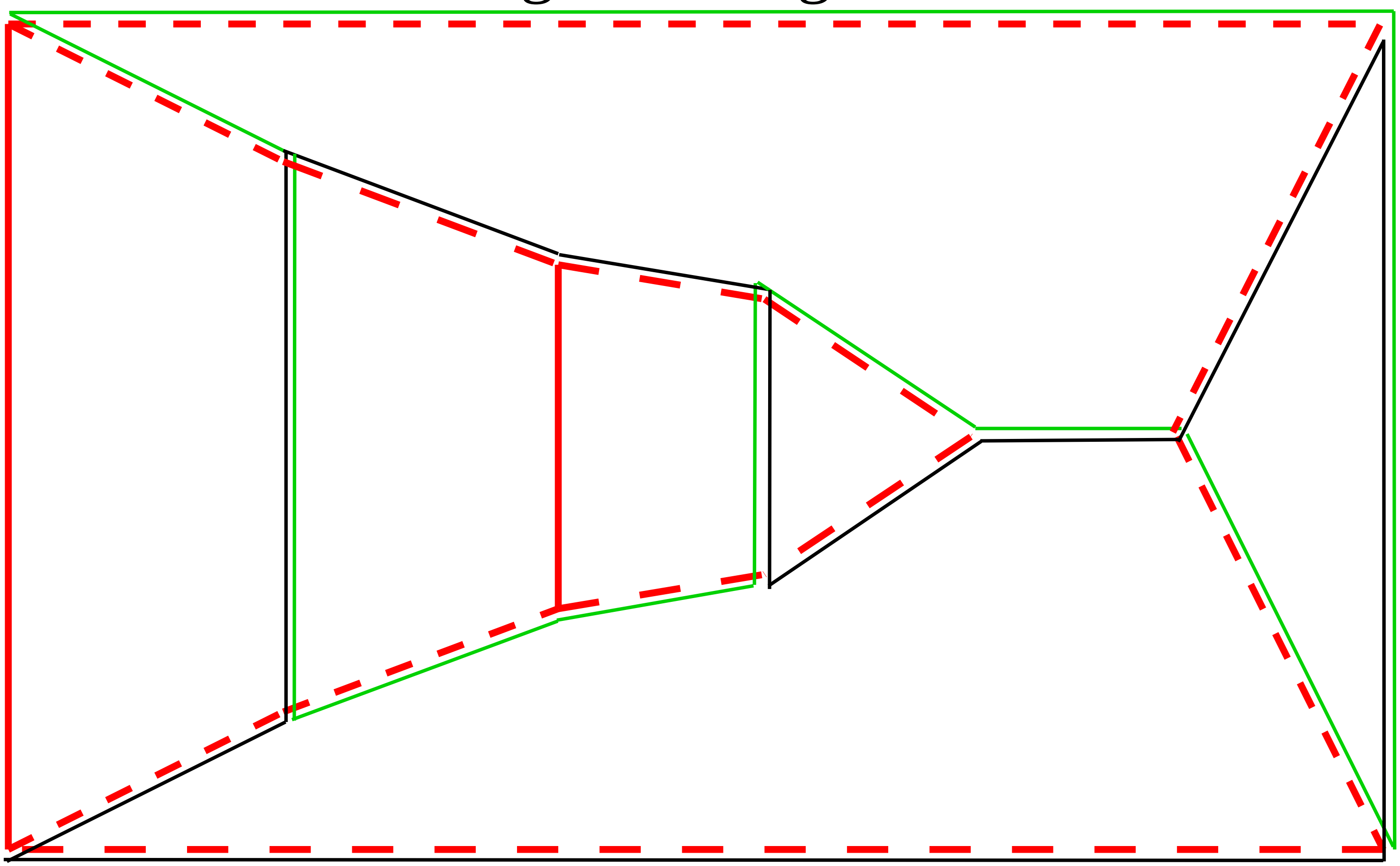
Zigzags

A self−intersecting zigzag



Zigzags

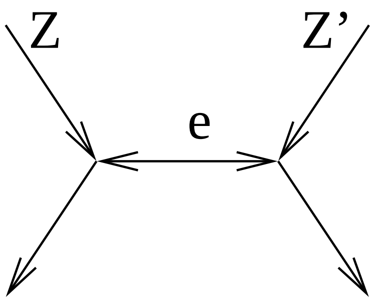
A double covering of 18 edges: 10+10+16



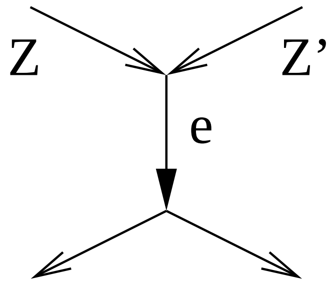
z−vector $z = 10^2, 16_{2,0}$

Intersection Types

Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge of intersection $Z \cap Z'$ is called of type I or type II, if Z and Z' traverse e in opposite or same direction, respectively



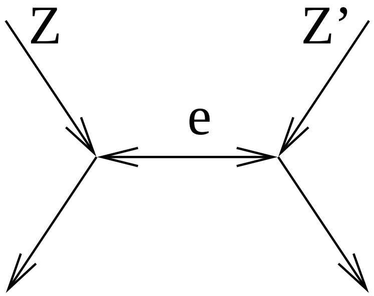
Type I



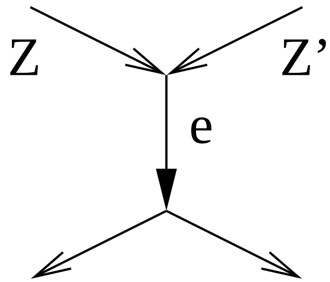
Type II

Intersection Types

Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge of intersection $Z \cap Z'$ is called of type I or type II, if Z and Z' traverse e in opposite or same direction, respectively

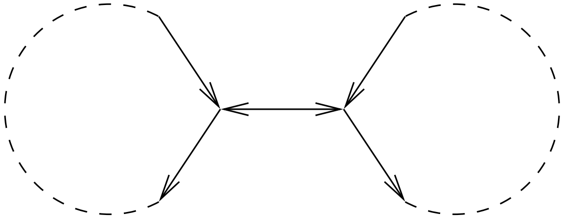


Type I

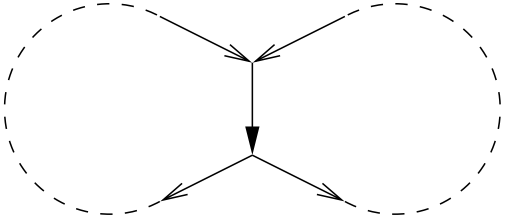


Type II

The types of self-intersection depends on orientation chosen on zigzags except if $Z = Z'$:



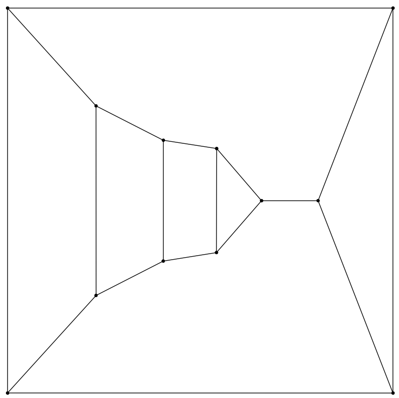
Type I



Type II

Zigzag parameters

- The signature of a zigzag Z is the pair (α_1, α_2), where α_1 and α_2 are the numbers of its edges of self-intersection of type I and type II, respectively.
- The intersection vector $Int(Z)$ lists pairs of intersection (α_1, α_2) with all other zigzags.
- z-vector of G is the vector enumerating lengths (numbers of edges) of all its zigzags with their signature as subscript.



2 zigzags with $Int = (1,3), (3,3)$
1 self-intersecting with $Int = (3,3)^2$

Duality and types

Theorem
The zigzags of a plane graph G *are in one-to-one correspondence with zigzags of* G^*. *The length is preserved, but intersection of type I and II are interchanged.*

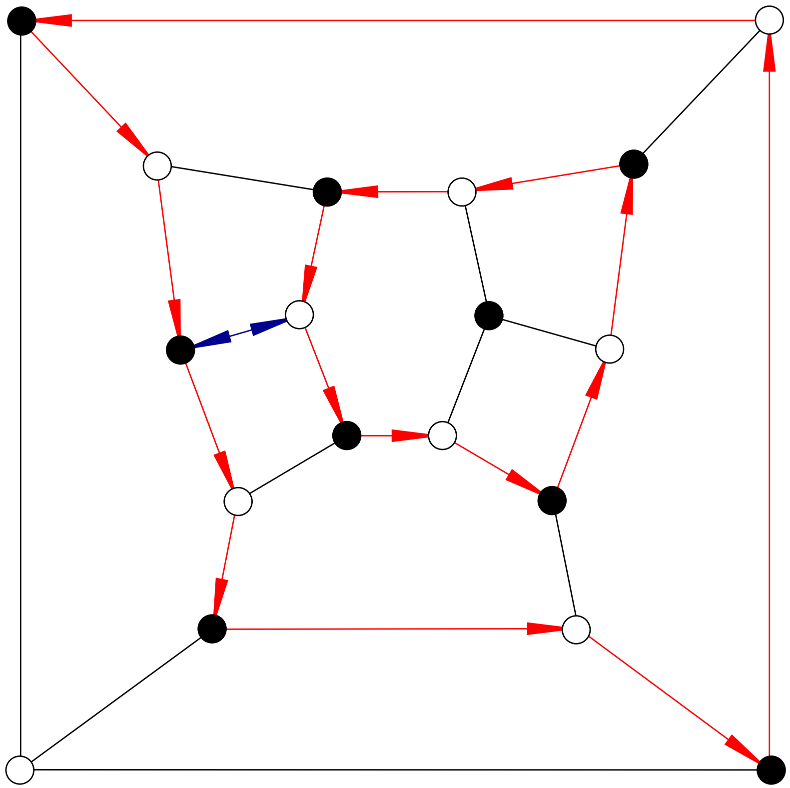
Theorem
Let G *be a plane graph; for any orientation of all zigzags of* G, *we have:*
(i) The number of edges of type II, which are incident to any fixed vertex, is even.
(ii) The number of edges of type I, which are incident to any fixed face, is even.

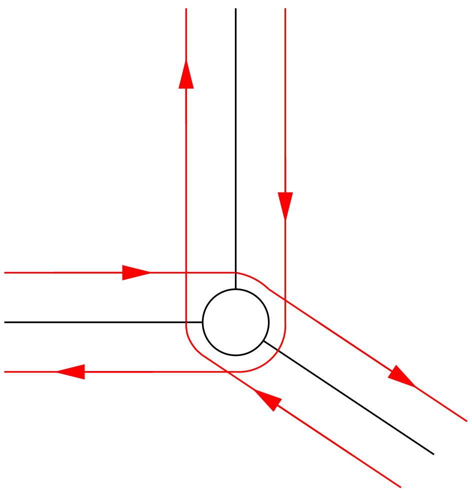
Bipartite graphs

Remark *A plane graph is bipartite if and only if its faces have even gonality.*

Theorem *(Shank-Shtogrin)*

For any planar bipartite graph G there exist an orientation of zigzags, with respect to which each edge has type I.

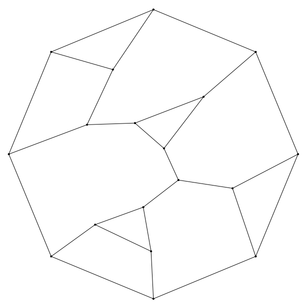




Zigzag properties of a graph

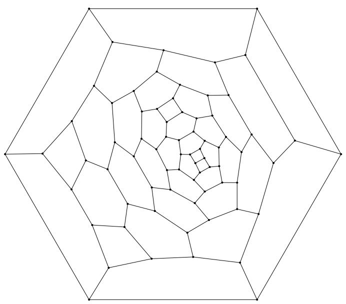
- z-uniform: all zigzags have the same length and signature,
- z-transitive: symmetry group is transitive on zigzags,
- z-knotted: there is only one zigzag,
- z-balanced: all zigzags of the same length and signature, have identical intersection vectors.

All known z-uniform 3-valent graphs are z-balanced.



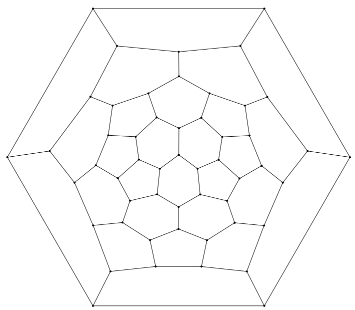
18-vertices (C_2),

$z = 8^3; 30_{2,5}$



$4_{72}(C_1)$,

$z = 30_{1,0}, 54^2_{4,0}, 78_{13,0}$



$5_{52}(D_{2d})$,

$z = 16^4; 92_{12,12}$

Smallest (among all 3-valent, all 4_n, all 5_n) z-unbalanced 3-valent graphs

Zigzags of reg. and semireg. polyhedra

# edges	polyhedron	z-vector	int. vector
6	Tetrahedron	4^3	$(1,1)^2$
12	Cube, Octahedron	6^4	$(0,2)^3$
30	Dodecahedron, Icosahedron	10^6	$(0,2)^5$
24	Cuboctahedron	8^6	$(0,2)^4, (0,0)$
60	Icosidodecahedron	10^{12}	$(0,2)^5, (0,0)^6$
48	Rhombicuboctahedron	12^8	$(0,2)^6, (0,0)$
120	Rhombicosidodecahedron	20^{12}	$(0,2)^{10}, (0,0)$
72	Truncated Cuboctahedron	18^8	$(0,6), (0,2)^6$
180	Truncated Icosidodecahedron	30^{12}	$(0,10), (0,2)^{10}$
18	Truncated Tetrahedron	12^3	$(3,3)^2$
36	Truncated Octahedron	12^6	$(0,4), (0,2)^4$

36	Truncated Cube	18^4	$(2,4)^3$
90	Truncated Icosahedron	18^{10}	$(0,2)^9$
90	Truncated Dodecahedron	30^6	$(2,4)^5$
60	Snub Cube	$30^4_{3,0}$	$(4,4)^3$
150	Snub Dodecahedron	$50^6_{5,0}$	$(4,4)^5$
3m	$Prism_m, m \equiv 0 \pmod 4$	$(\frac{3m}{2})^4$	$(0, \frac{m}{2})^3$
3m	$Prism_m, m \equiv 2 \pmod 4$	$(3m_{\frac{m}{2},0})^2$	$(0, 2m)$
3m	$Prism_m, m \equiv 1,3 \pmod 4$	$6m_{m,2m}$	
4m	$APrism_m, m \equiv 0 \pmod 3$	$(2m)^4$	$(0, \frac{2m}{3})^3$
4m	$APrism_m, m \equiv 1,2 \pmod 3$	$2m; 6m_{0,2m}$	
84	Klein map(oriented, genus 3 surface)	8^{21}	$(0,1)^8, 0^{12}$
48	Dyck map(oriented, genus 3 surface)	6^{16}	$(0,1)^6, 0^9$

First generalizations of zigzags

Above Table contains plane graphs, which are not 3-valent, and non-planar graphs.

In fact, the notion of zigzag can be easily generalized on any plane graph and on a graph, embedded in any oriented surface.

Moreover, this notion, being local, can be generalized even for non-oriented surfaces.

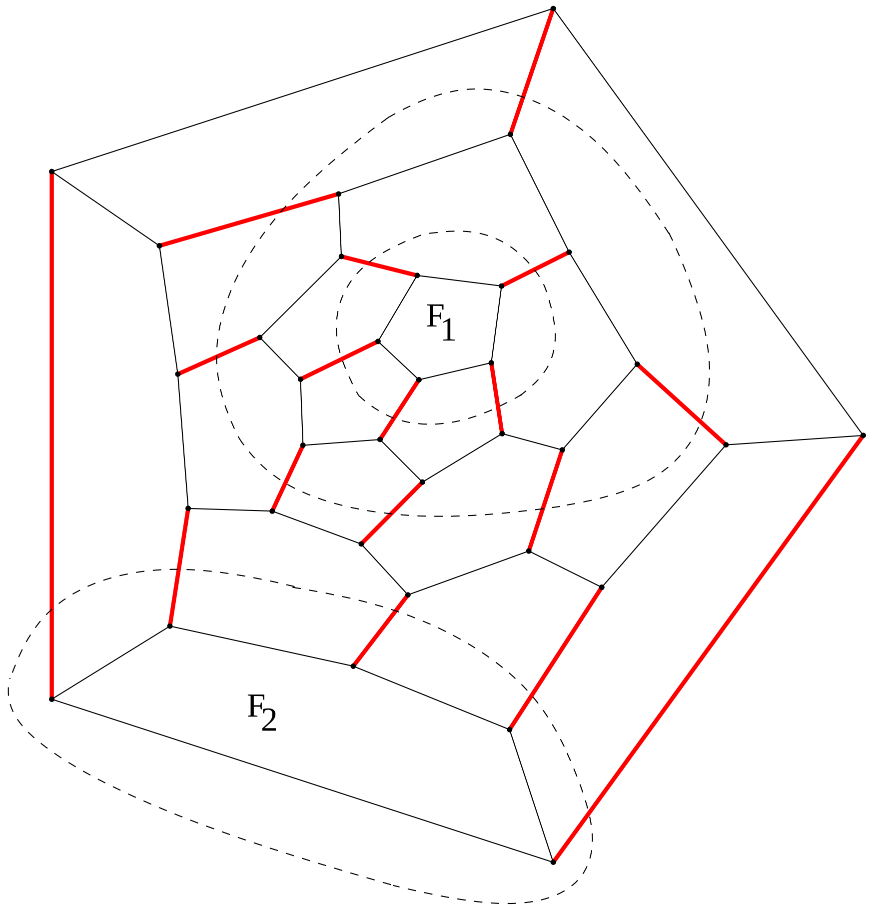
Perfect matching on 5_n graphs

Let G be a z-knotted graph 5_n.

(i) $z = n_{\alpha_1,\alpha_2}$ with $\alpha_1 \geq \frac{n}{2}$. If $\alpha_1 = \frac{n}{2}$ then the edges of type I form a perfect matching PM

(iii) every face incident to two or zero edges of PM

(iv) two faces, F_1 and F_2 are incident to zero edges of PM, PM is organized around them in concentric circles.

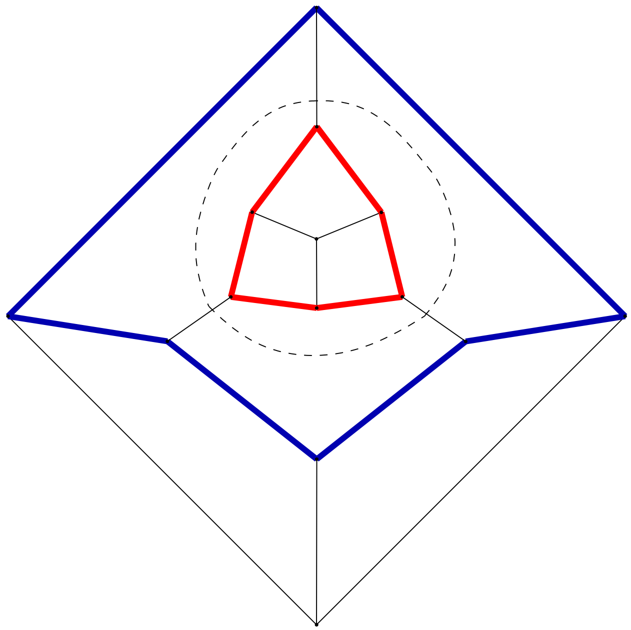


M. Deza, M. Dutour and P.W. Fowler, *Zigzags, Railroads and Knots in Fullerenes*, (2002).

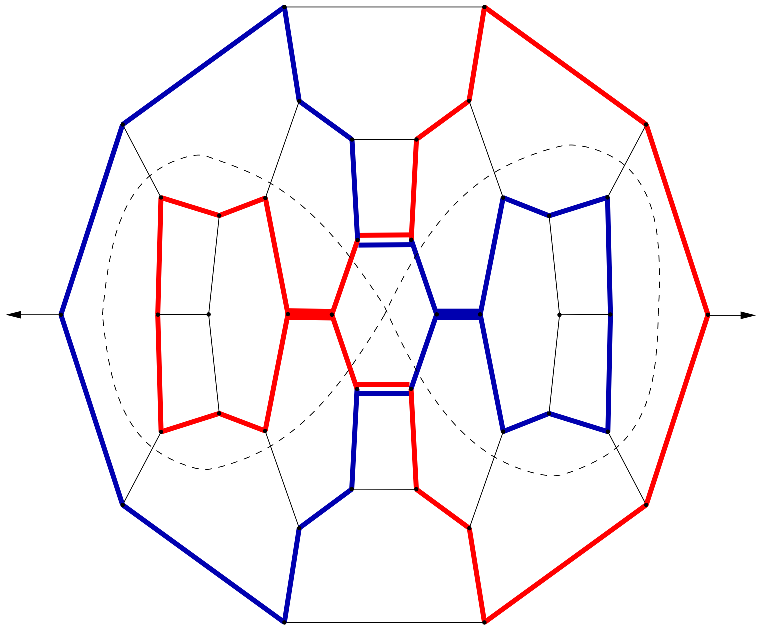
III. railroad structure of graphs q_n

Railroads

A railroad in graph q_n, $q = 3, 4, 5$ is a circuit of hexagonal faces, such that any of them is adjacent to its neighbors on opposite faces. Any railroad is bordered by two zigzags.



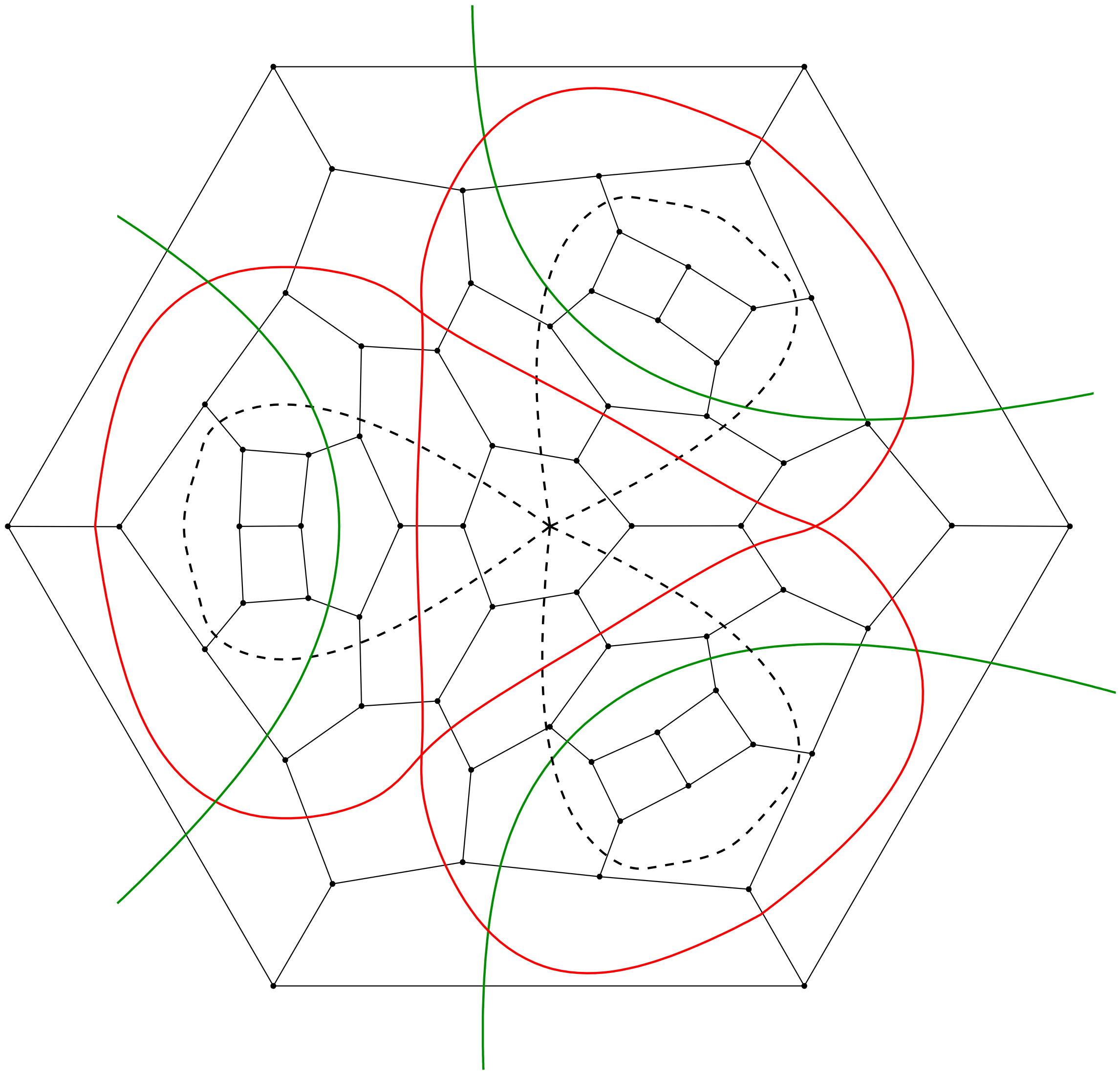
$4_{14}(D_{3h})$



$4_{42}(C_{2v})$

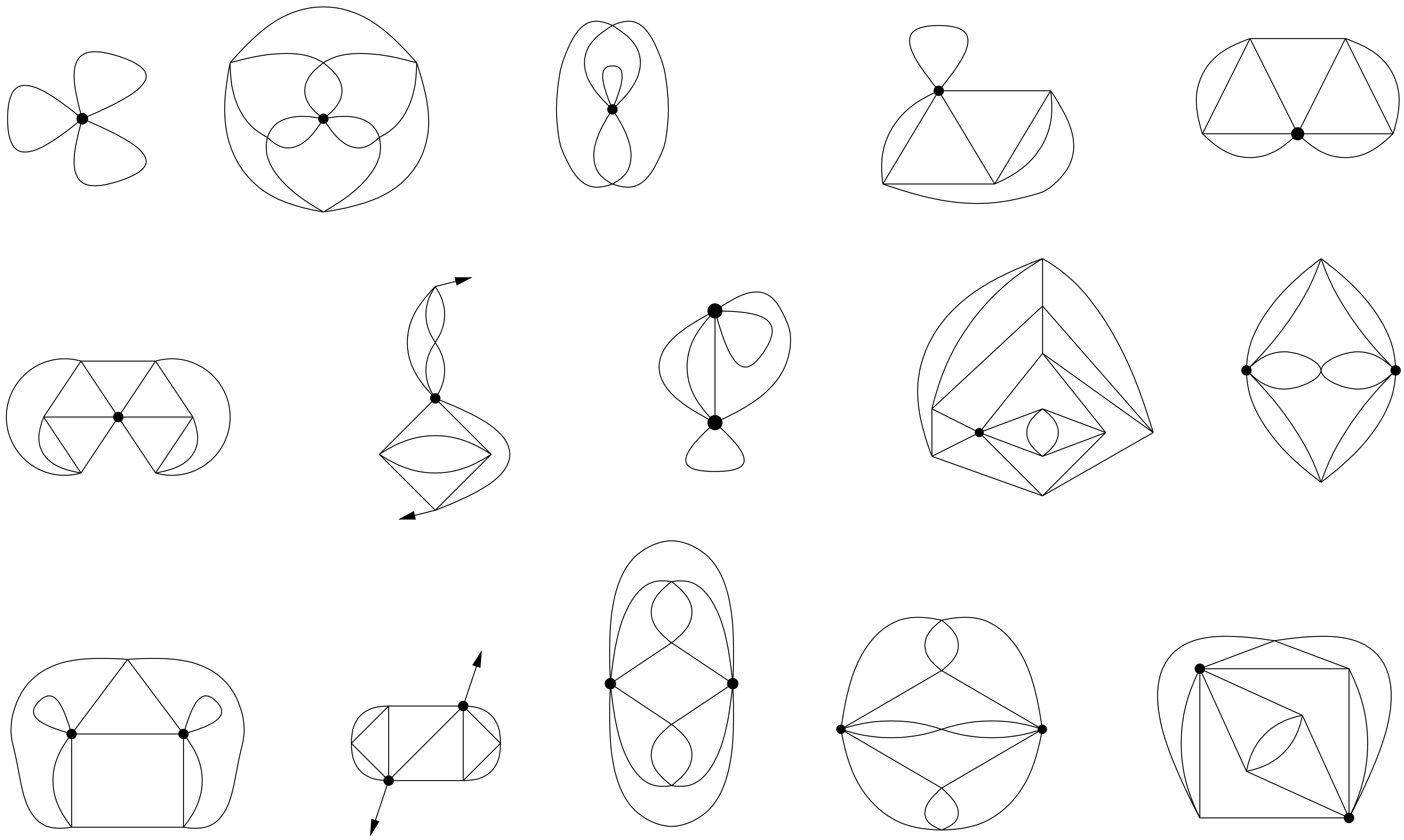
Railroads, as well as zigzags, can be self-intersecting (doubly or triply). A graph is called tight if it has no railroad.

$4_{66}(D_{3h})$ with triply self-int. railroad

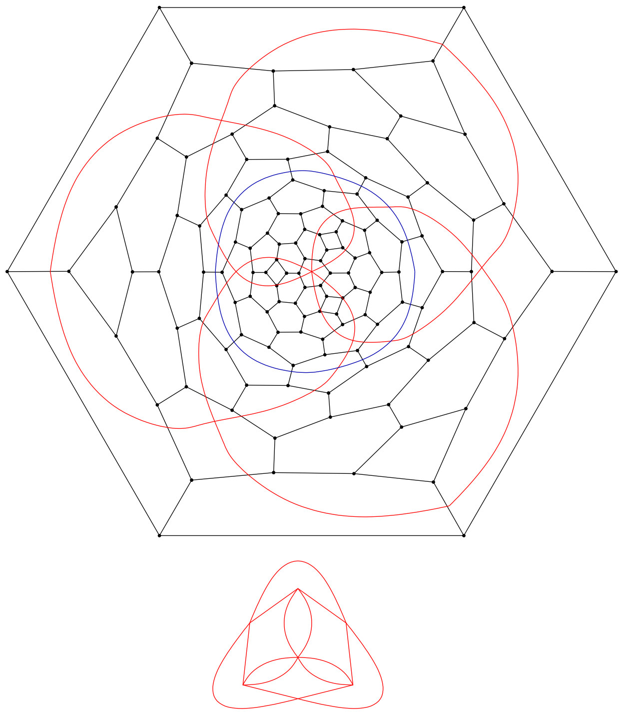


It is smallest such 4_n. Green railroad also triply self-int.

Railroads with triple points in small 4_n

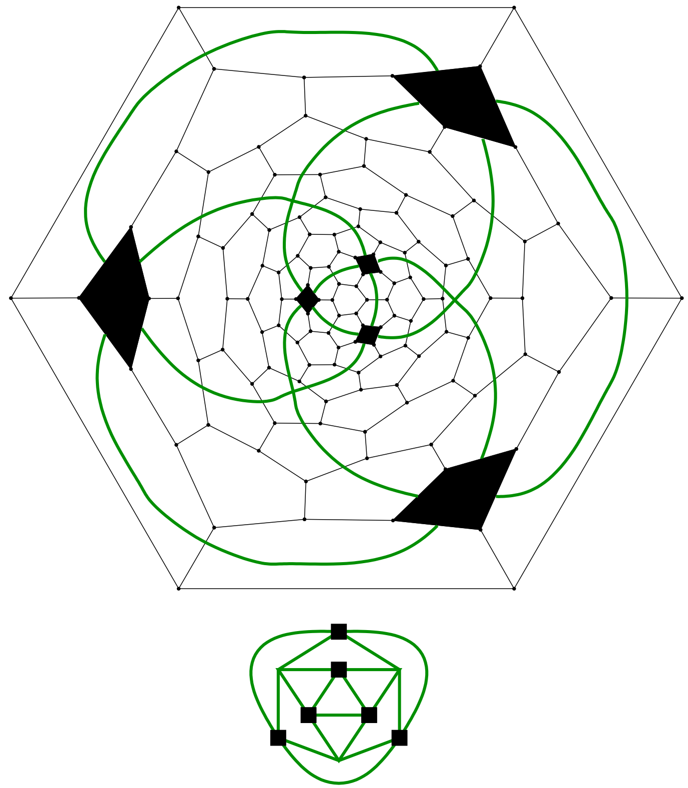


Railroads and pseudo-roads of $4_{126}(D_{3h})$



One of two self-intersecting railroads

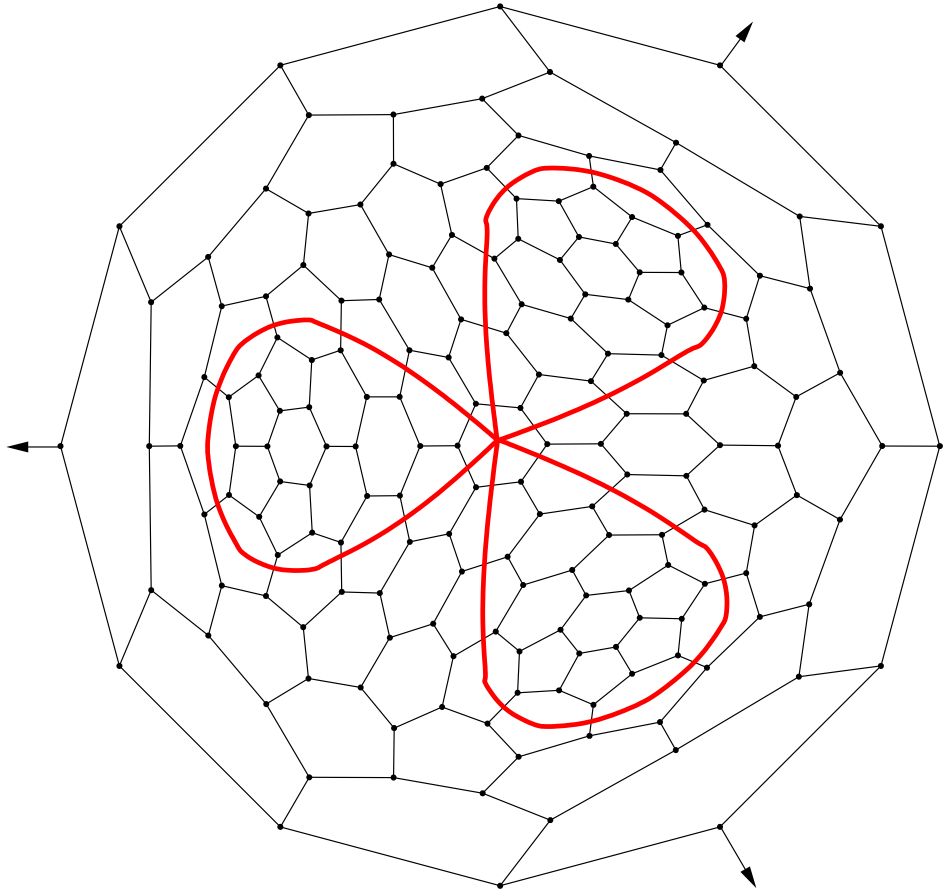
and the equatorial simple railroad



All twelve pseudo-roads

A pseudo-road between 4-gons b and c is a sequence of hexagons $a_1, \ldots, a_l$, s.t. if $a_0 = b$ and $a_{l+1} = c$, then any a_i, $1 \le i \le l$, is adjacent to a_{i-1} and a_{i+1} on opposite edges.

Triply intersecting railroad in $5_{176}(C_{3v})$



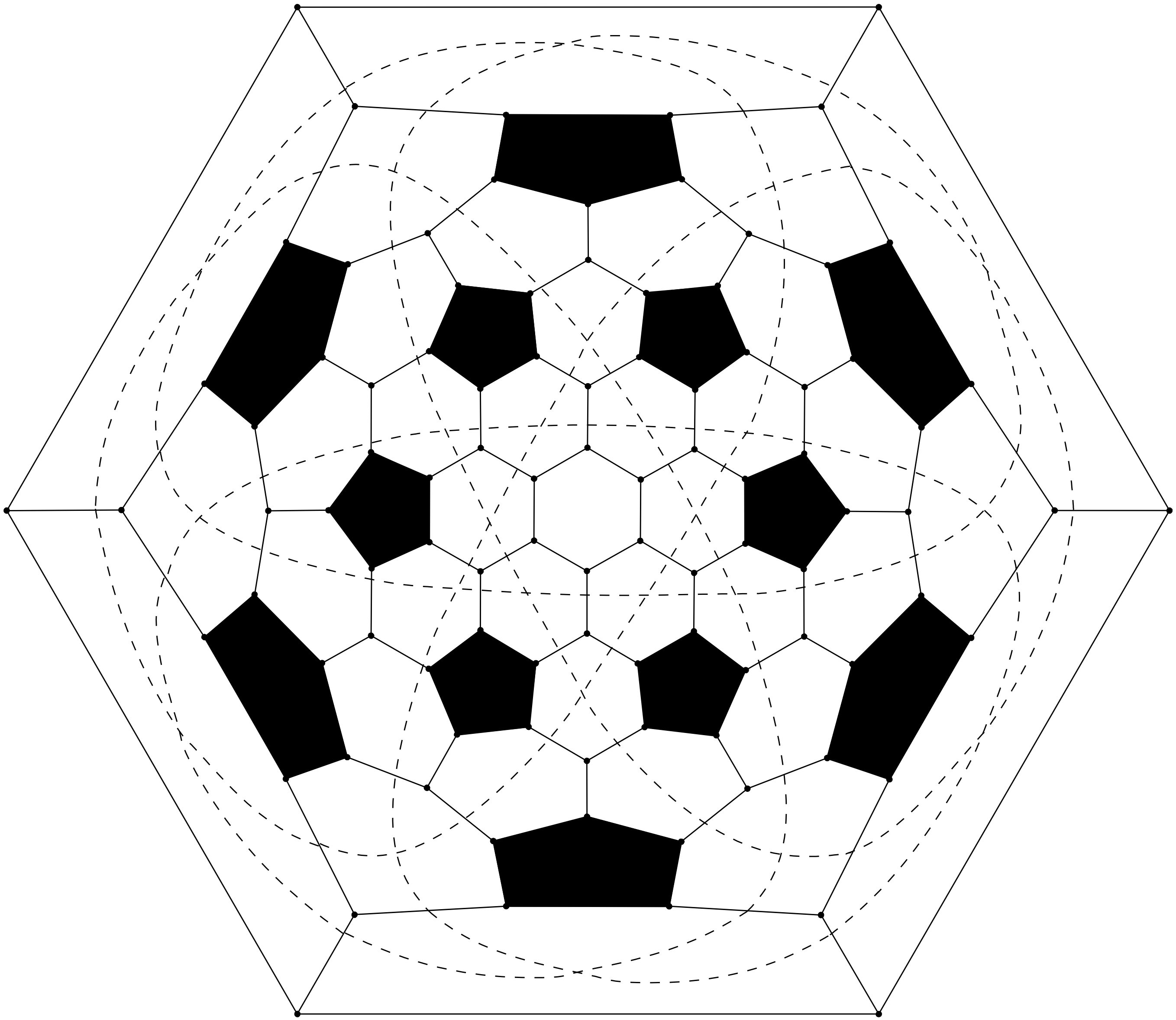
Conjecture: *a railroad-curve of any* 4_n *appears in some* 5_m.

Tight 5_n with only simple zigzags

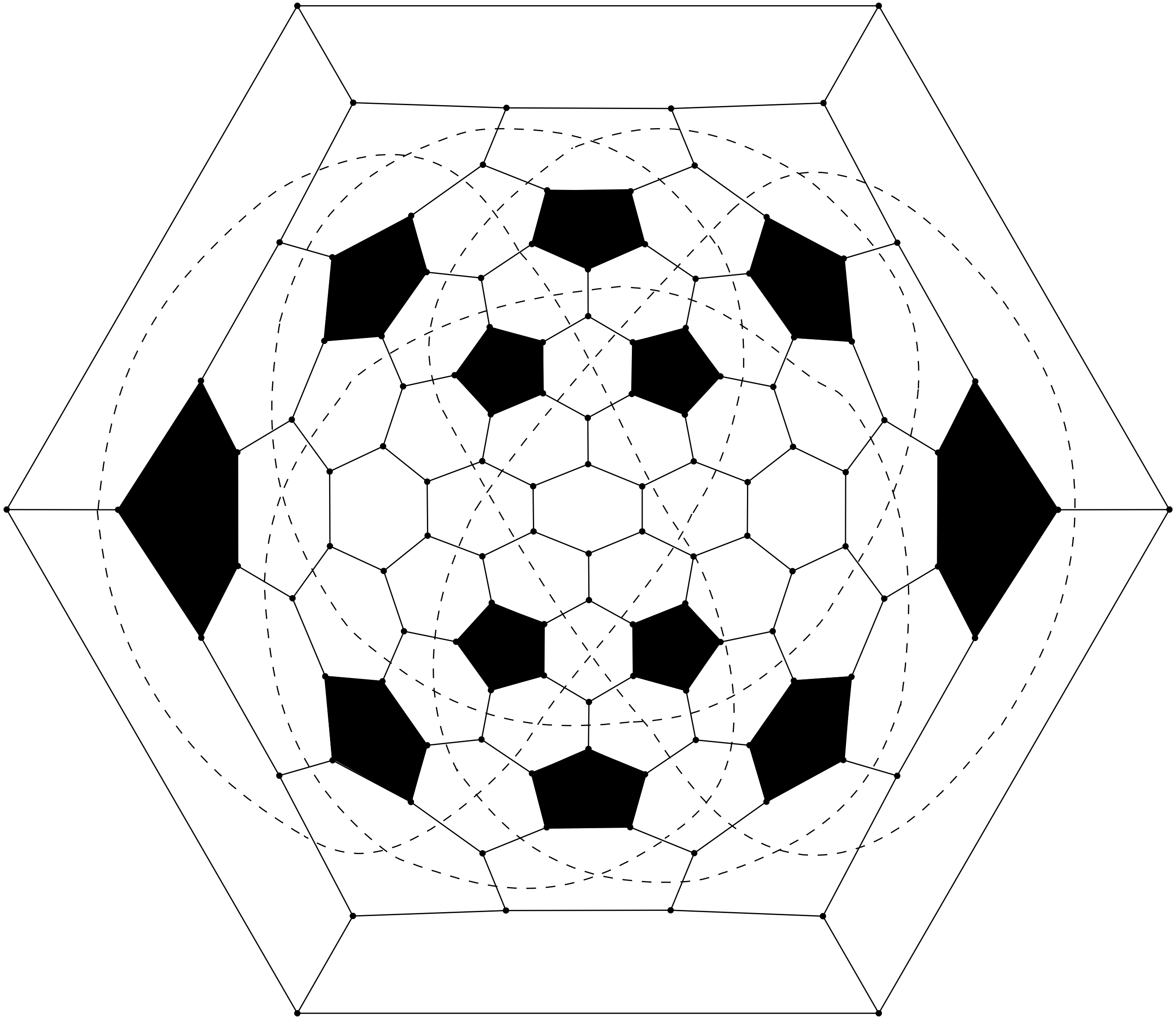
n	group	z-vector	orbit lengths	int. vector
20	I_h	10^6	6	2^5
28	T_d	12^7	3,4	2^6
48	D_3	16^9	3,3,3	2^8
60	I_h	18^{10}	10	2^9
60	D_3	18^{10}	1,3,6	2^9
76	D_{2d}	$22^4, 20^7$	1,2,4,4	$4, 2^9$ and 2^{10}
88	T	22^{12}	12	2^{11}
92	T_h	$22^6, 24^6$	6,6	2^{11} and $2^{10}, 4$
140	I	28^{15}	15	2^{14}

Conjecture: this list is complete (checked for $n \leq 200$).
It gives 7 Grünbaum arrangements of plane curves.

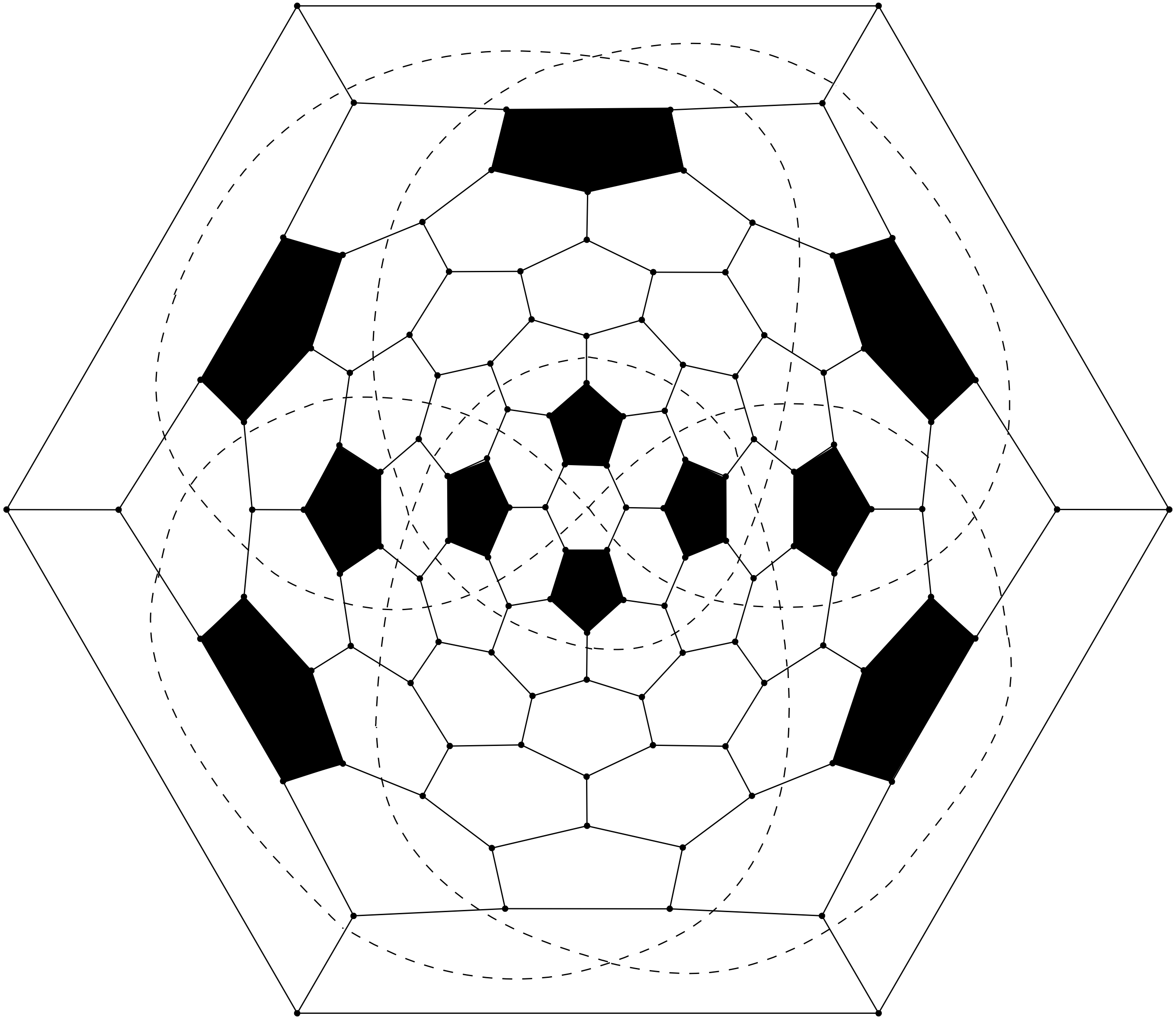
First IPR 5_n with self-intersect. railroad



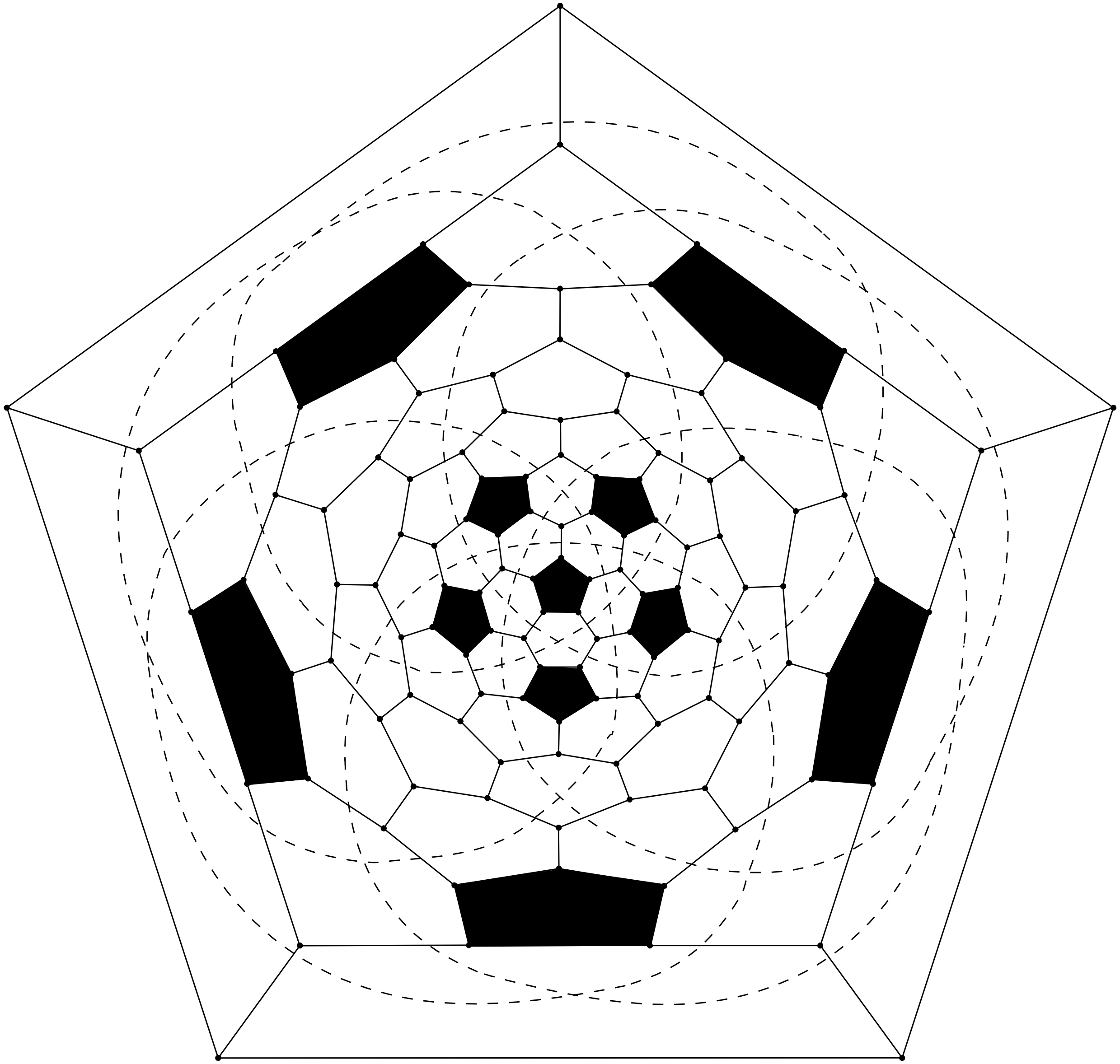
IPR $5_{120}(C_{2v})$



IPR $5_{120}(C_{2v})$

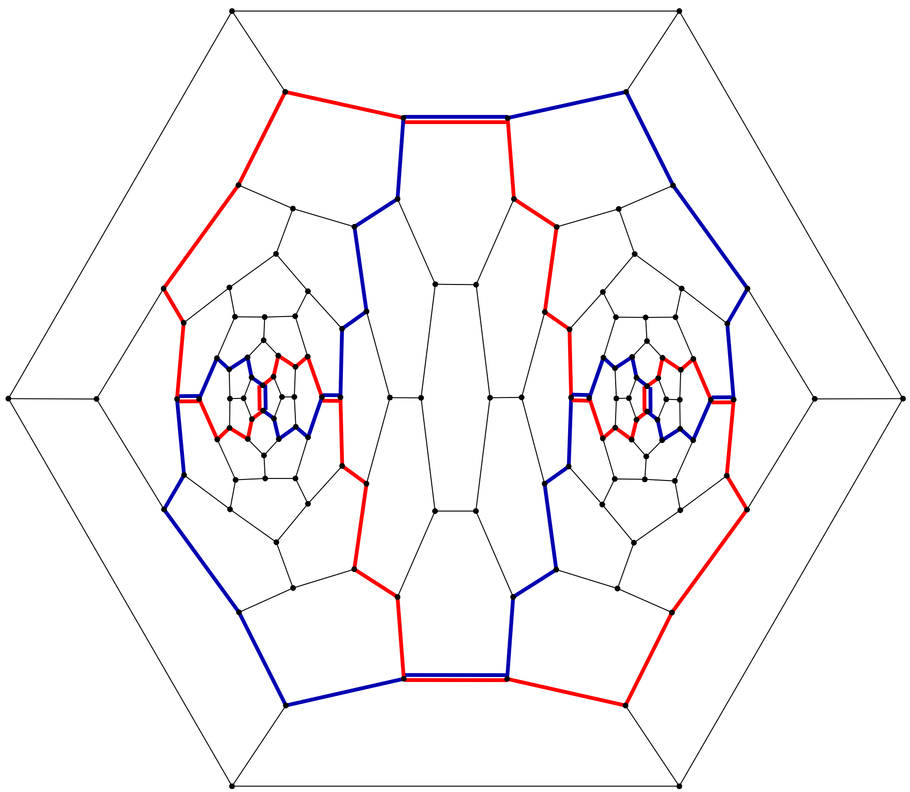


IPR $5_{120}(D_{5h})$



Comparing graphs q_n

q	3	4	5
max # of zigzags in tight	3	$8(?)$	$15(?)$
all tight with simple zigzags	all tight	Cube, Tr. Oct.	9 examples(?)
int. size of 2 simple zigzags	any even	$2, 4, 6$	any even



IV. parametrizing graphs q_n

Parametrizing graphs q_n

idea: the hexagons are of zero curvature, it suffices to give relative positions of faces of non-zero curvature.

- Goldberg (1937) All 3_n, 4_n or 5_n of symmetry (T, T_d), (O, O_h) or (I, I_h) are given by Goldberg-Coxeter construction $GC_{k,l}$.
- Fowler and al. (1988) All 5_n of symmetry D_5, D_6 or T are described in terms of 4 parameters.
- Graver (1999) All 5_n can be encoded by 20 integer parameters.
- Thurston (1998) The 5_n are parametrized by 10 complex parameters.
- Sah (1994) Thurston's result implies that the Nrs of 3_n, 4_n, $5_n \sim n, n^3, n^9$.

Goldberg-Coxeter construction

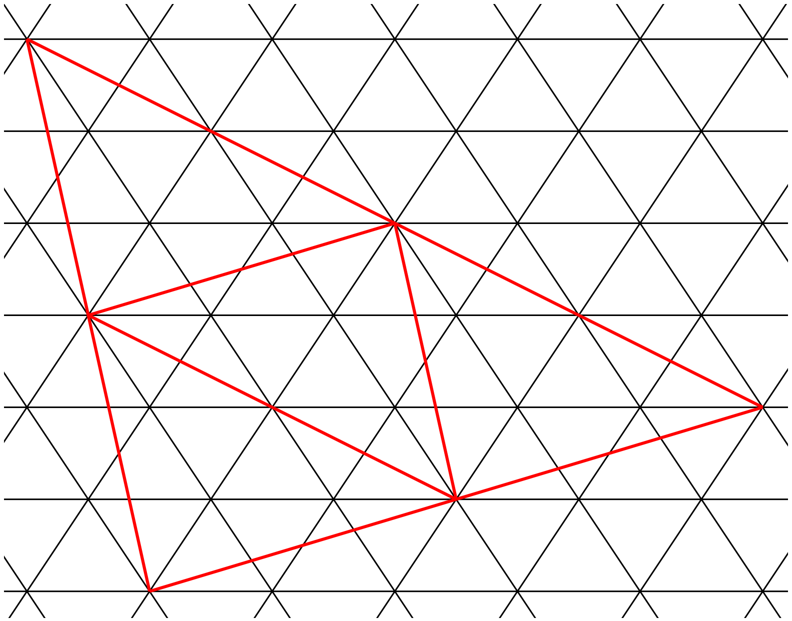
Given a 3-valent plane graph G, the zigzags of the Goldberg-Coxeter construction of $GC_{k,l}(G)$ are obtained by:

- Associating to G two elements L and R of a group called moving group,
- computing the value of the (k,l)-product $L \odot_{k,l} R$,
- the lengths of zigzags are obtained by computing the cycles structure of $L \odot_{k,l} R$.

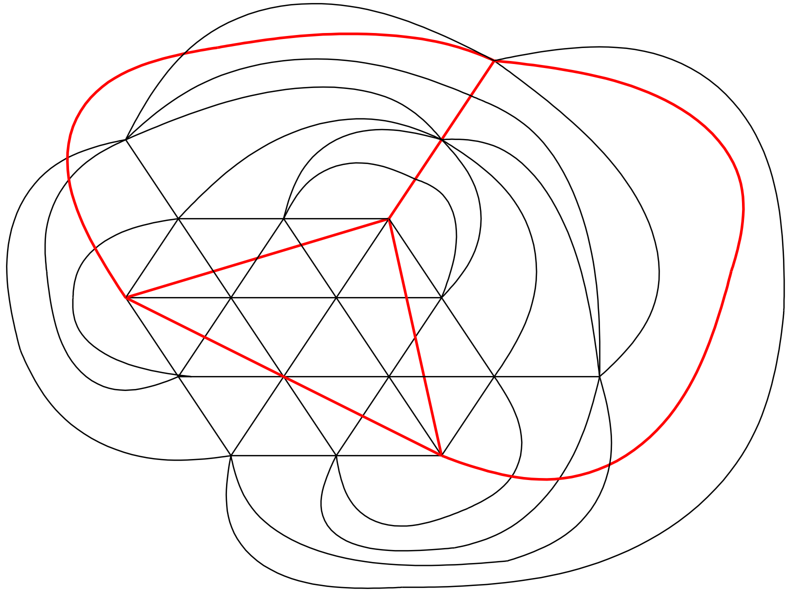
For tight 5_n of symmetry I or I_h this gives 6, 10 or 15 zigzags.

M. Dutour and M. Deza, *Goldberg-Coxeter construction for 3- or 4-valent plane graphs*, submitted

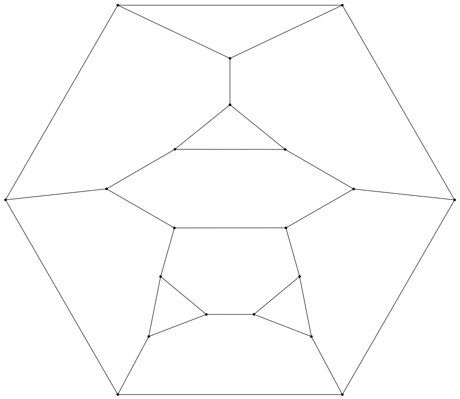
The structure of graphs 3_n



4 triangles in $Z[\omega]$



The corresponding triangulation



The graph $3_{20}(D_{2d})$

z- and railroad-structure of graphs 3_n

All zigzags and railroads are simple.

- The z-vector is of the form

$$(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3} \quad \text{with} \quad s_i m_i = \frac{n}{4};$$

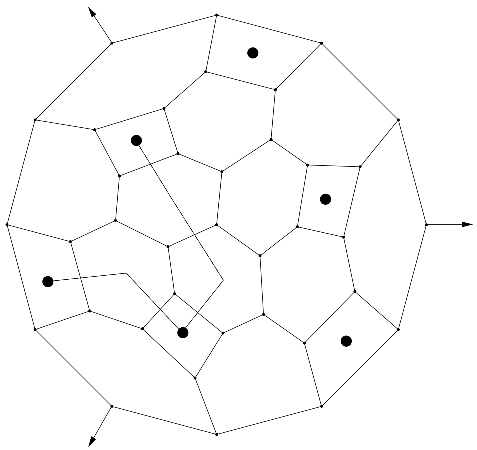
  the number of railroads is $m_1 + m_2 + m_3 - 3$.
- G has ≥ 3 zigzags with equality if and only if it is tight.
- If G is tight, then $z(G) = n^3$ (so, each zigzag is a Hamiltonian circuit).
- All 3_n are tight if and only if $\frac{n}{4}$ is prime.
- There exists a tight 3_n if and only if $\frac{n}{4}$ is odd.

Conjecture on $4_n(D_3)$, 4 parameters

- For tight graphs 4_n of symmetry D_3, D_{3d} or D_{3h} the z-vector is of the form

$$a^k \text{ with } k \in \{1,2,3,6\}$$
$$\text{or } a^k, b^l \text{ with } k,l \in \{1,3\}$$

- A knotted 4_n of such symmetry has symmetry D_3.
- if there is a knotted 4_n of symmetry D_3, then $\frac{n}{2}$ is the product of at most 2 primes



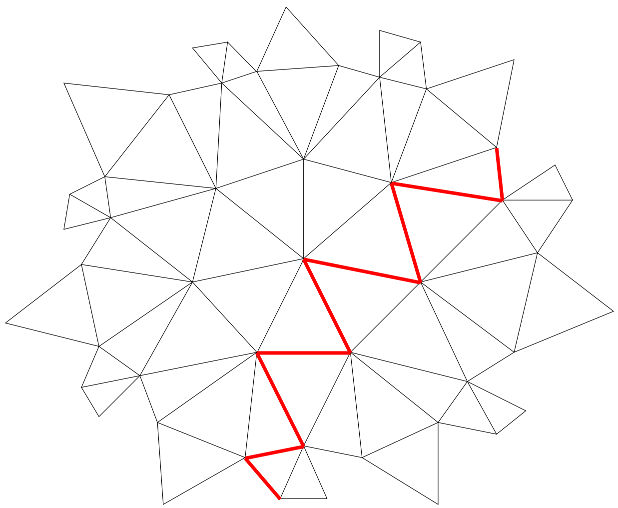
First z-knotted 4_n of symmetry D_3.

V. Zigzags

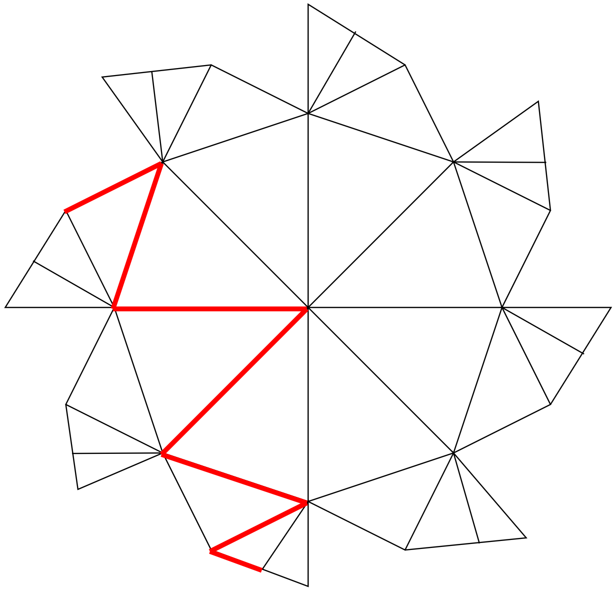
on

surfaces

Klein and Dyck map



Klein map: $z = 8^{21}$



Dyck map: $z = 6^{16}$

Zigzag, being a local notion, is defined on any surface, even on non-orientable ones.

Regular maps

A flag-transitive map is called regular.
Zigzags of regular maps are simple.

map	n	rot. group	z	$z(GC_{k,l})/k^2+kl+l^2$
Dod. $\{5^3\}$	20	A_5	10^6	10^6 or 6^{10} or 4^{15}
Klein* $\{7^3\}$	56	$PSL(2,7)$	8^{21}	8^{21} or 6^{28}
Dyck* $\{8^3\}$	32	$(*)$	6^{16}	6^{16} or 8^{12}
$\{11^3\}$	220	$PSL(2,11)$	10^{66}	10^{66} or 6^{110} or 12^{55}

$(*)$ is a solvable group of order 96 generated by two elements R, S subject to the relations
$R^3=S^8=(RS)^2=(S^2R^{-1})^3=1$.

Folding a surface

Let G be a map on a surface S and f a fixed-point free involution on S; denote by $\tilde{G}$ the corresponding map on the folded surface $\tilde{S}$.

- Zigzags of G, which are invariant under f, are mapped to zigzags of half-length and half-signature in $\tilde{G}$.
- If $Z_2 = f(Z_1)$ with $Z_2 \neq Z_1$, then we put compatible orientation on Z_i. Then, the Z_i are mapped to a zigzag $\tilde{Z}$ of $\tilde{G}$ with the signature of Z_1 plus the half of the intersection between Z_1 and Z_2.

Example: Petersen graph embedded on the projective plane is a folding of the Dodecahedron by central inversion.

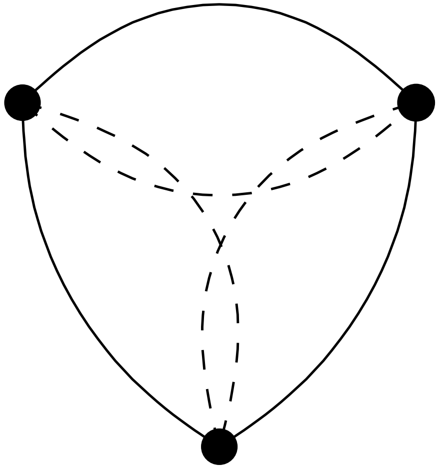
Lins trialities

$(v, f, z) \to$	our notation	notation in [1]	notation in [2]
(v, f, z)	$\mathcal{M}$	gem	$\mathcal{M}$
(f, v, z)	$\mathcal{M}^*$	dual gem	$\mathcal{M}^*$
(z, f, v)	$phial(\mathcal{M})$	phial gem	$p((p(\mathcal{M}))^*)$
(f, z, v)	$(phial(\mathcal{M}))^*$	skew-dual gem	$(p(\mathcal{M}))^*$
(v, z, f)	$skew(\mathcal{M})$	skew gem	$p(M)$
(z, v, f)	$(skew(\mathcal{M}))^*$	skew-phial gem	$p(\mathcal{M}^*)$

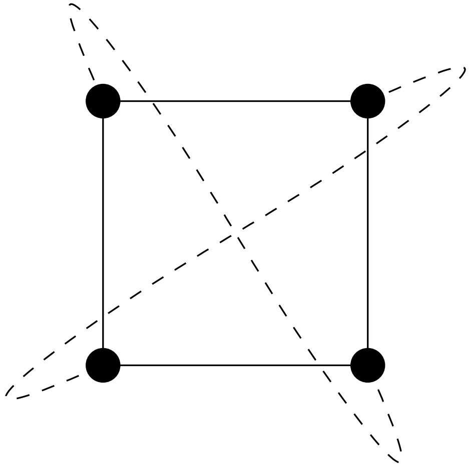
Jones, Thornton (1987): those are only “good” dualities.

1. S. Lins, *Graph-Encoded Maps*, J. Combinatorial Theory Ser. B **32** (1982) 171–181.
2. K. Anderson and D.B. Surowski, *Coxeter-Petrie Complexes of Regular Maps*, European J. of Combinatorics **23-8** (2002) 861–880.

Example: Tetrahedron



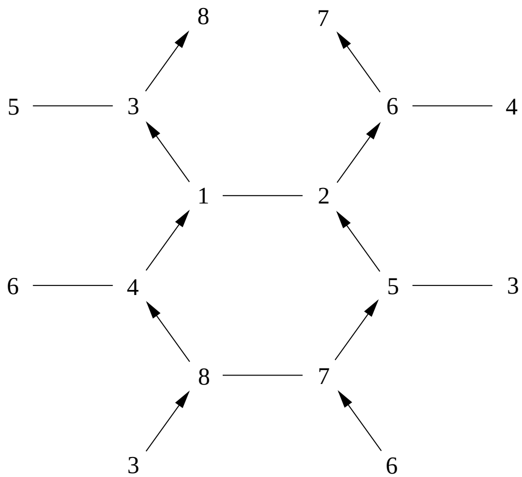
$phial(Tetrahedron)$

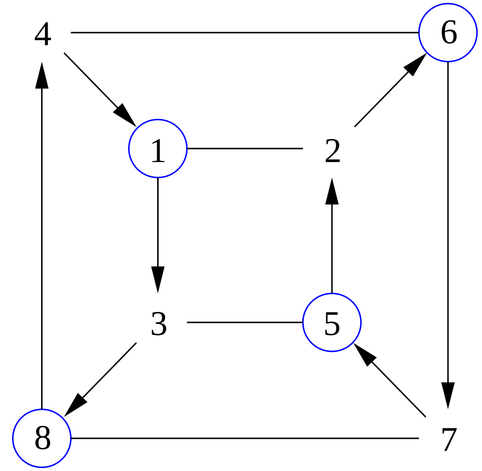


$skew(Tetrahedron)$

two Lins maps on projective plane.

Bipartite skeleton case





Two representation of $\color{red}{skew}(Cube)$: on Torus and as a Cube with cyclic orientation of vertices (marked by ◯) reversed.

Theorem

For bipartite graph embedded in oriented surface, the skew operation is, in fact, reversing orientation of one of the part of the bipartition.

Prisms and antiprisms

Let χ denotes the Euler characteristic.
We conjecture:

- $skew(Prism_m)$ has $\chi = gcd(m, 4) - m$ and is oriented iff m is even;
- $phial(Prism_m)$ has $\chi = 2 + gcd(m, 4) - 2m$ and is non-oriented.

- $skew(APrism_m)$ has $\chi = 1 + gcd(m, 3) - 2m$ and is non-oriented;
- $phial(APrism_m)$ has $\chi = 3 + gcd(m, 3) - 2m$ and is oriented.

VI. Zigzags on n-dimensional complexes

Zigzags on n-dimensional polytopes

A flag $u = (f_0, \dots, f_{n-1})$ is a sequence of faces f_i (of polytope P) of dimension i with $f_i \subset f_{i+1}$.
Given a flag u, there exist an unique flag $\sigma_i(u)$, which differs from u only in position i.

A zigzag z is a circuit of flags $(u_j)_{1 \le j \le l}$, such that $u_j = \sigma_n \dots \sigma_1(u_{j-1})$; the number of flags is called its length.

The zigzags partition the flag-set of P.
z-vector of P is a vector, listing zigzags with their lengths.

Proposition
If the dimension of polytope is odd, then the length of any zigzag is even.

Zigzag of reg. and semireg. d-polytopes

d	d-polytope	z-vector
3	Dodecahedron	10^6
4	24-cell	12^{48}
4	600-cell	30^{240}
d	d-simplex=α_d	$(n+1)^{n!/2}$
d	d-cross-polytope=β_d	$(2n)^{2^{n-2}(n-1)!}$
4	octicosahedric polytope	45^{480}
4	snub 24-cell	20^{144}
4	0_{21}=Med(α_4)	15^{12}
5	1_{21}=Half-5-Cube	12^{240}
6	2_{21}=Schläfl polytope (in E_6)	18^{4320}
7	3_{21}=Gosset polytope (in E_7)	90^{48384}
8	4_{21} (240 roots of E_8)	$36^{29030400}$

Reg.-faced and Conway's polytopes

d	d-polytope	z-vector
4	$Pyr(Icosahedron)$	25^{12}
4	$BPyr(Icosahedron)$	40^{12}
4	$0_{21} + Pyr(\beta_3)$	42^6
d	$Pyr(\beta_{d-1})$, $d \geq 4$	$(\frac{2(d^2-1)}{gcd(d,2)})^x$
d	$BPyr(\alpha_{d-1})$, $d \geq 5$	$(\frac{2d^2}{gcd(d,2)})^y$
4	Grand Antiprism	$30^{20}, 50^{40}, 90^{20}$
4	$C_p \times C_q$	$(\frac{2pq}{t})^{2t}, (\frac{4pq}{t})^{2t}$
	(put $t = gcd(p, q)$)	if both, p and q, are odd
		$(\frac{2pq}{t})^{6t}$, otherwise