Zigzags and Central Circuits for 3- or 4-valent plane graphs and generalizations

Michel Deza

ENS/CNRS, Paris and ISM, Tokyo

Mikhail Shtogrin

Steklov Institute, Moscow

Mathieu Dutour

Hebrew University, Jerusalem

and Patrick Fowler

Exeter University

I. k-valent two-faced polyhedra

Polyhedra and planar graphs

A graph is called k-connected if after removing any set of $k-1$ vertices it remains connected.

The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face of P.

Theorem *(Steinitz)*
(i) A graph G is the skeleton of a 3-polytope if and only if it is planar and 3-connected.
(ii) P and P' are in the same combinatorial type if and only if $G(P)$ is isomorphic to $G(P')$.

A planar graph is represented as Schlegel diagram, the program used for this is CaGe by G. Brinkmann, O. Delgado, A. Dress and T. Harmuth.

k-valent two-faced polyhedra

The Euler formula for plane graphs $V - E + F = 2$,
take the following form for k-valent graphs:

$$\begin{array}{rll} 12 = \sum_i (6-i)p_i & \text{if} & k=3 \\ \text{and} \;\; 8 = \sum_i (4-i)p_i & \text{if} & k=4 \end{array}$$

With p_i the number of faces of gonality i.
A k-valent plane graph is called two-faced if the gonality of its faces has only two possible values a and b.

- 3-valent plane graphs with n vertices and faces of gonality q and 6 (classes q_n),
- 4-valent plane graphs with n vertices and faces of gonality 3 or 4 (octahedrites)

Classes and their generation

k	(a, b)	Polyhedra	Exist if and only if	p_a	n
3	$(3, 6)$	3_n	$p_6/2 \in N - \{1\}$	$p_3 = 4$	$4 + 2p_6$
3	$(4, 6)$	4_n	$p_6 \in N - \{1\}$	$p_4 = 6$	$8 + 2p_6$
3	$(5, 6)$	5_n (fullerenes)	$p_6 \in N - \{1\}$	$p_5 = 12$	$20 + 2p_6$
4	$(3, 4)$	octahedrite	$p_4 \in N - \{1\}$	$p_3 = 8$	$6 + p_4$

Generation programs

1. 3-valent: CPF for two-faced maps on the sphere by T. Harmuth
   CGF for two-faced maps on surfaces of genus g by T. Harmuth
2. 4-valent: ENU by T. Heidemeier
3. General: plantri by G. Brinkmann and B. McKay

Point groups

(point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group)

Theorem *(Mani, 1971)*
Given a 3-connected planar graph G, there exist a 3-polytope P, whose group of isometries is isomorphic to $Aut(G)$ and $G(P) = G$.

- For octahedrites: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}, S_4), (D_2, D_{2d}, D_{2h}), (D_3, D_{3d}, D_{3h}), (D_4, D_{4d}, D_{4h}), (O, O_h). (Deza and al.)
- For 3_n: (D_2, D_{2h}, D_{2d}), (T, T_d) (Fowler and al.)
- For 4_n: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}), (D_2, D_{2d}, D_{2h}), (D_3, D_{3d}, D_{3h}), (D_6, D_{6h}), (O, O_h) (Deza and al.)
- For 5_n: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}, S_4), (C_3, C_{3v}, C_{3h}, S_6), (D_2, D_{2h}, D_{2d}), (D_3, D_{3h}, D_{3d}), (D_5, D_{5h}, D_{5d}), (D_6, D_{6h}, D_{6d}), (T, T_d, T_h), (I, I_h) (Fowler and al.)

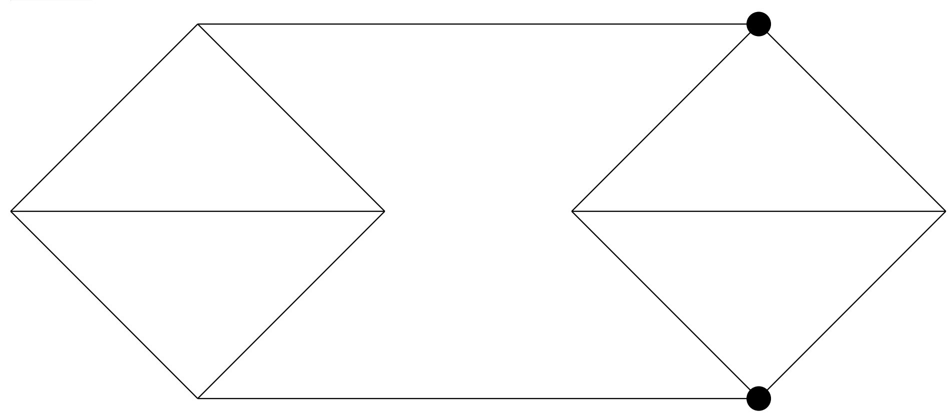
k-connectedness

Theorem

(i) *Any octahedrite is 3-connected.*

(ii) *Any 3-valent plane graph without (> 6)-gonal faces is 2-connected.*

(iii) *Moreover, any 3-valent plane graph without (> 6)-gonal faces is 3-connected except of the following serie G_n:*



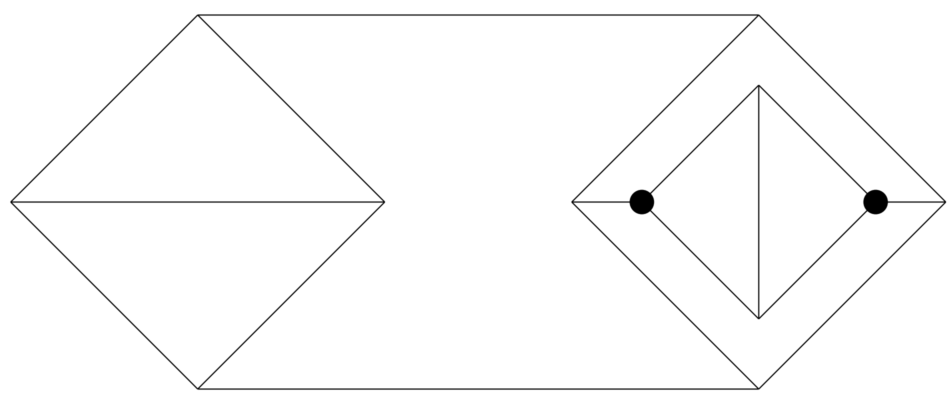
k-connectedness

Theorem

(i) Any octahedrite is 3-connected.

(ii) Any 3-valent plane graph without (> 6)-gonal faces is 2-connected.

(iii) Moreover, any 3-valent plane graph without (> 6)-gonal faces is 3-connected except of the following serie G_n:



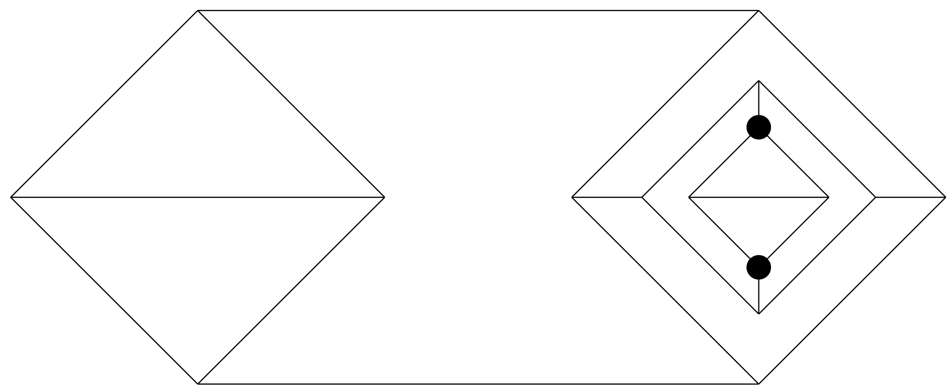
k-connectedness

Theorem

(i) Any octahedrite is 3-connected.

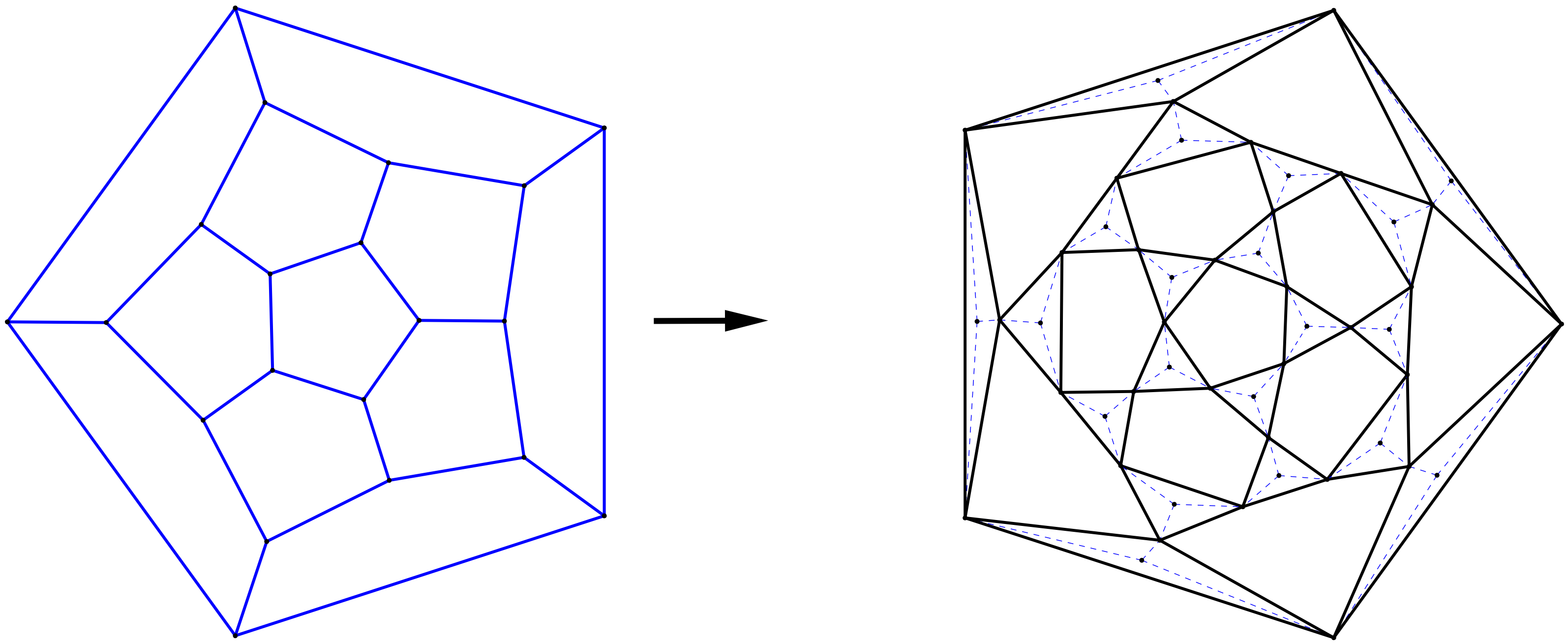
(ii) Any 3-valent plane graph without (> 6)-gonal faces is 2-connected.

(iii) Moreover, any 3-valent plane graph without (> 6)-gonal faces is 3-connected except of the following serie G_n:



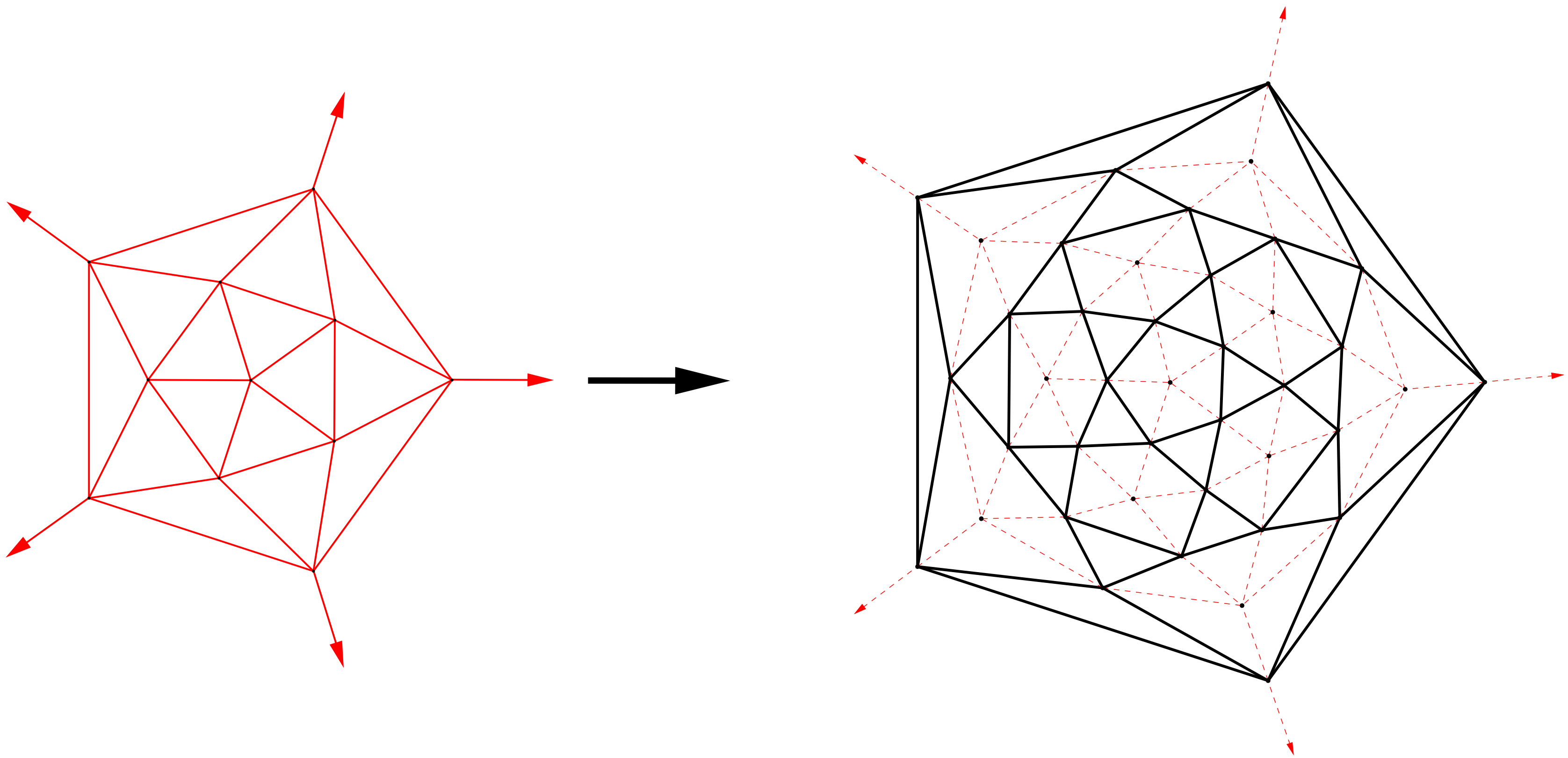
Medial Graph

Given a plane graph G, the 4-valent plane graph $Med(G)$ is defined as the graph having as vertices the edges of G with two vertices adjacent if and only if they share a vertex and belong to a common face.



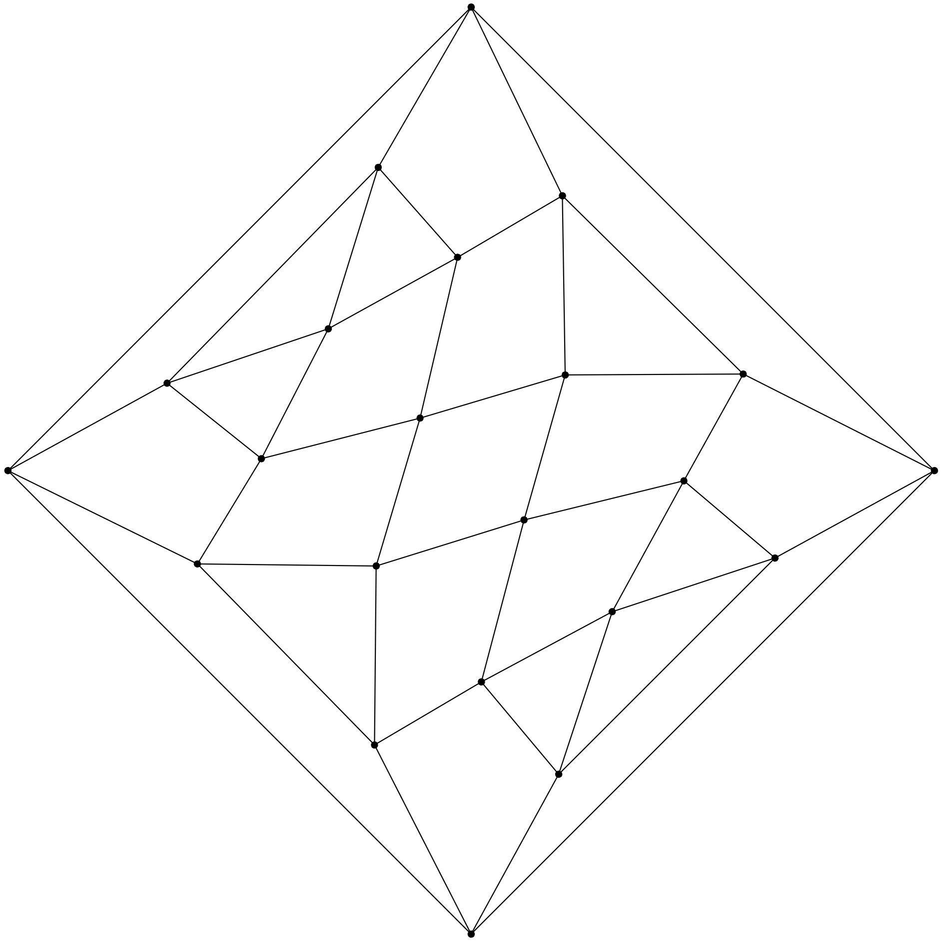
Medial Graph

Given a plane graph G, the 4-valent plane graph $Med(G)$ is defined as the graph having as vertices the edges of G with two vertices adjacent if and only if they share a vertex and belong to a common face. $Med(G) = Med(G^*)$



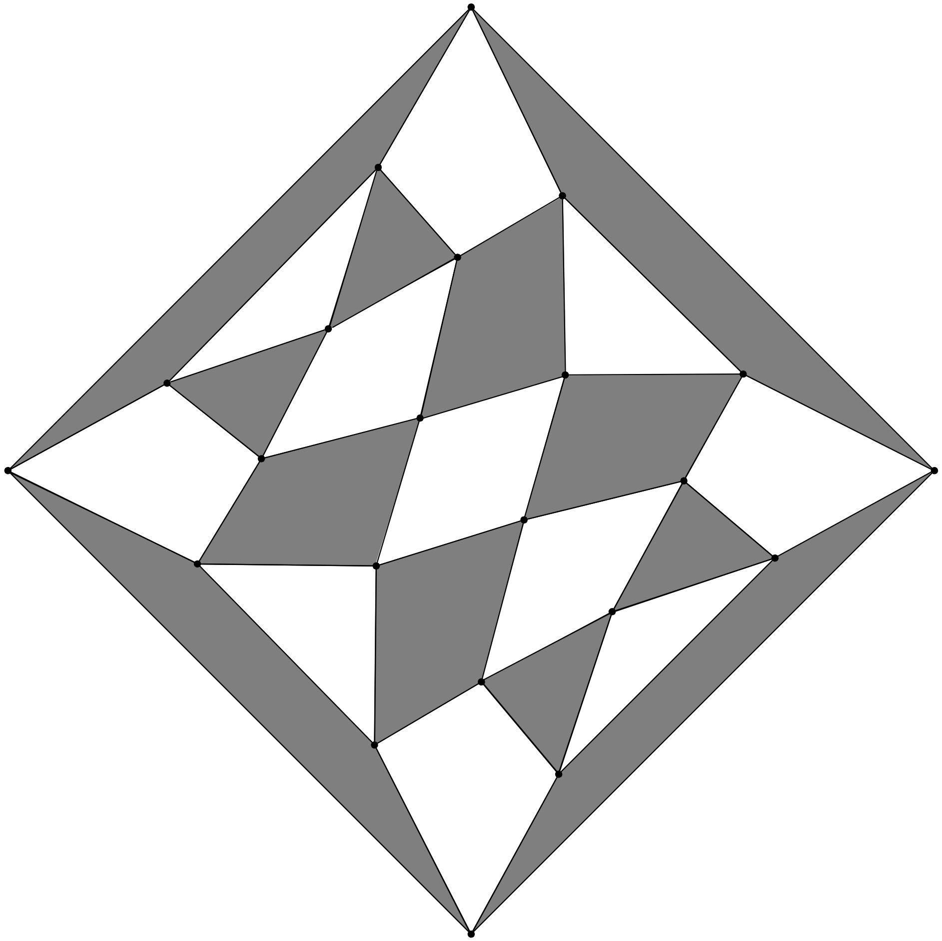
Inverse medial graph

If G is a 4-valent plane graph, then there exist exactly two graphs H_1 and H_2 such that $G = Med(H_1) = Med(H_2)$.



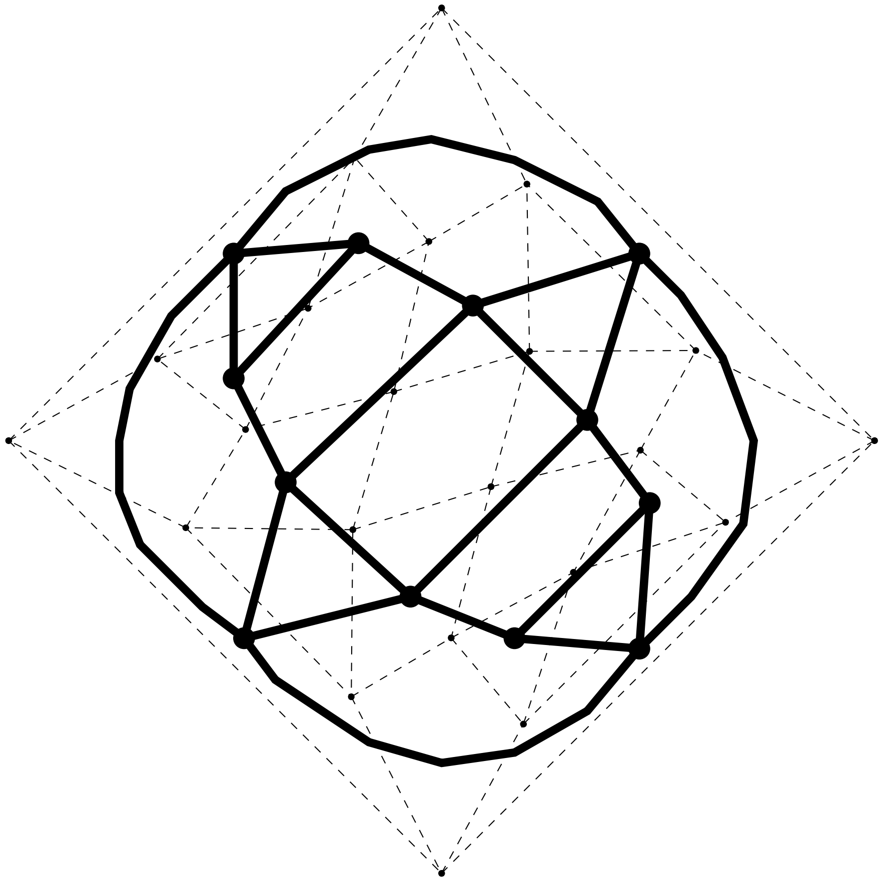
Inverse medial graph

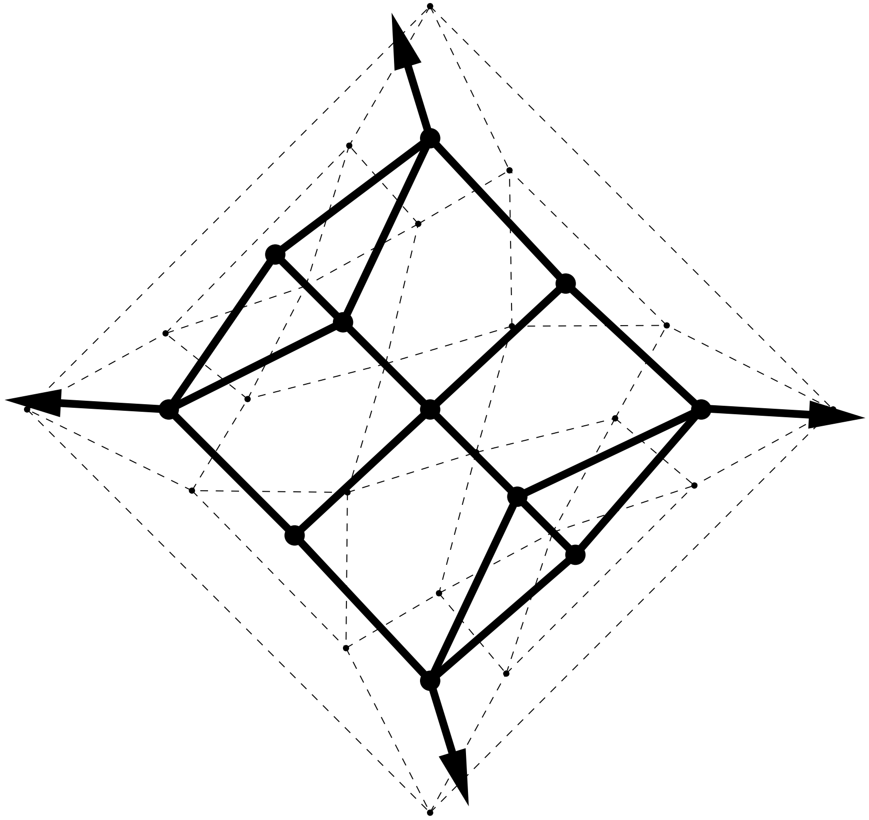
If G is a 4-valent plane graph, then there exist exactly two graphs H_1 and H_2 such that $G = Med(H_1) = Med(H_2)$.



Inverse medial graph

If G is a 4-valent plane graph, then there exist exactly two graphs H_1 and H_2 such that $G = Med(H_1) = Med(H_2)$.





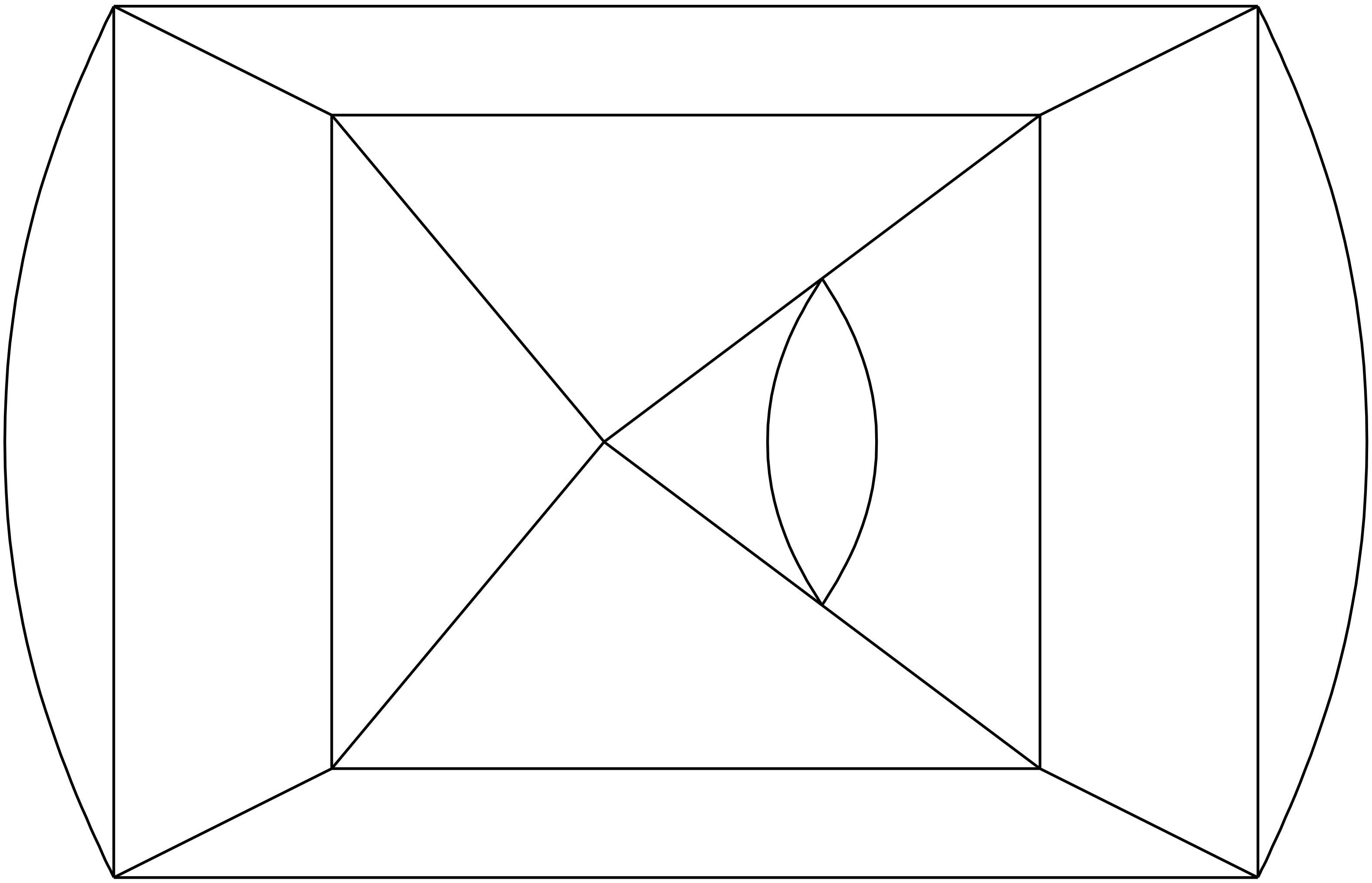
II. Zigzags

and

Central circuits

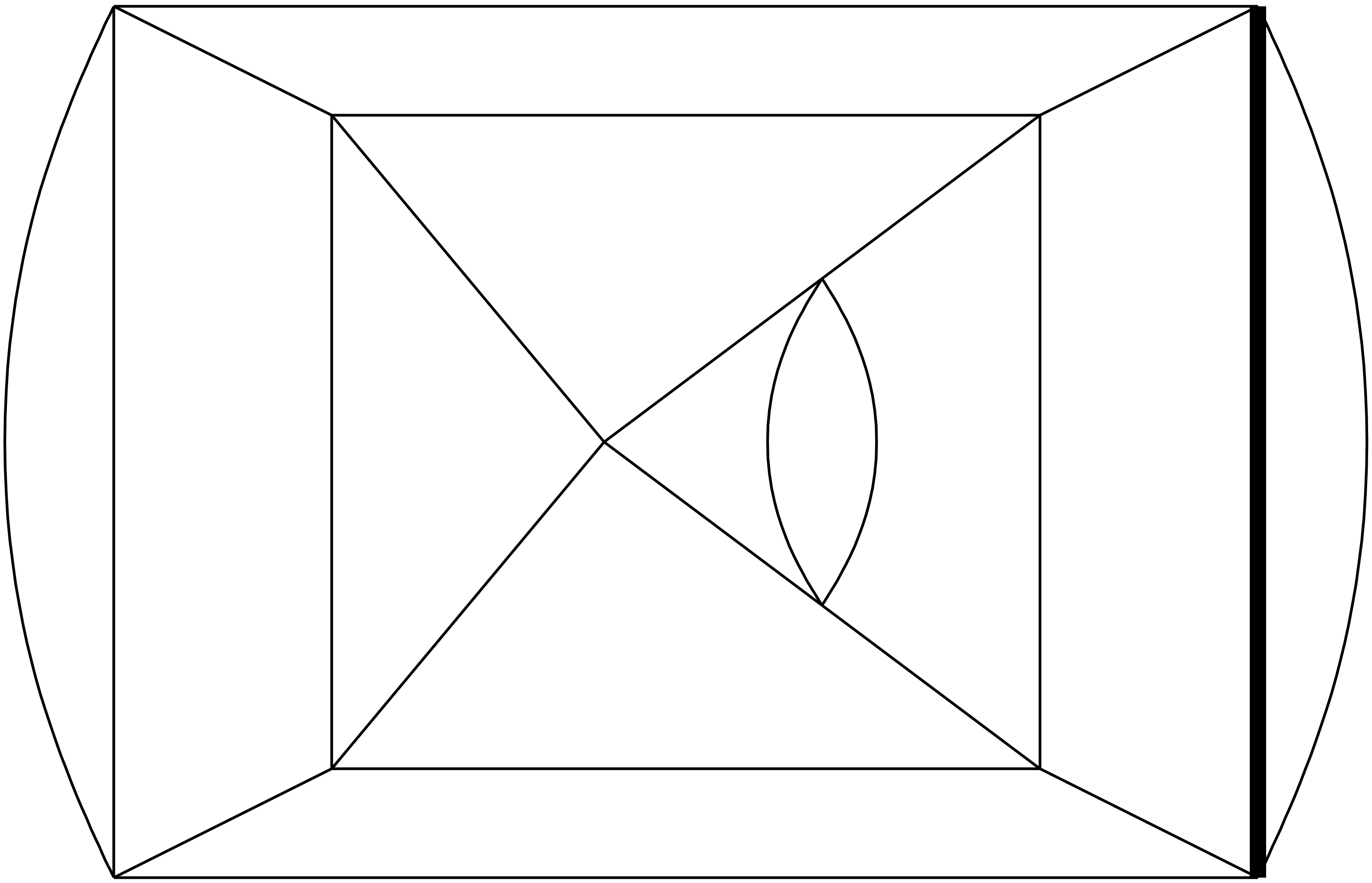
Central circuits

A 4−valent plane graph G



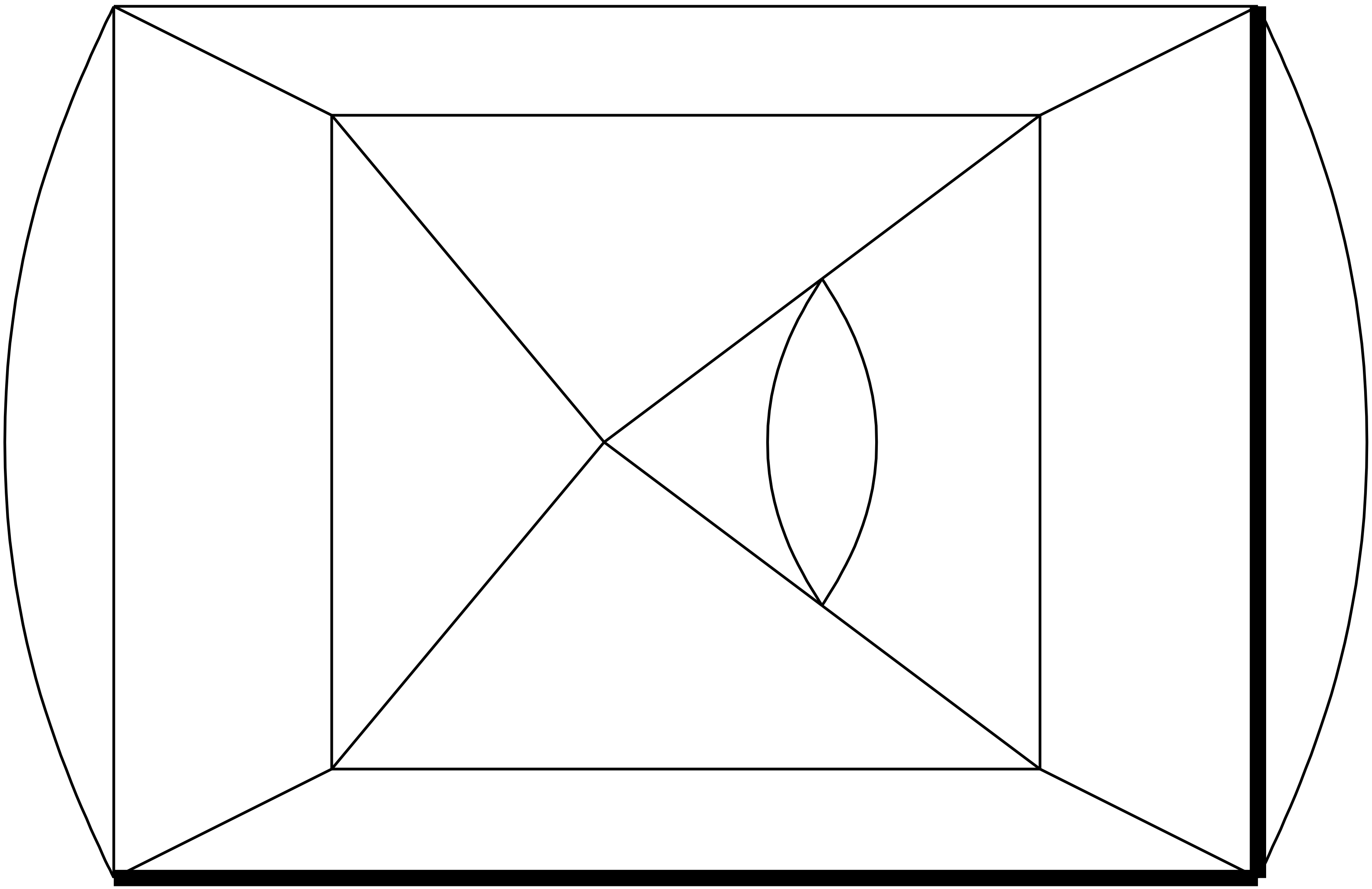
Central circuits

Take an edge of G



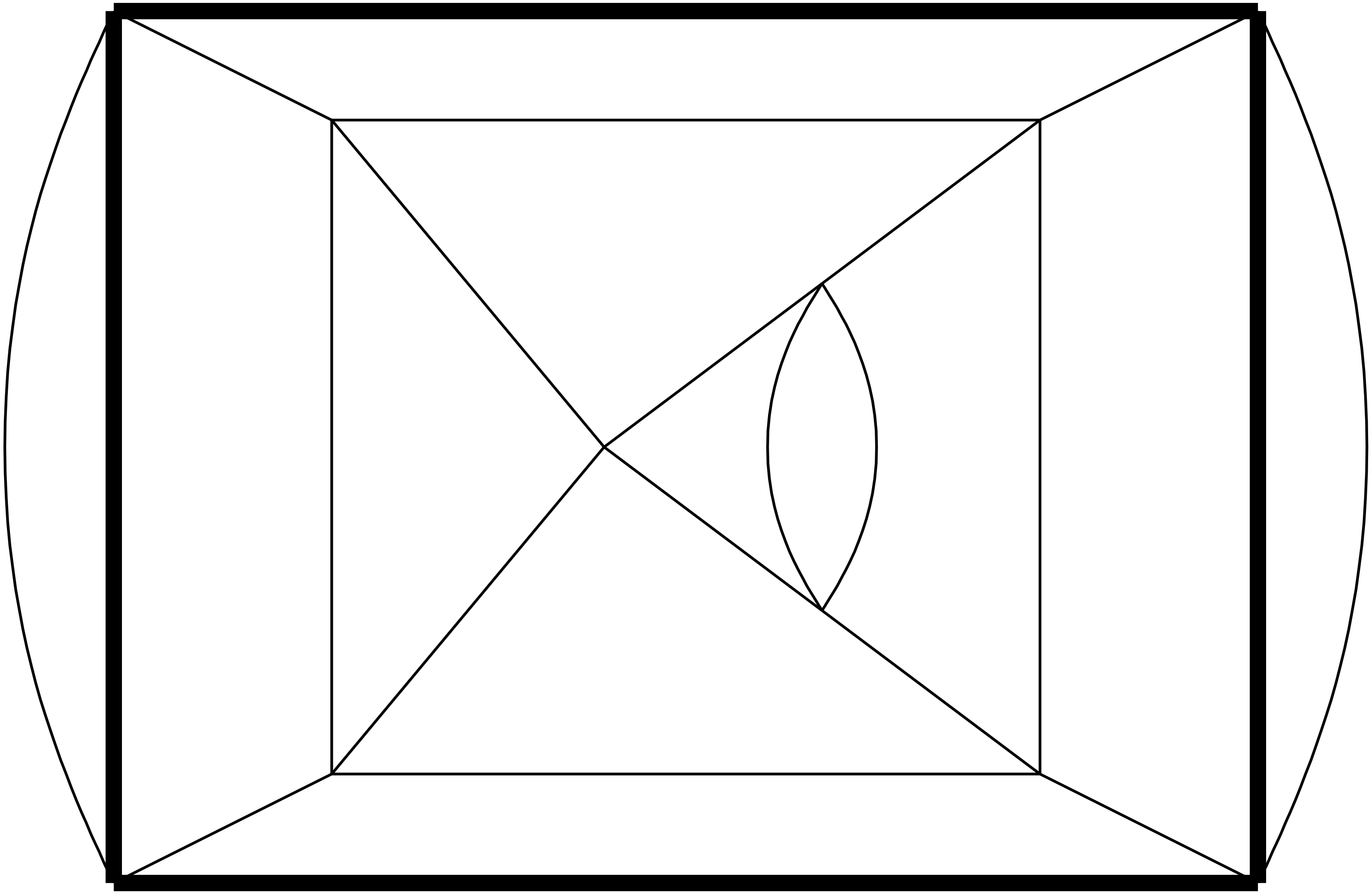
Central circuits

Continue it straight ahead ...



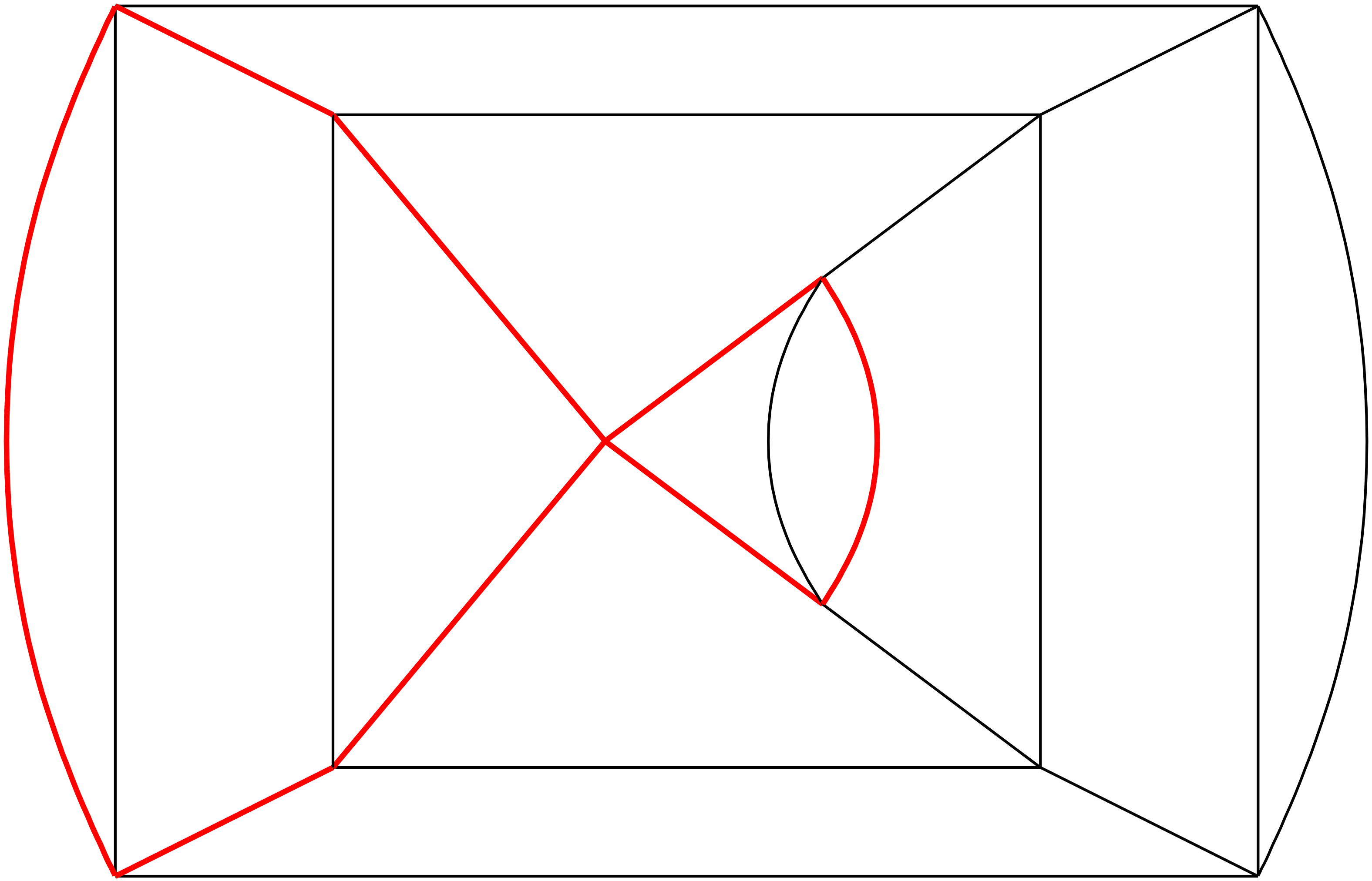
Central circuits

... until the end



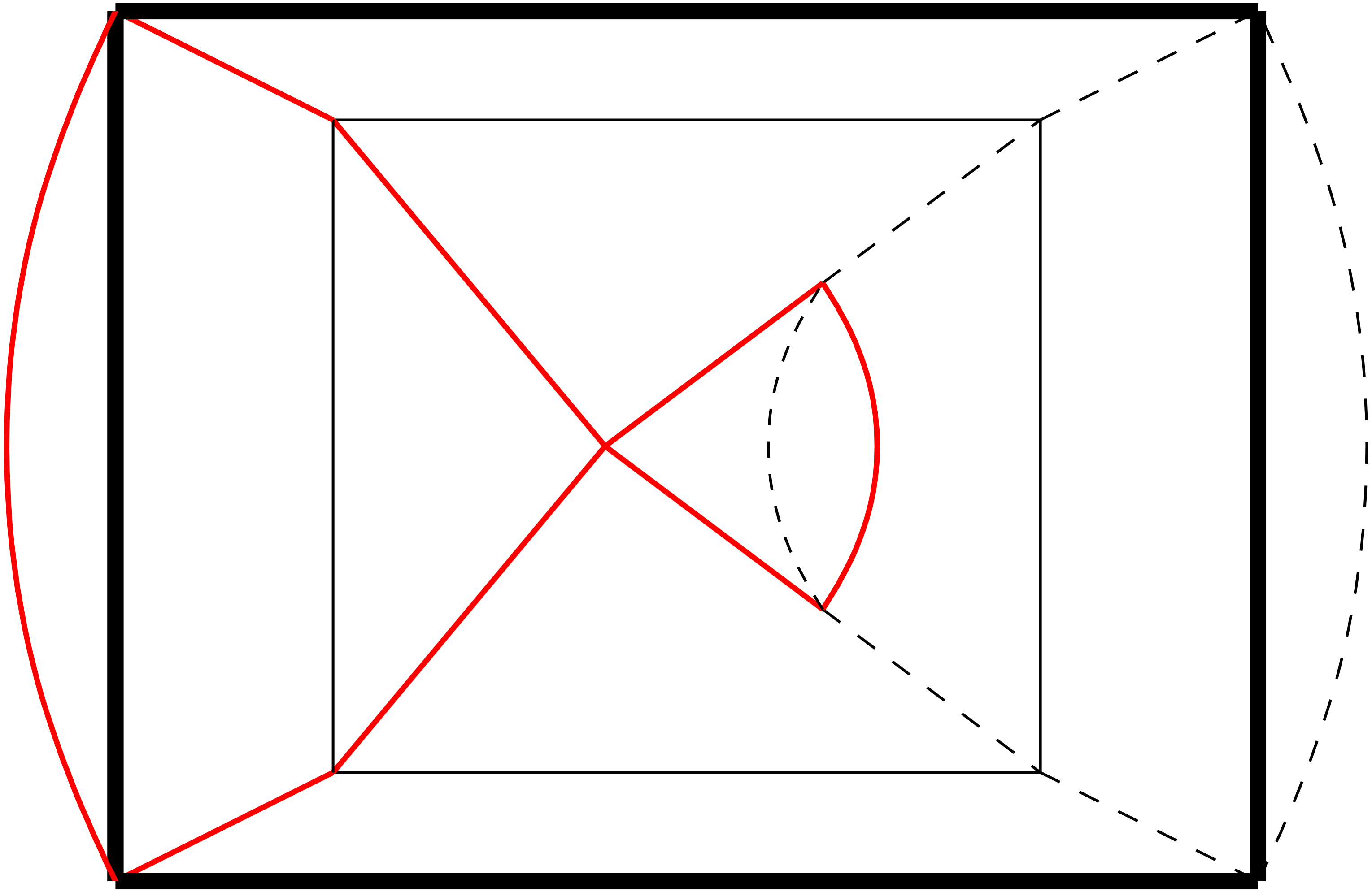
Central circuits

A self−intersecting central circuit



Central circuits

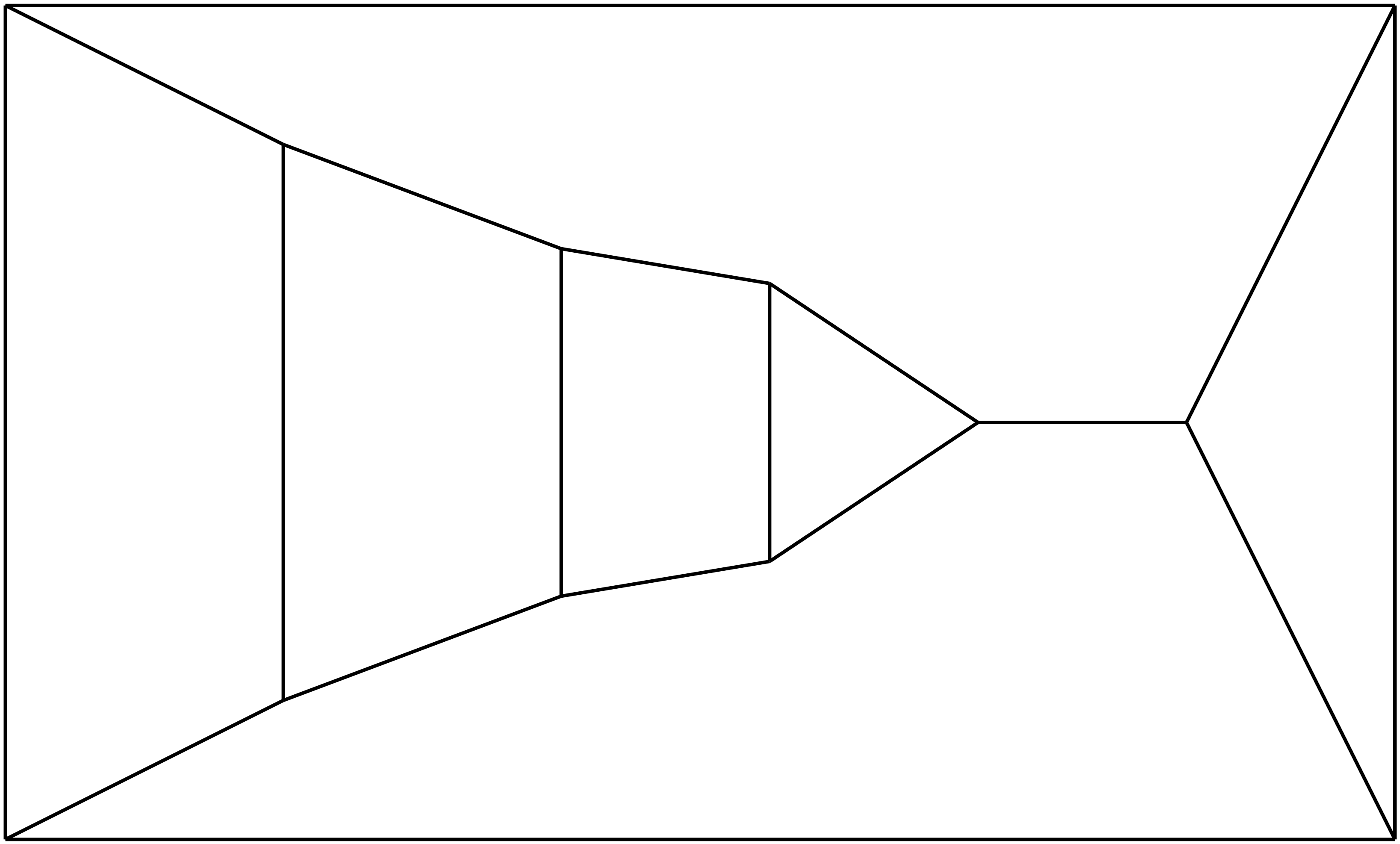
A partition of edges of G



CC=4^2, 6, 8

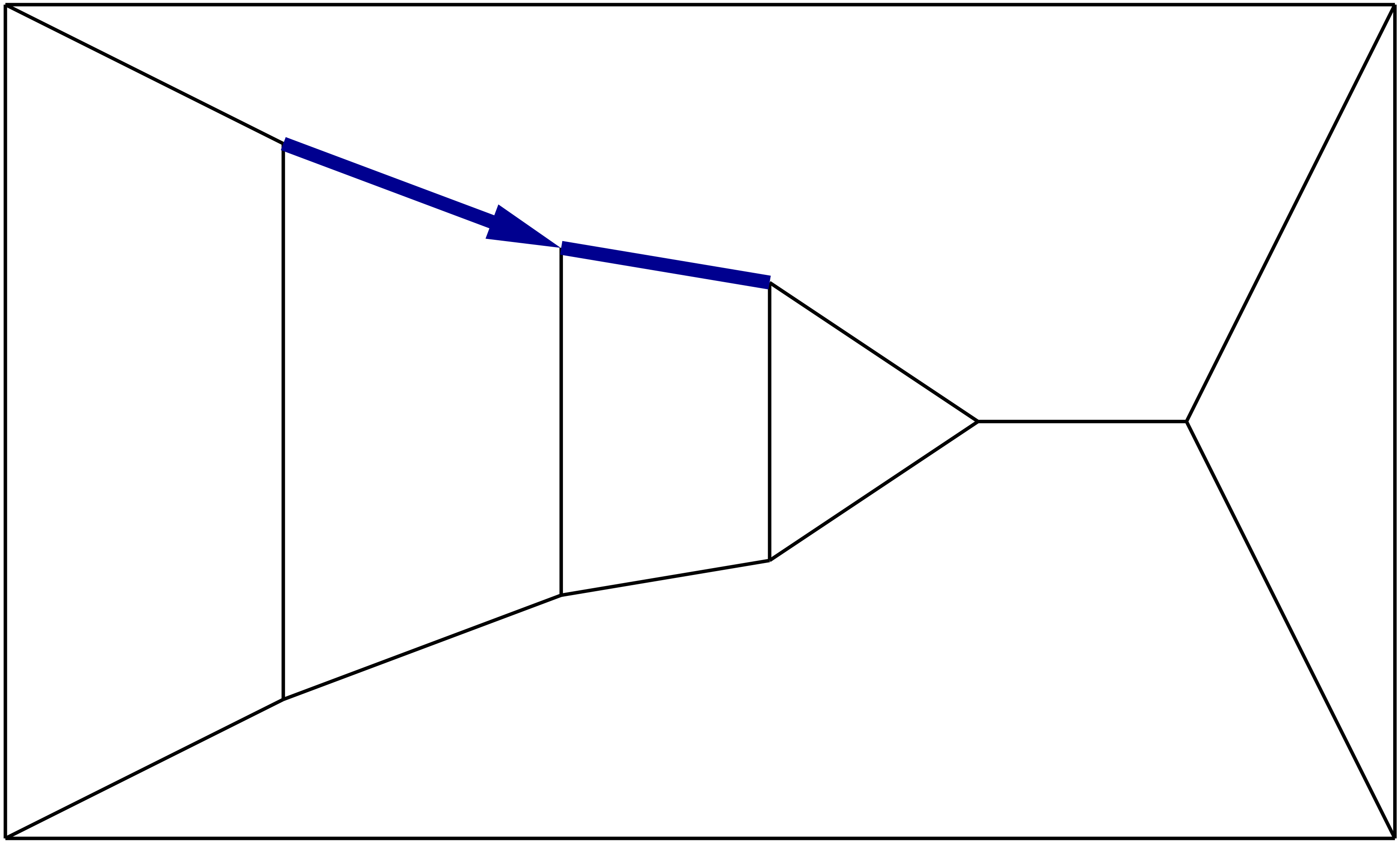
Zigzags

A plane graph G



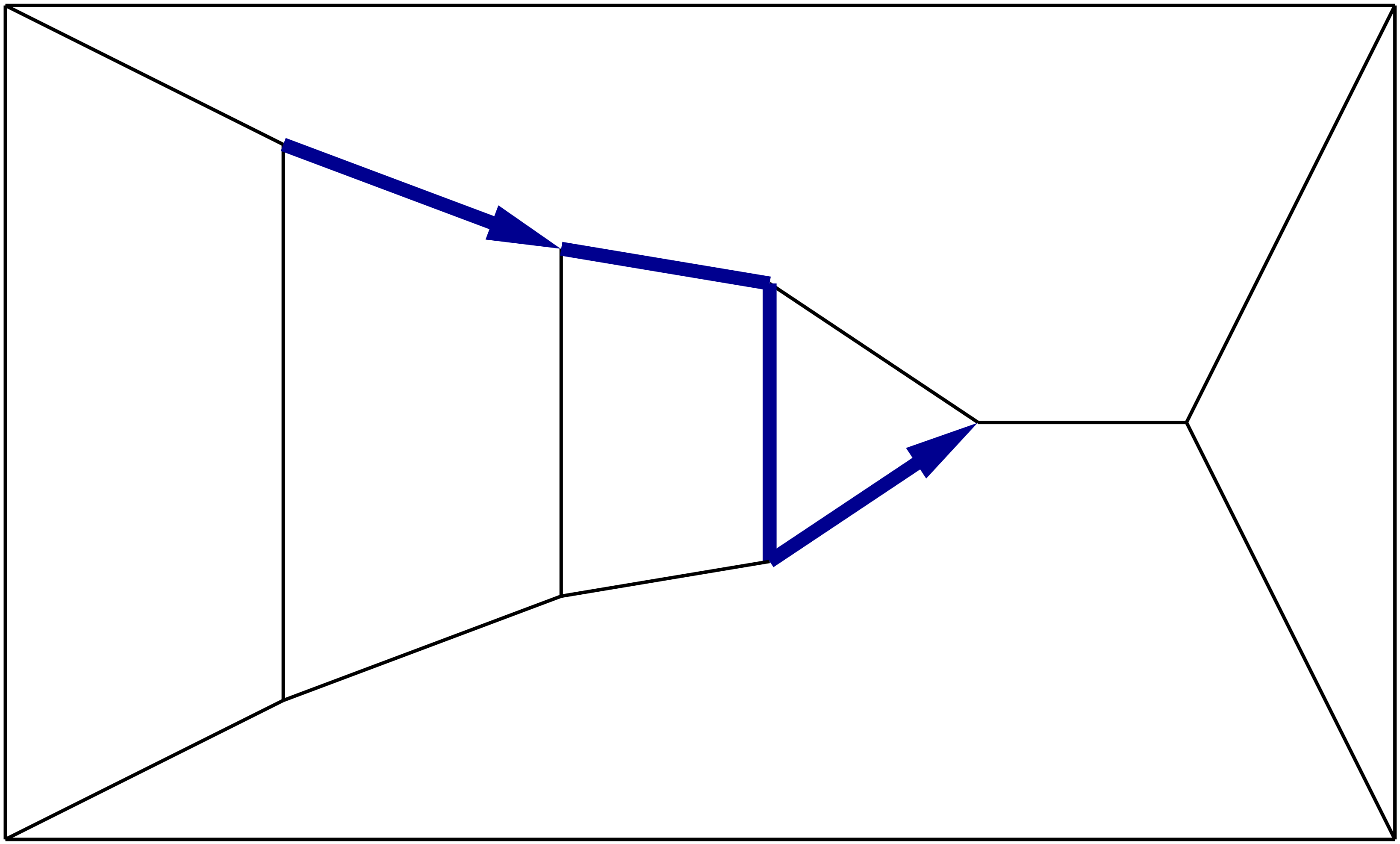
Zigzags

Take two edges



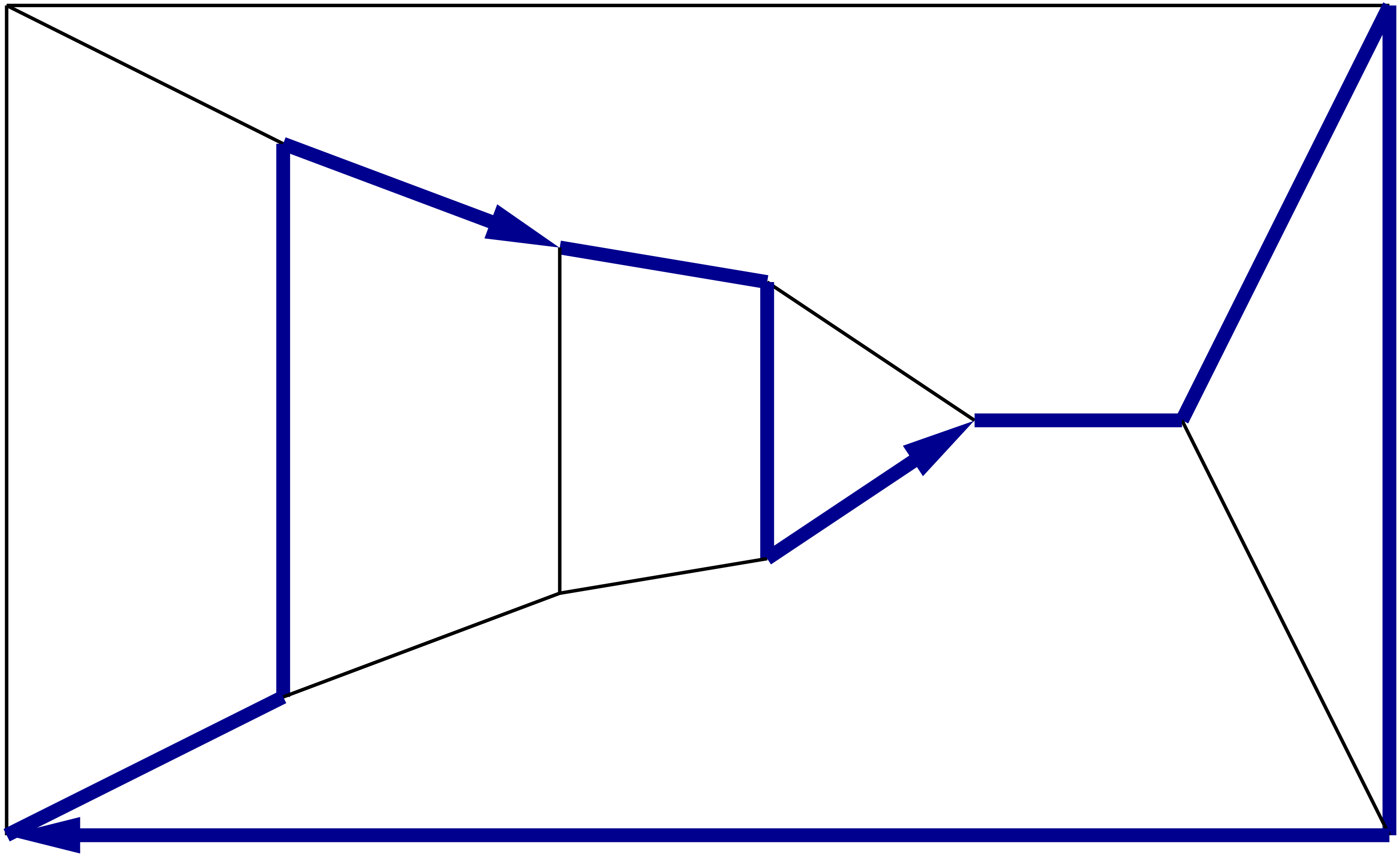
Zigzags

Continue it left−right alternatively ....



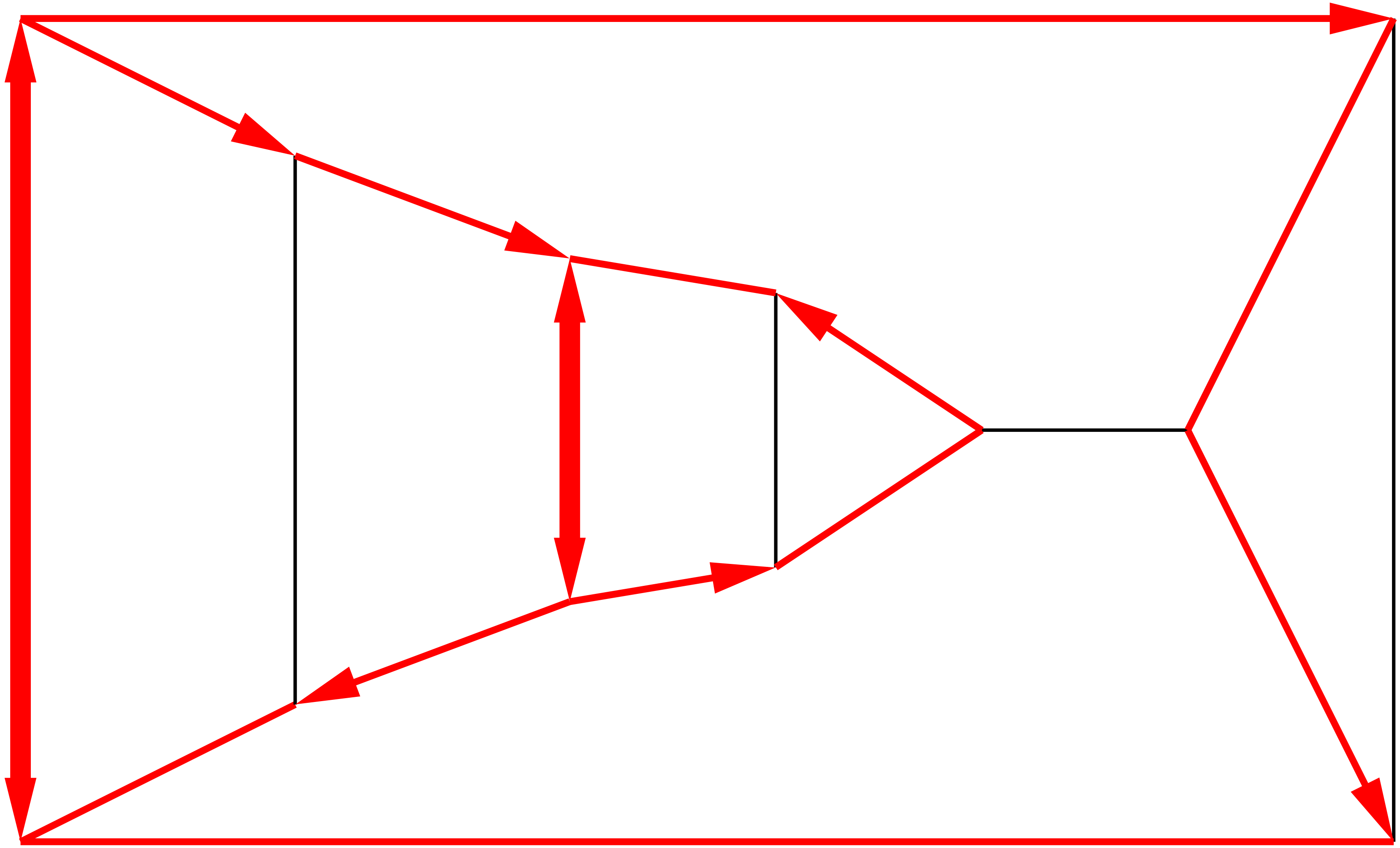
Zigzags

... until we come back



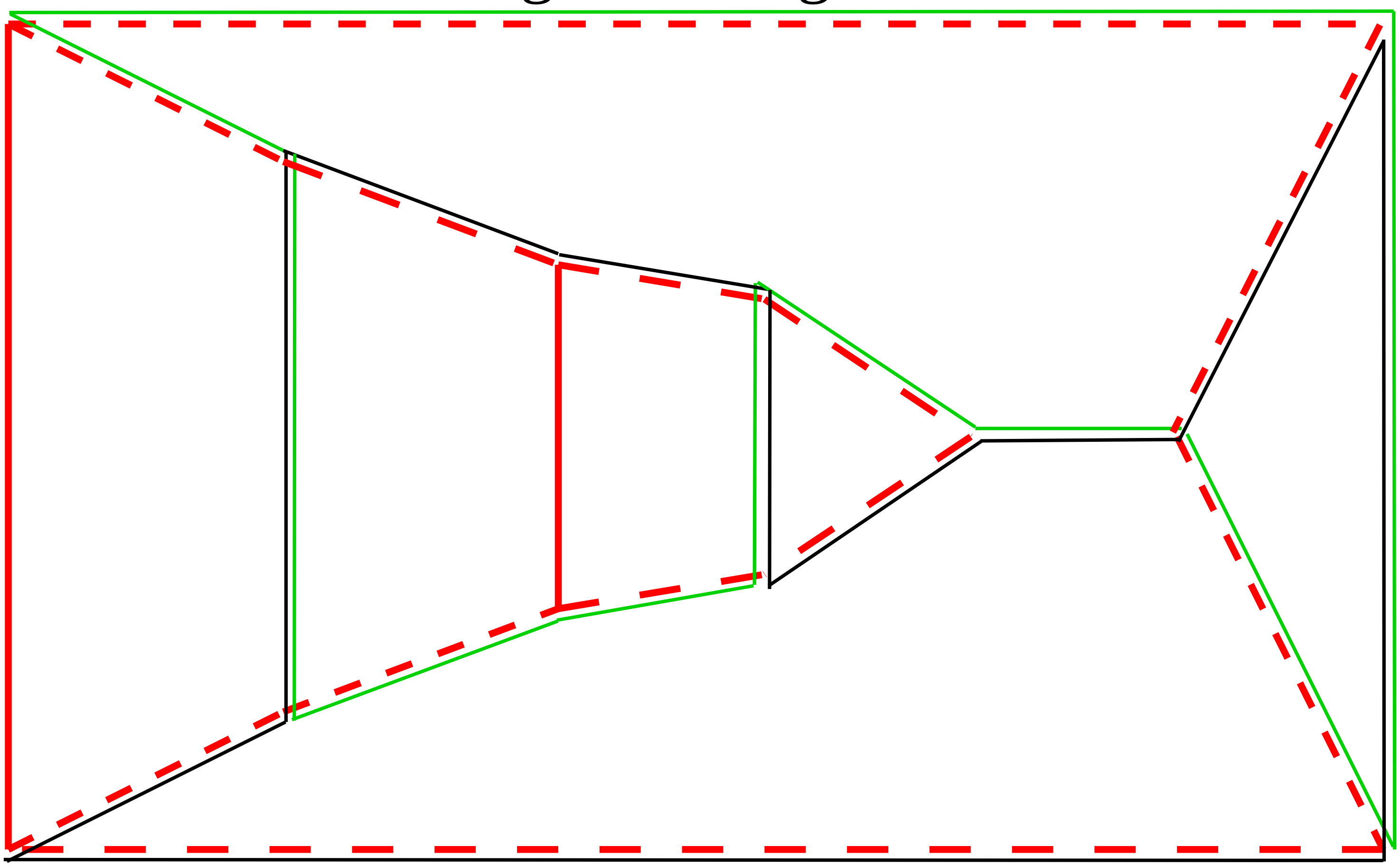
Zigzags

A self−intersecting zigzag



Zigzags

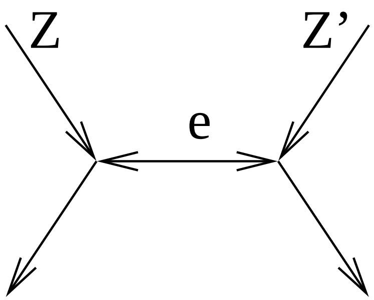
A double covering of 18 edges: 10+10+16



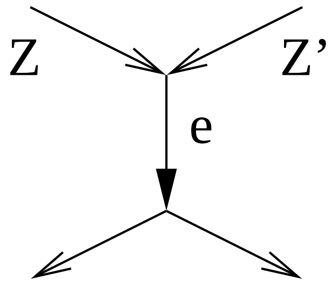
z−vector $z = 10^2, \; 16_{2,0}$

Intersection Types for zigzags

Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge of intersection $Z \cap Z'$ is called of type I or type II, if Z and Z' traverse e in opposite or same direction, respectively



Type I

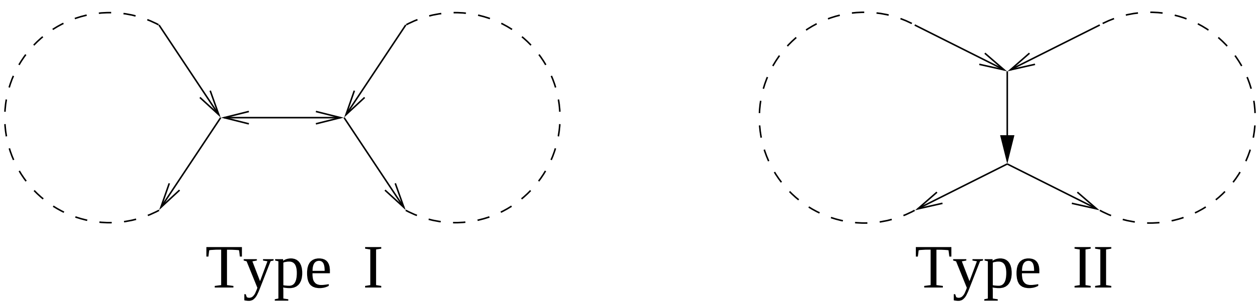


Type II

Intersection Types for zigzags

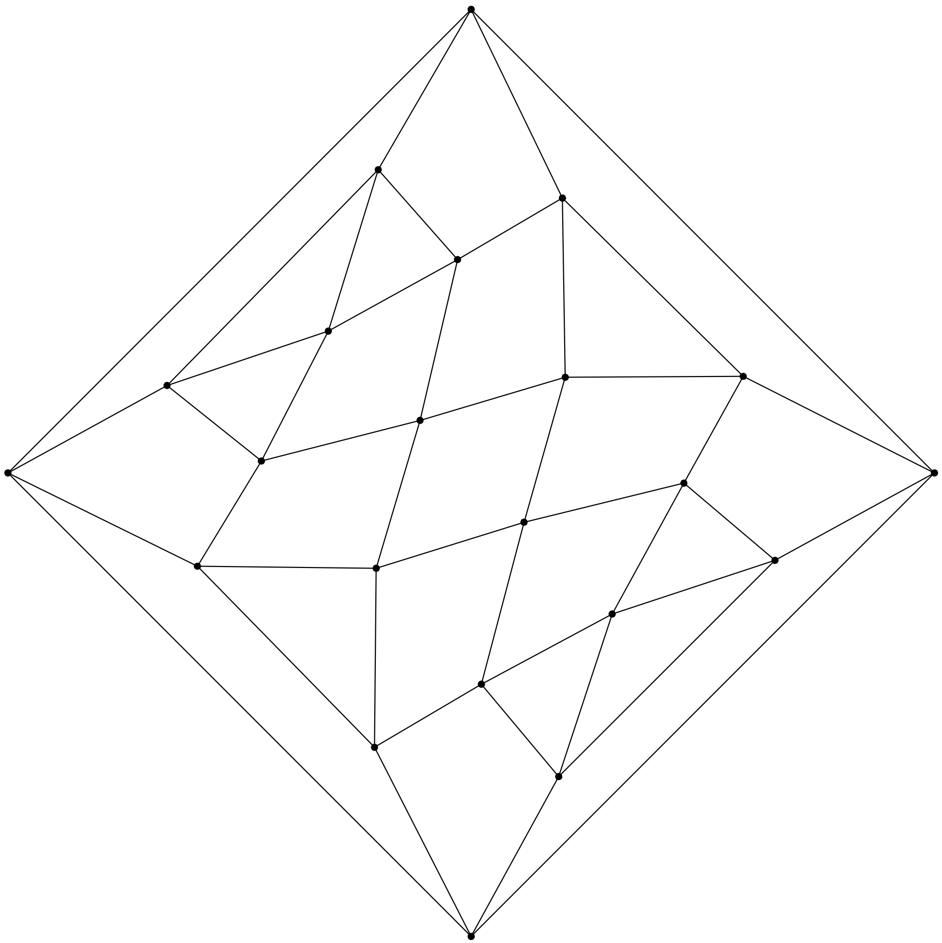
Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge of intersection $Z \cap Z'$ is called of type I or type II, if Z and Z' traverse e in opposite or same direction, respectively

The types of self-intersection depends on orientation chosen on zigzags except if $Z = Z'$:



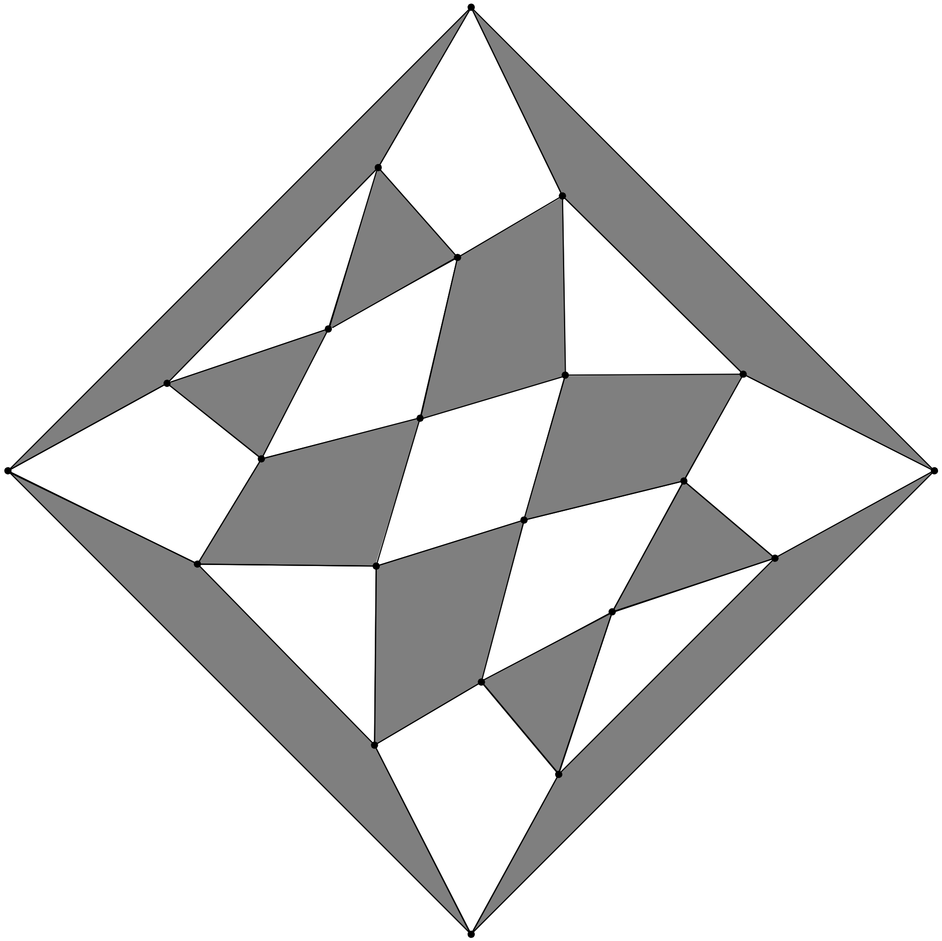
Intersection Types for central circuits

Let G be a 4-valent plane graph



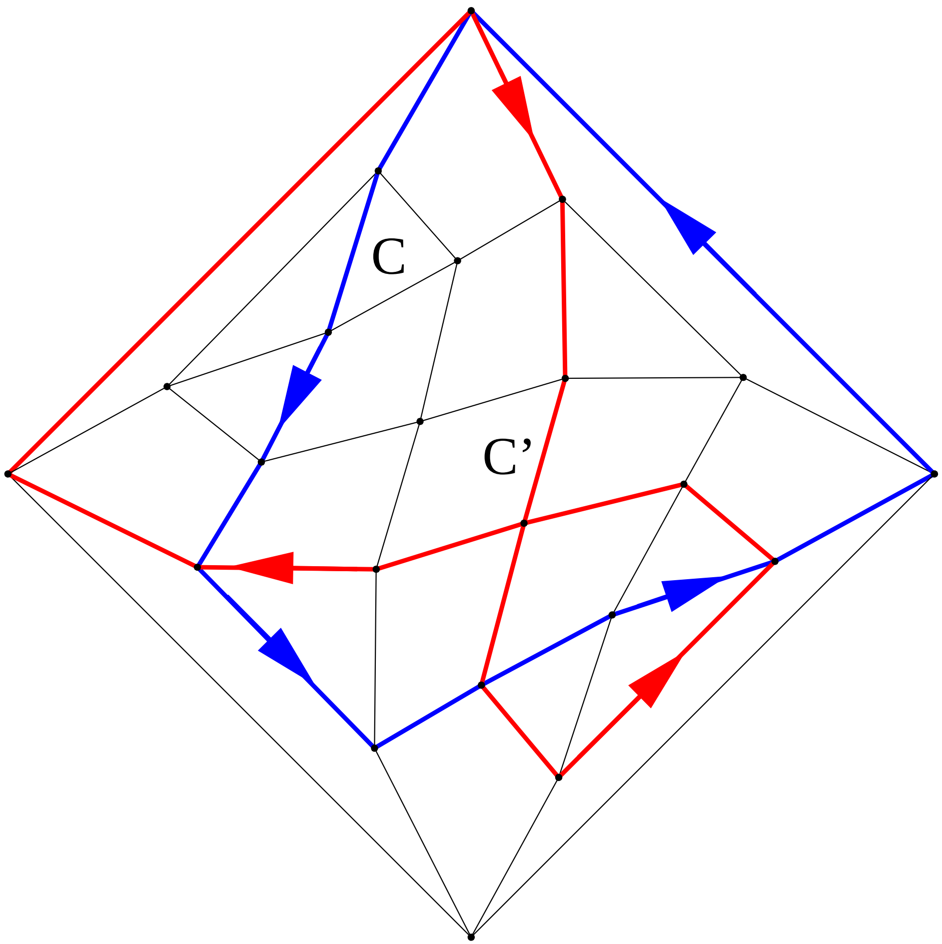
Intersection Types for central circuits

Take $\mathcal{C}_1$(▇), $\mathcal{C}_2$(□) a bipartition of the face-set of G



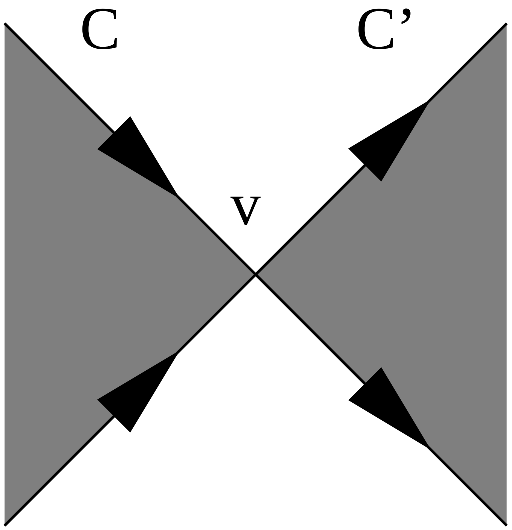
Intersection Types for central circuits

Let C and C' be two central circuits of G and let an orientation be selected on them.

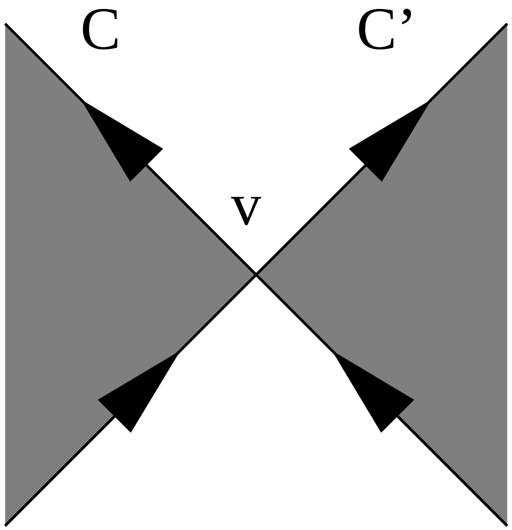


Intersection Types for central circuits

Local View on a vertex v and type



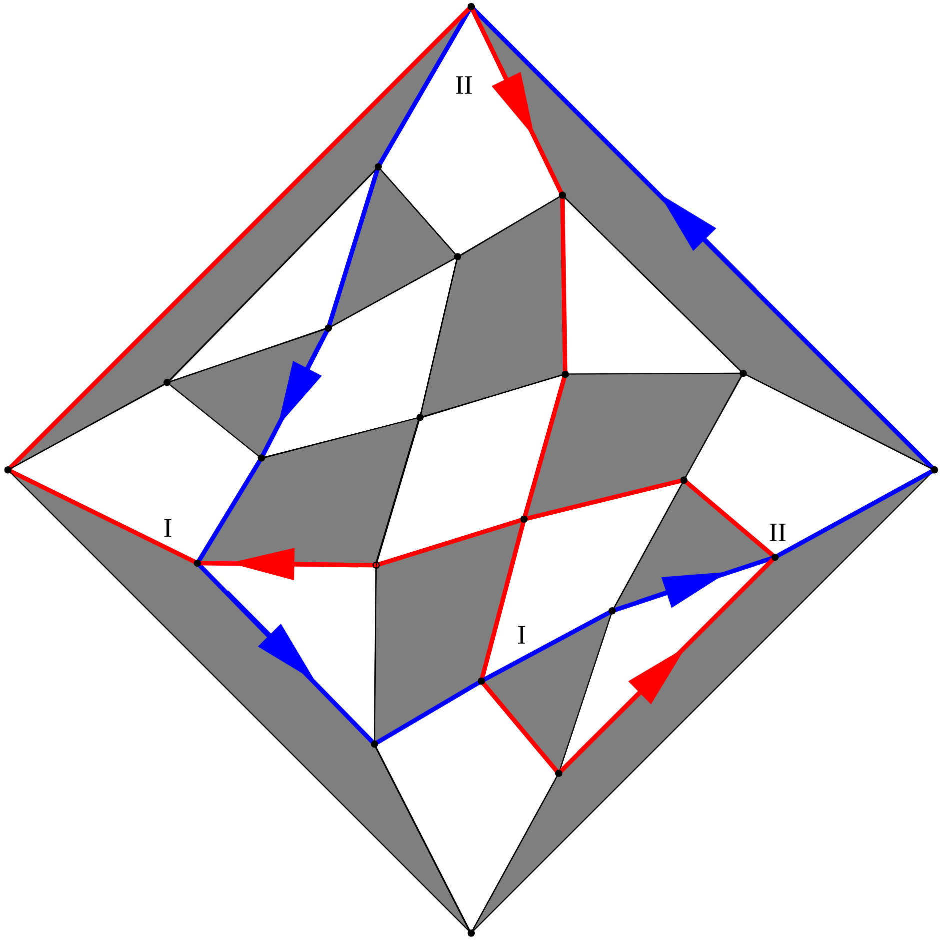
Type I



Type II

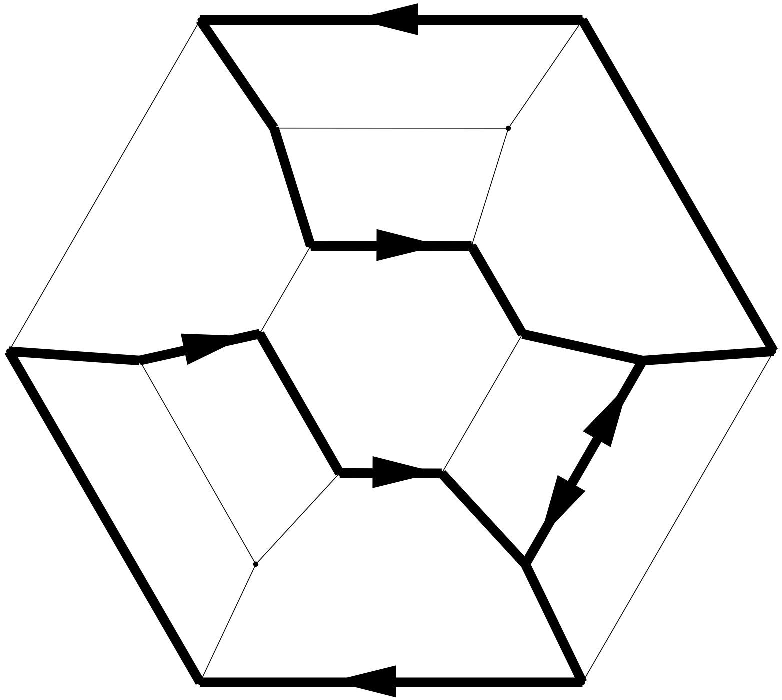
Intersection Types for central circuits

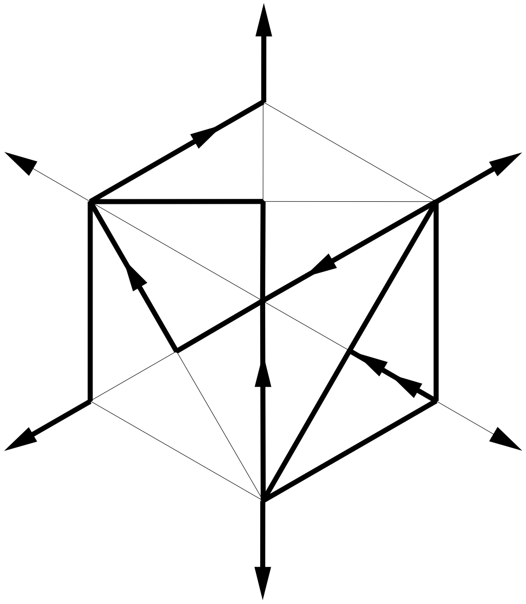
C and C' have 2 intersections I and 2 intersection II



Medial, zigzags and central circuits

Zigzags of a plane graph G are in one-to-one correspondence with zigzags of G^*.

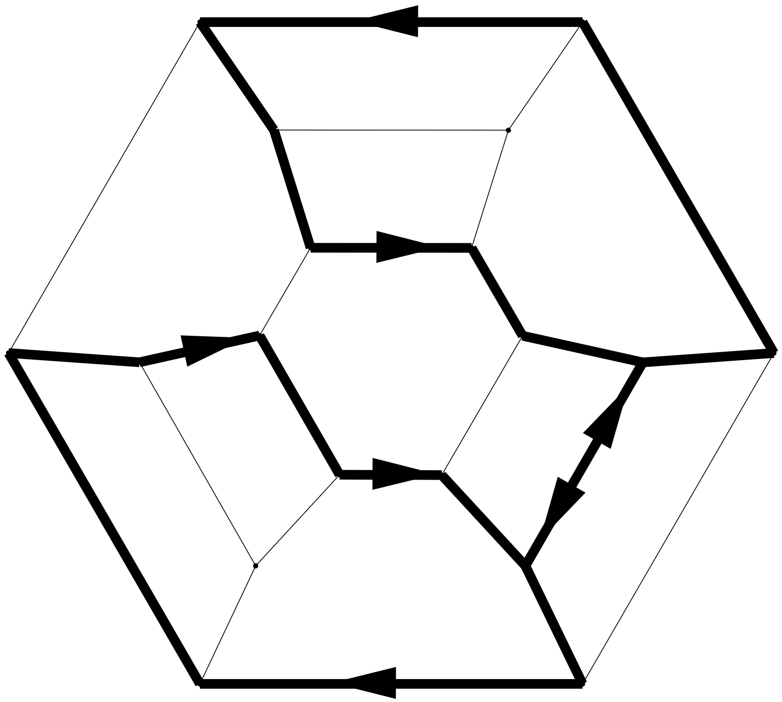


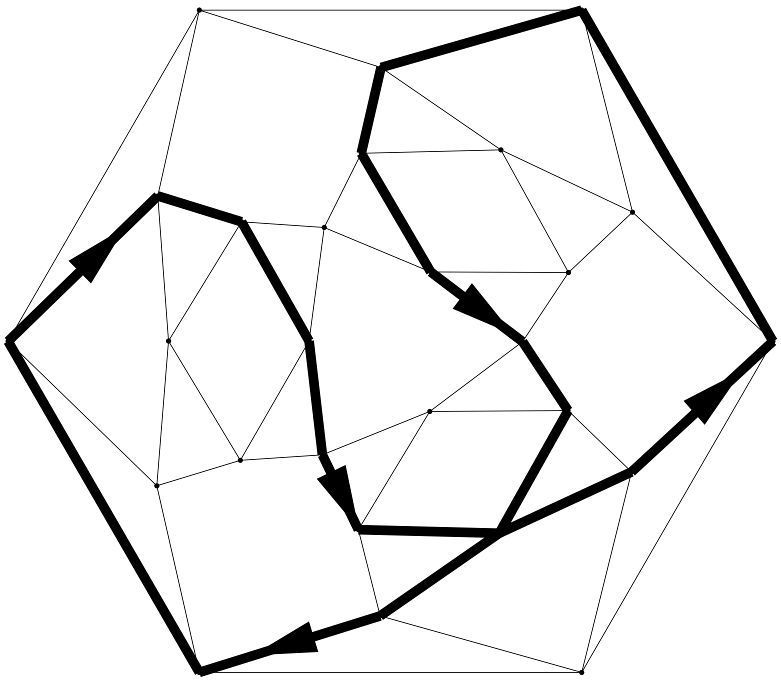


Types are interchanged

Medial, zigzags and central circuits

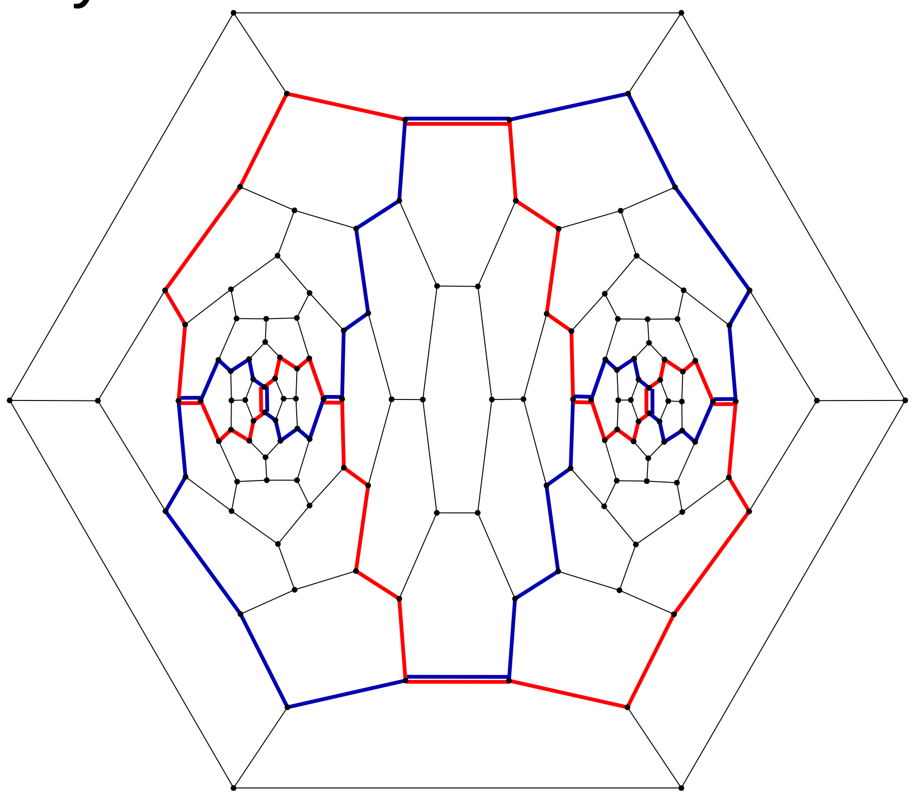
Zigzags of a plane graph G are in one to one correspondence with central circuits of $Med(G)$.





Intersection two simple ZC-circuits

- For the class of graph 4_n the size of the intersection of two simple zigzags belongs to $\{0, 2, 4, 6\}$.
- For classes of octahedrites, graph 3_n or graph 5_n the size of the intersection of two simple ZC-circuits can be any even number.



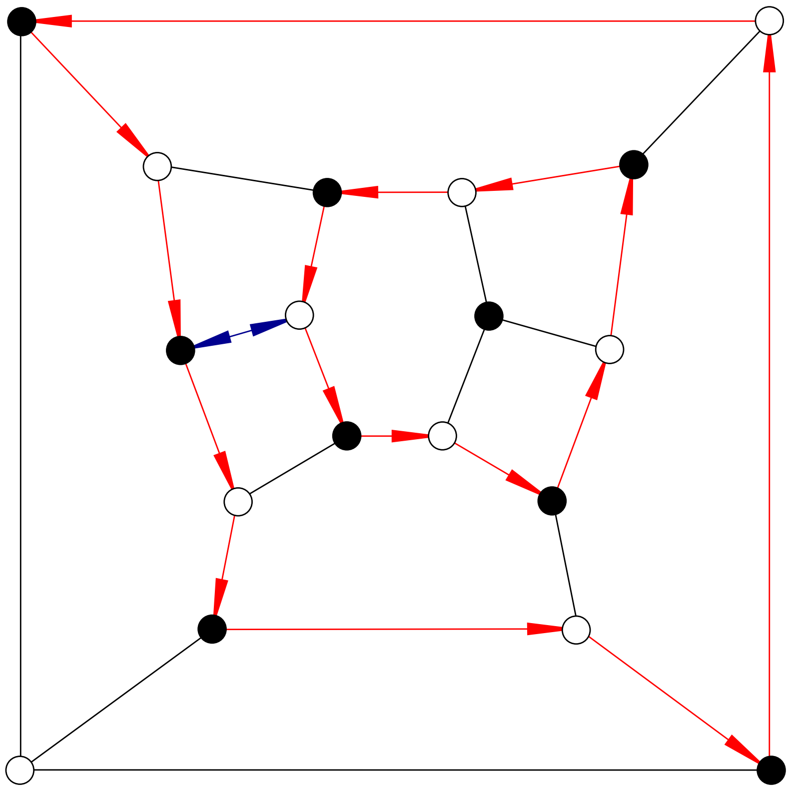
Two simple zigzags of a graph 5_n with $|Z \cap Z'| = 8$.

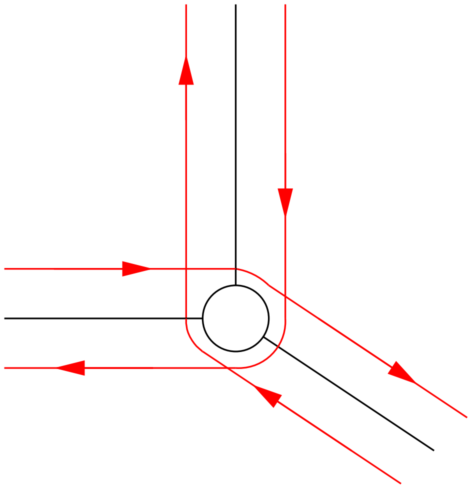
On surfaces of genus $g \geq 1$, the intersection can be odd.

Bipartite graphs

Remark *A plane graph is bipartite if and only if its faces have even gonality.*

Theorem *(Shank-Shtogrin)*
For any planar bipartite graph G there exist an orientation of zigzags, with respect to which each edge has type I.





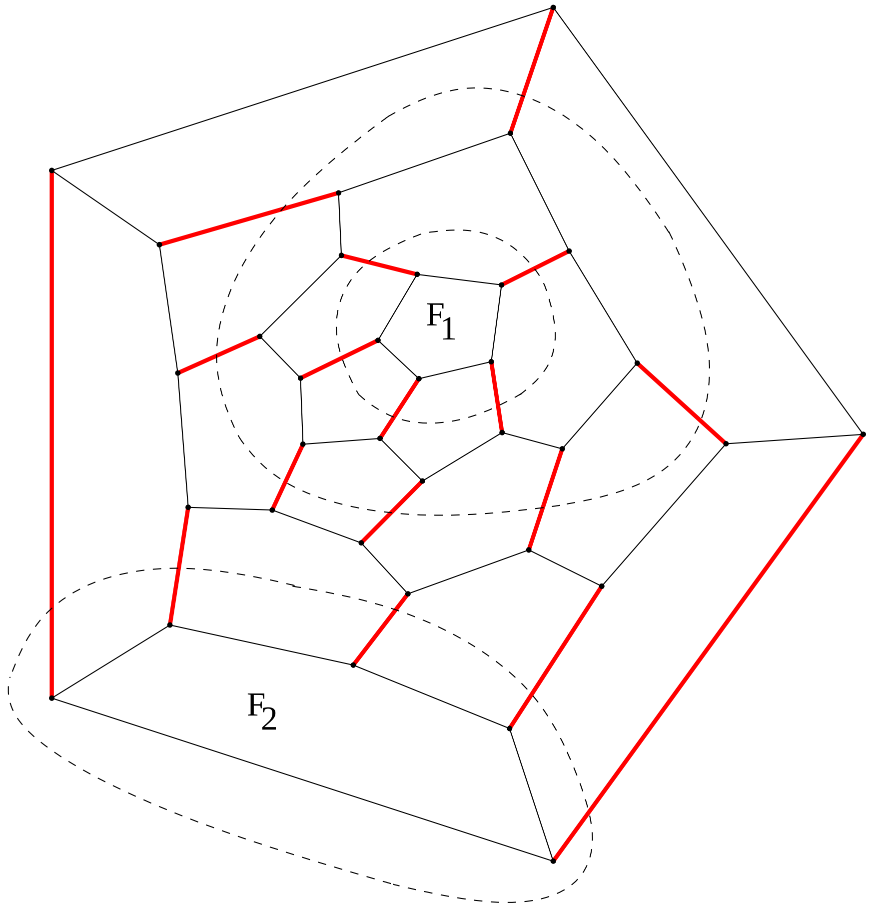
Perfect matching on 5_n graphs

Let G be a graph 5_n with one zigzag with self-intersection numbers (α_1, α_2).

(i) $\alpha_1 \geq \frac{n}{2}$. If $\alpha_1 = \frac{n}{2}$ then the edges of self-intersection of type I form a perfect matching PM

(ii) every face incident to 0 or 2 edges of PM

(iii) two faces, F_1 and F_2 are free of PM, PM is organized around them in concentric circles.

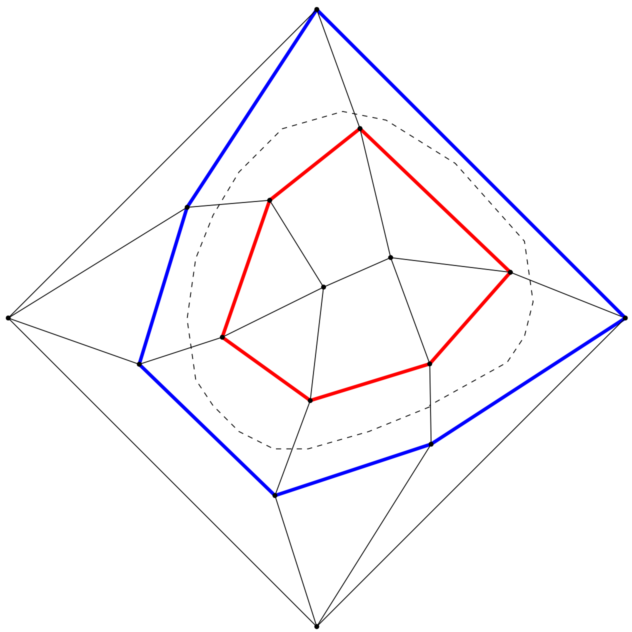


M. Deza, M. Dutour and P.W. Fowler, *Zigzags, Railroads and Knots in Fullerenes*, Journal of Chemical Information and Computer Sciences, in press.

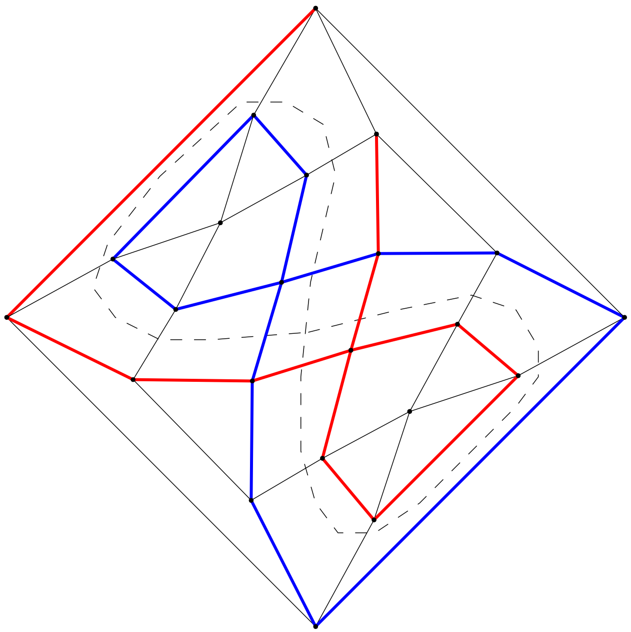
III. railroad structure and tightness

Railroads, 4-valent case

A railroad in an octahedrite is a circuit of square faces, such that any of them is adjacent to its neighbors on opposite faces. Any railroad is bordered by two central circuits



$oc_{16}(D_2)$

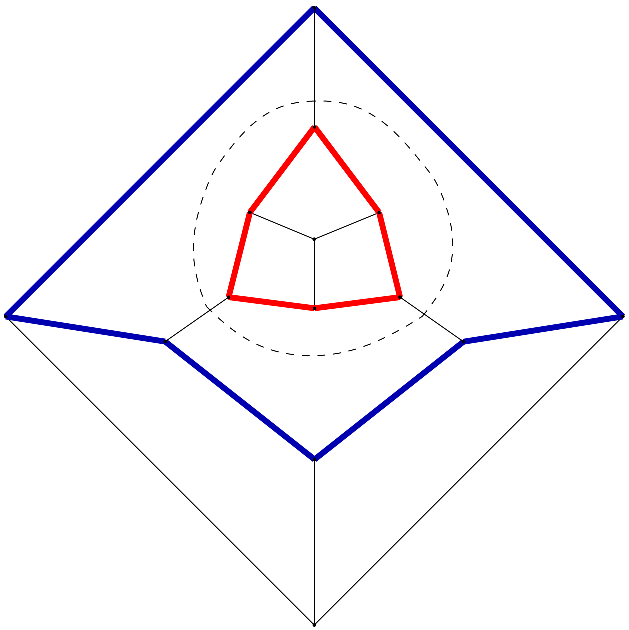


$oc_{22}(C_{2v})$

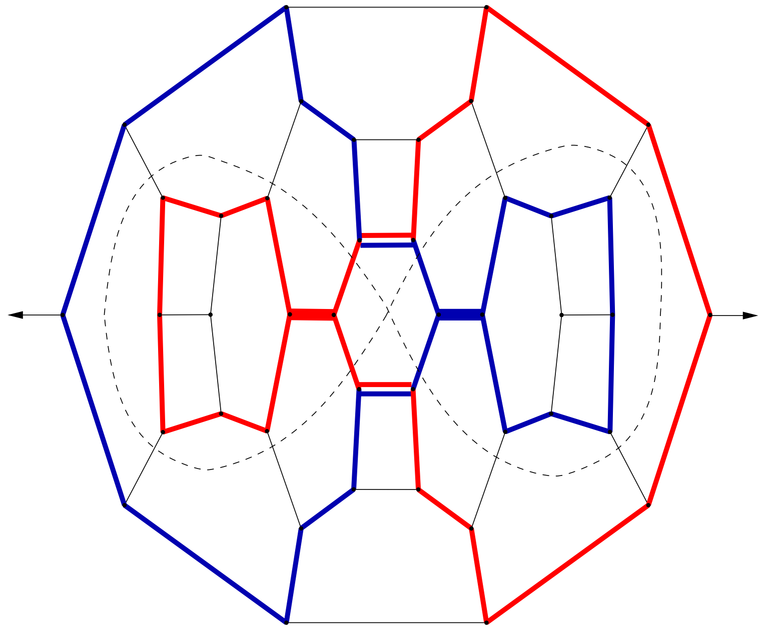
Railroads, as well as central circuits, can be self-intersecting. A graph is called tight if it has no railroad.

Railroads, 3-valent case

A railroad in graph q_n, $q = 3, 4, 5$ is a circuit of hexagonal faces, such that any of them is adjacent to its neighbors on opposite faces. Any railroad is bordered by two zigzags.



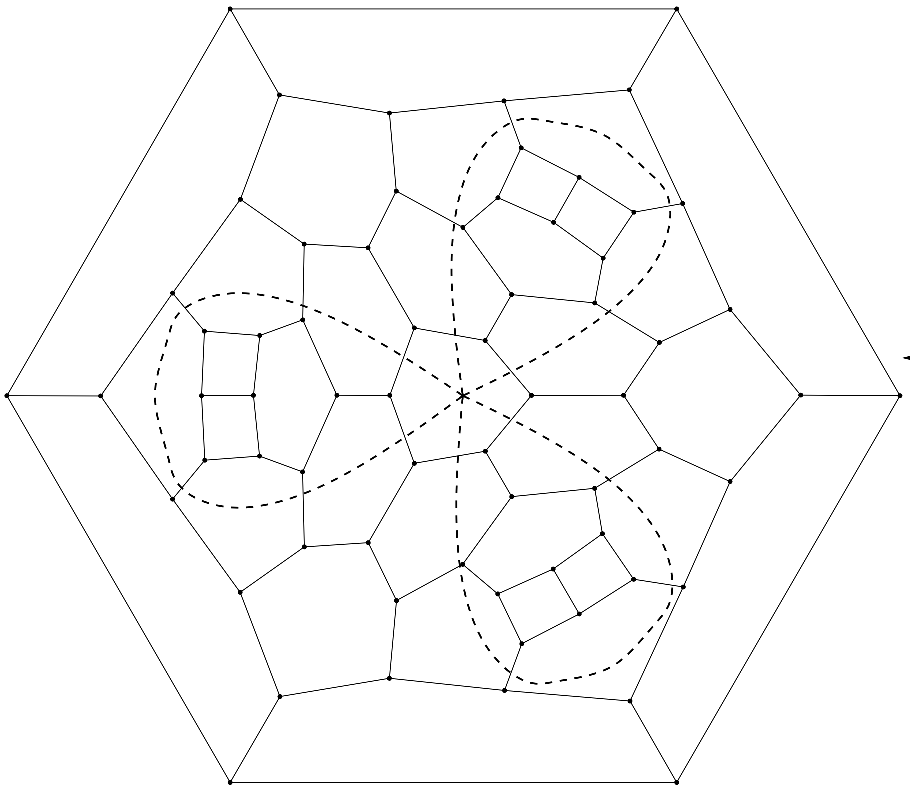
$4_{14}(D_{3h})$



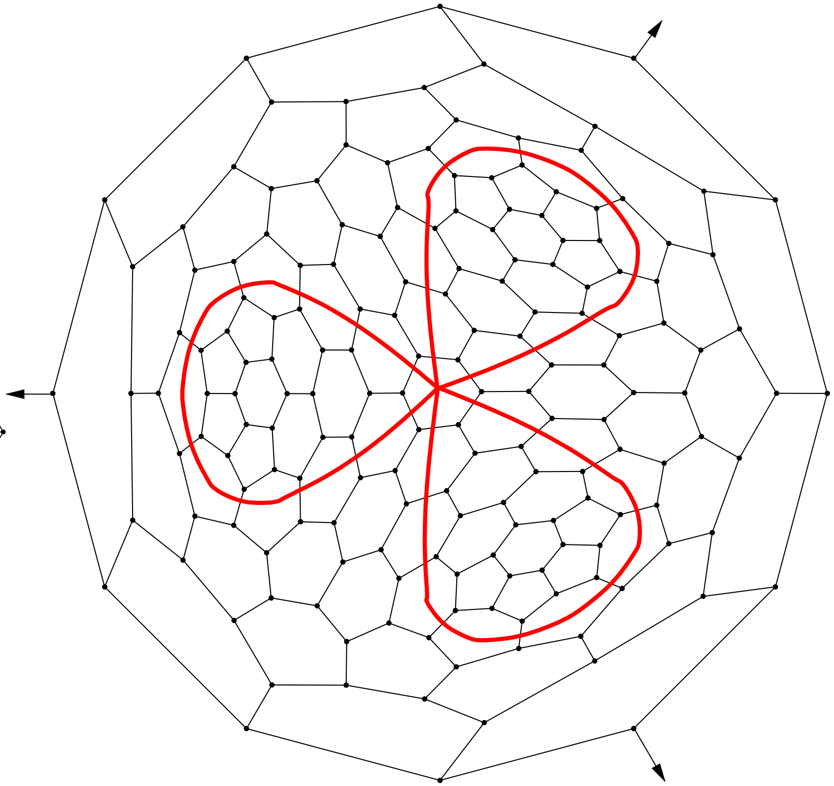
$4_{42}(C_{2v})$

Railroads, as well as zigzags, can be self-intersecting (doubly or triply). A graph is called tight if it has no railroad.

Triple self-intersection



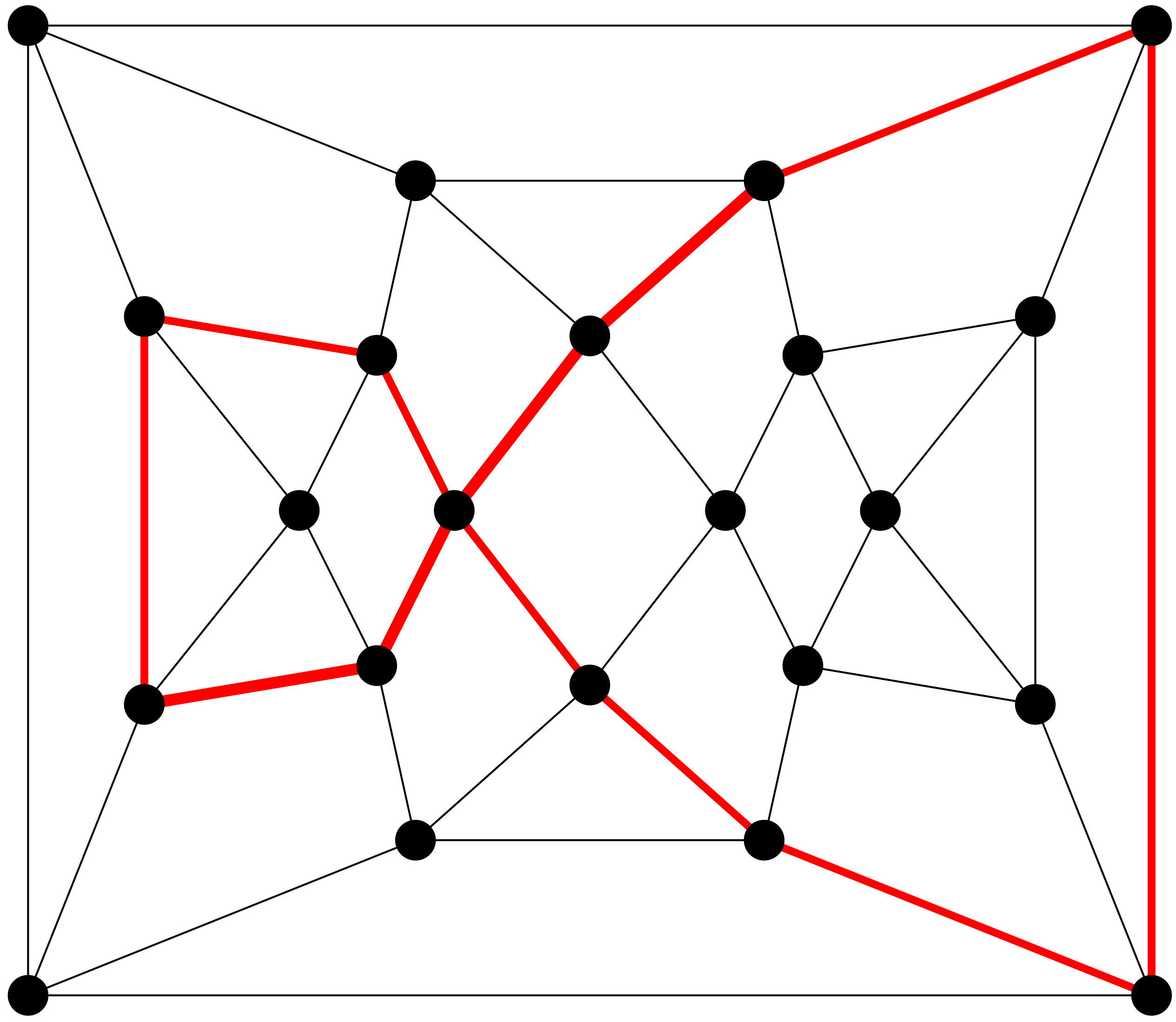
$4_{66}(D_{3h})$
It is smallest such 4_n.



$5_{176}(C_{3v})$
Conjecture: It is smallest such 5_n

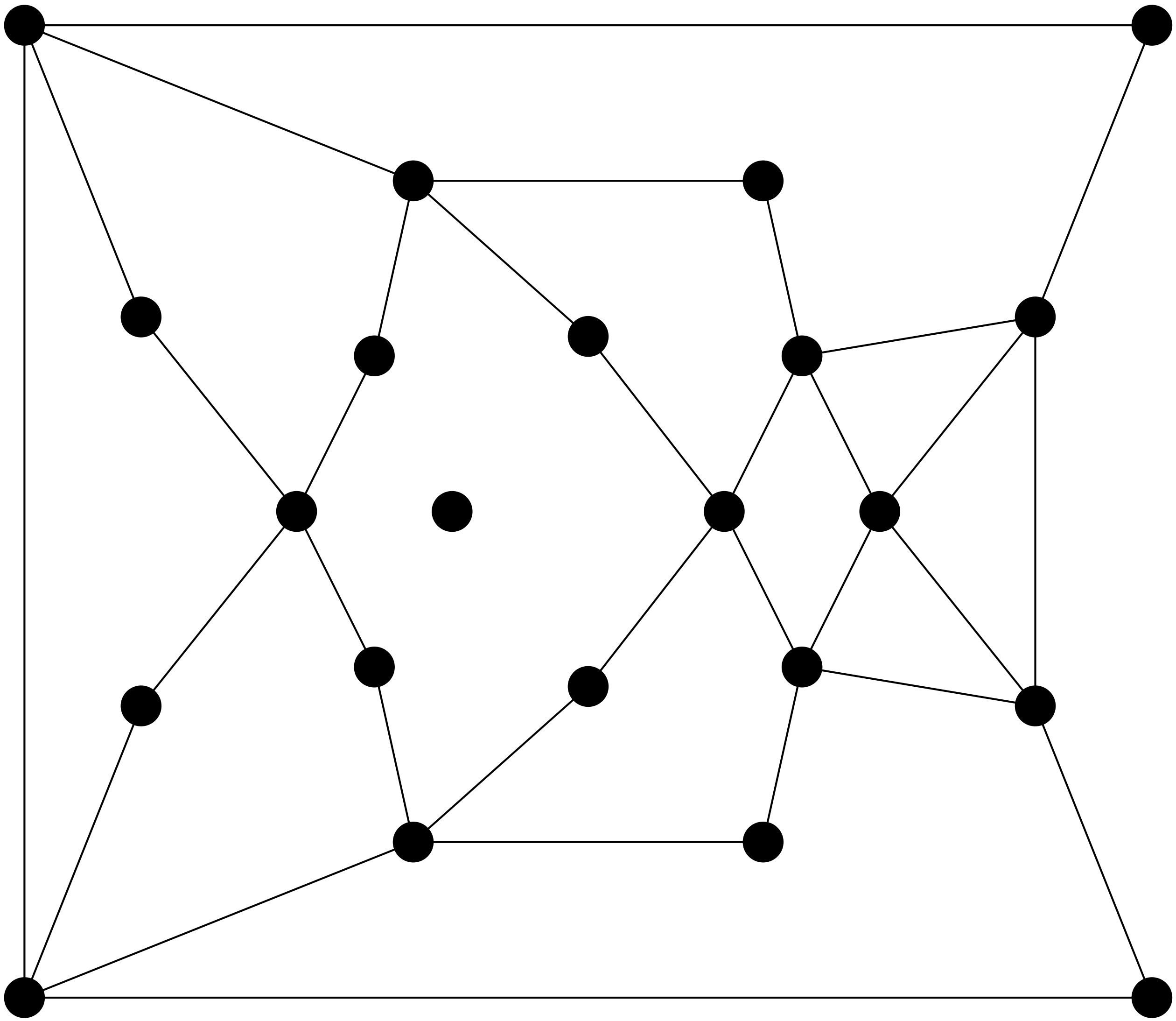
Removing central circuits

Take a 4-valent plane graph G and a central circuit



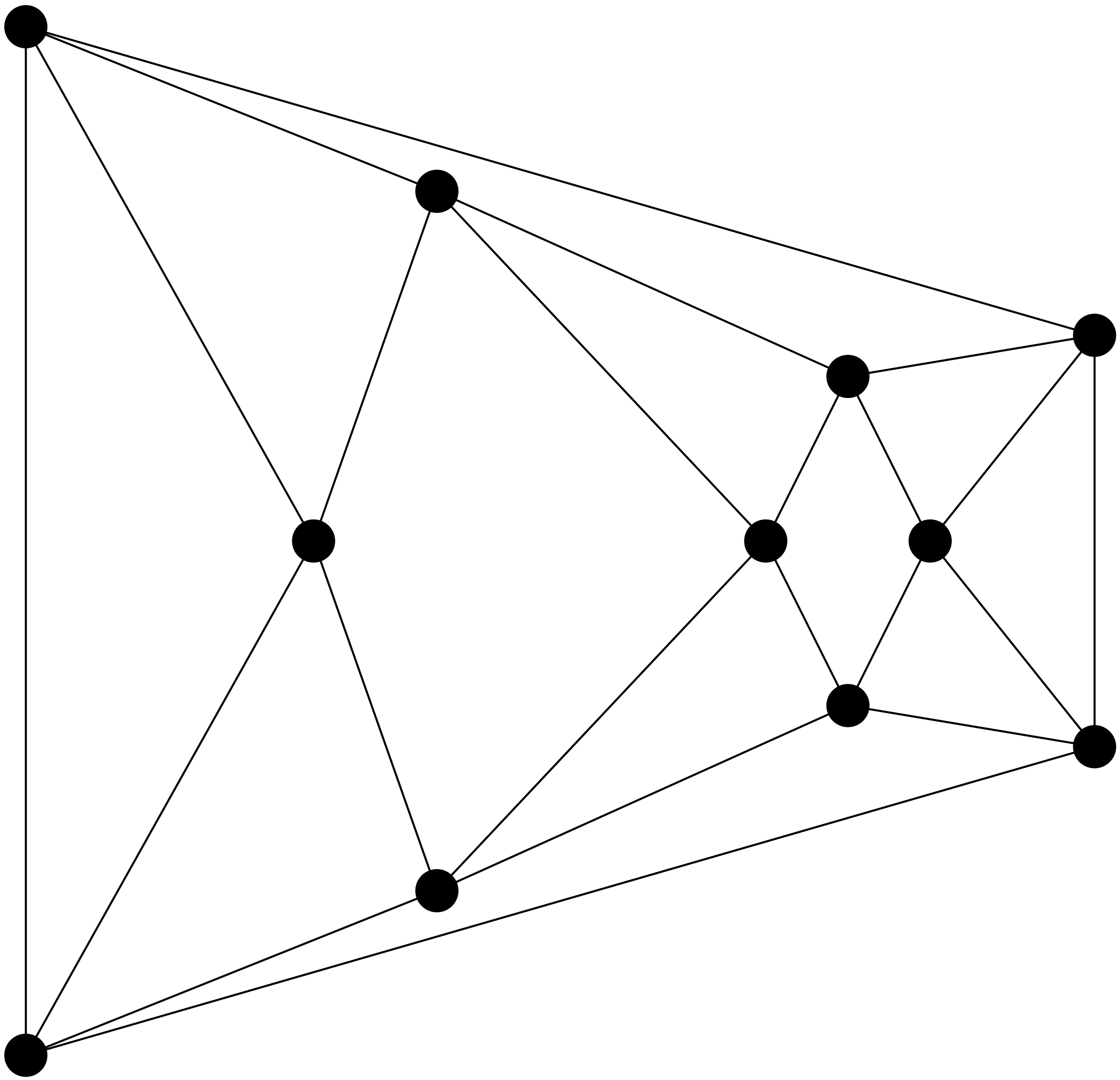
Removing central circuits

Remove the edges of the central circuit



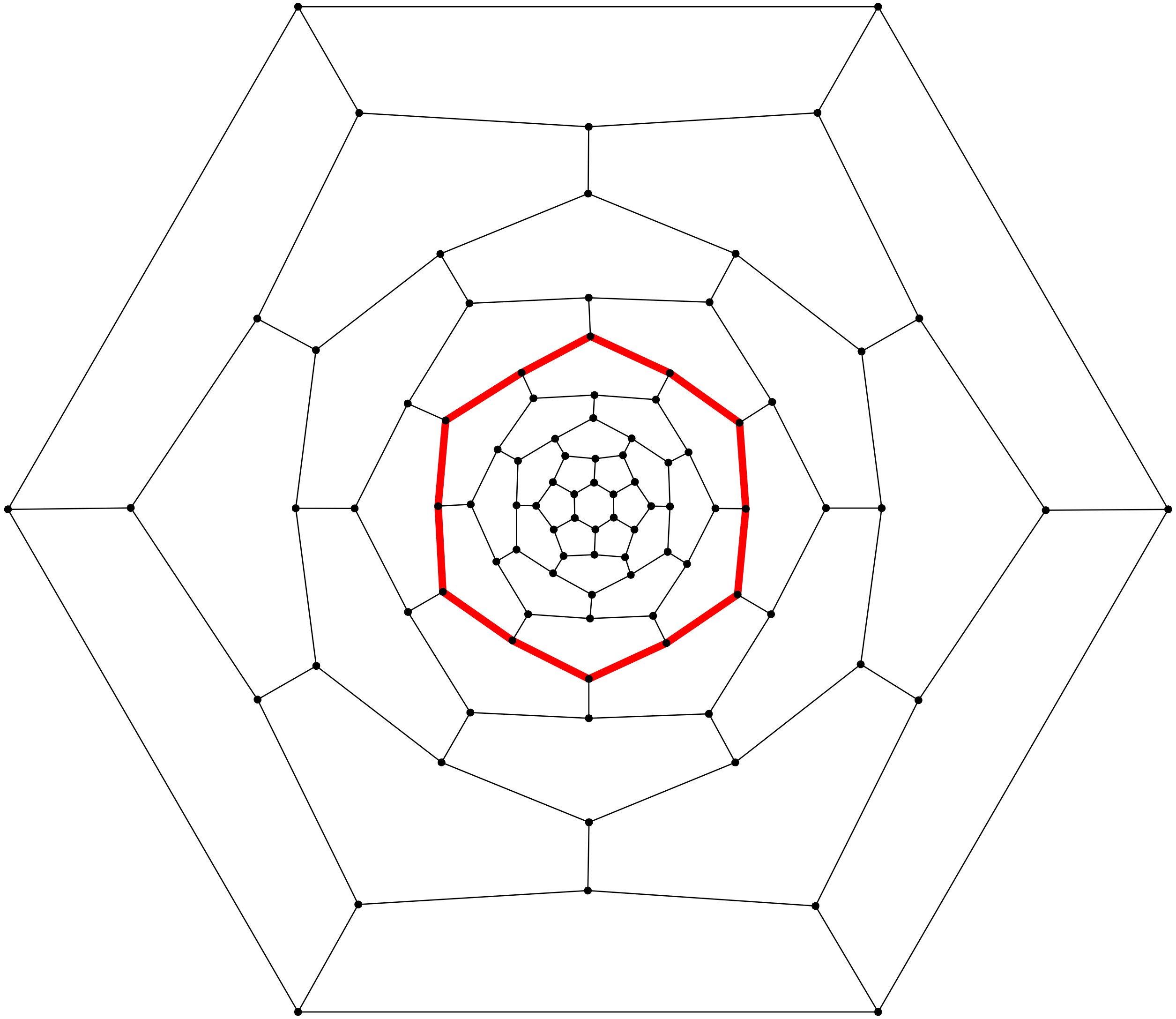
Removing central circuits

Remove the vertices of degree 0 or 2



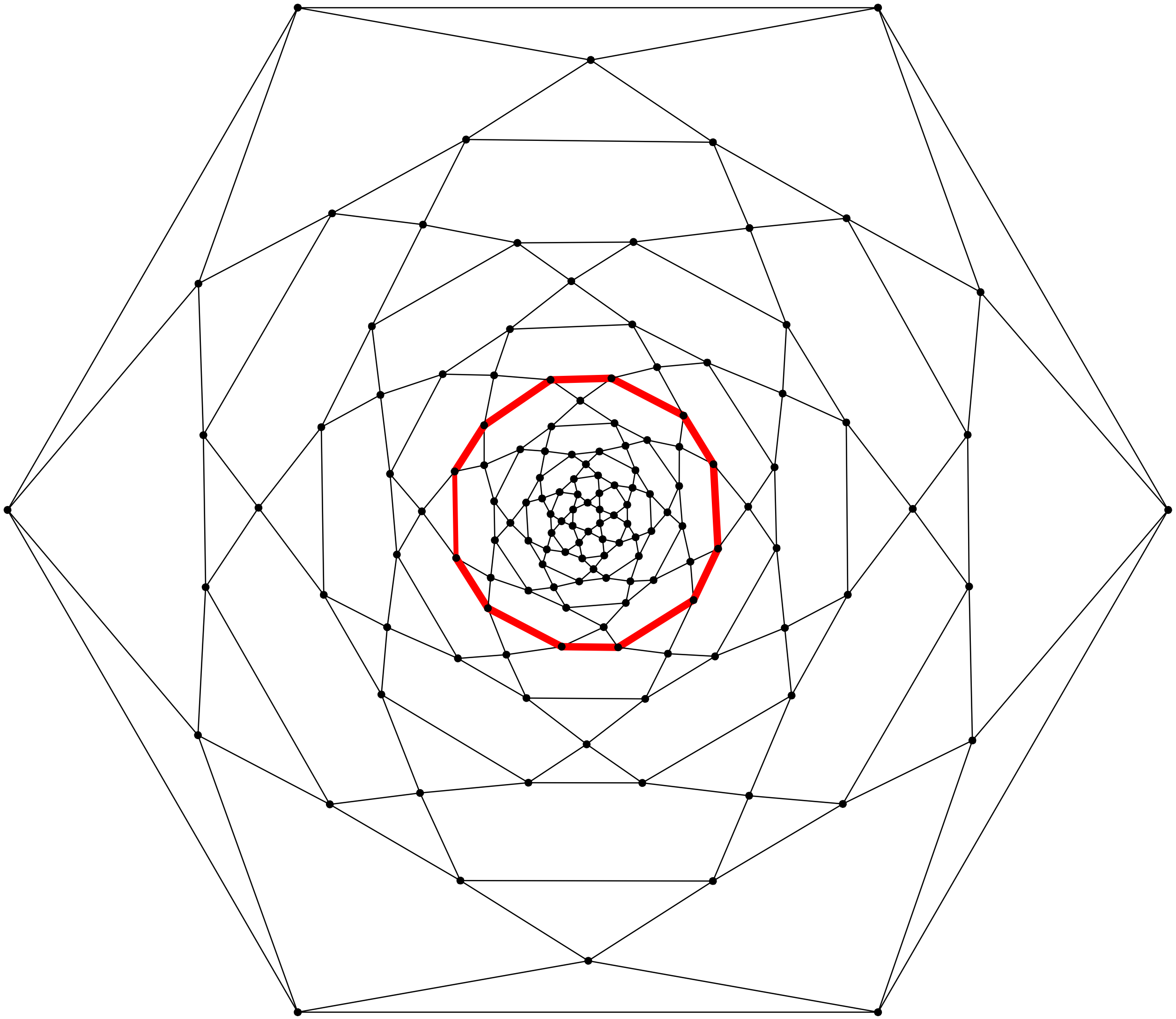
Removing zigzags

Take a plane graph G and a zigzag



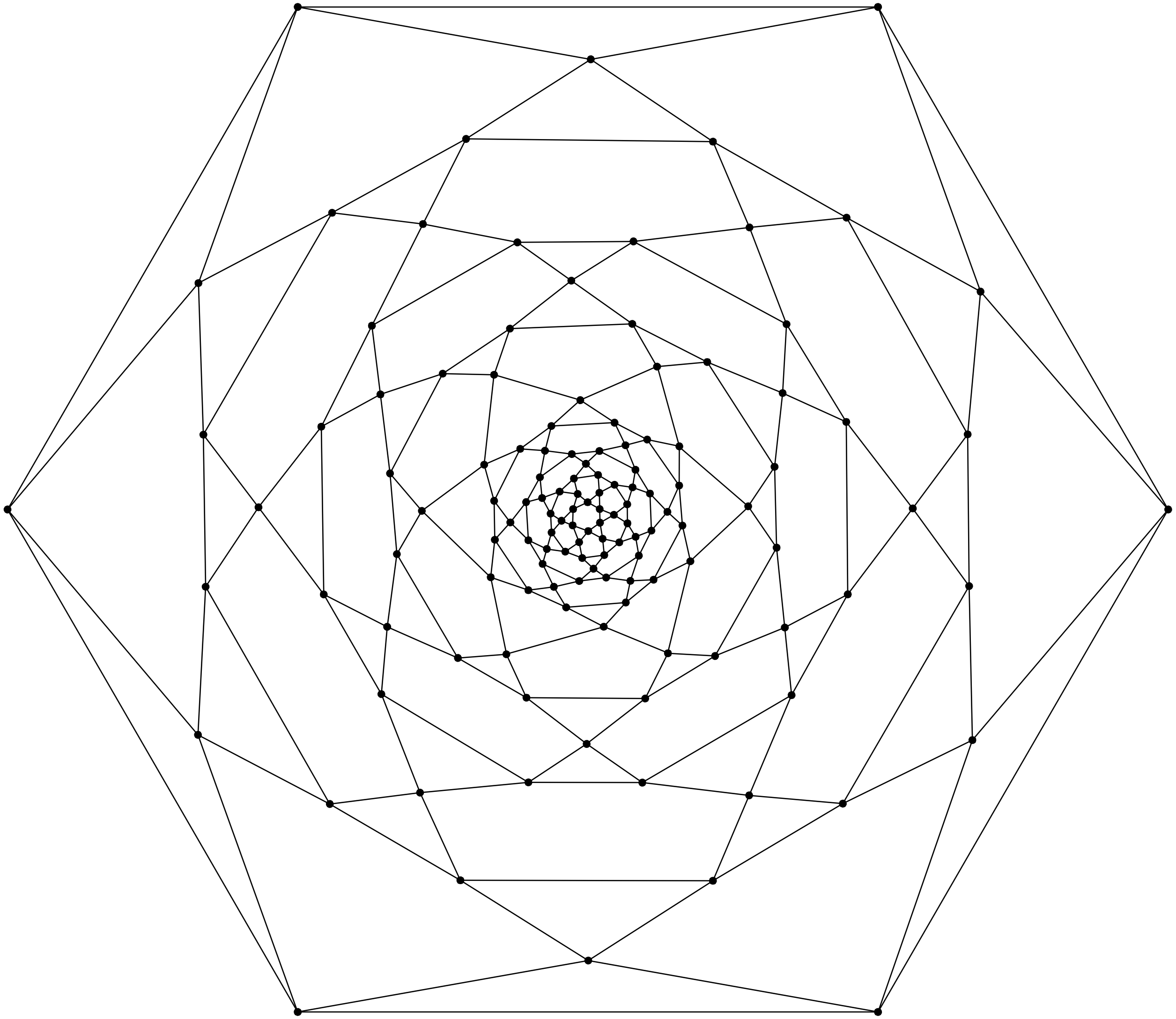
Removing zigzags

Go to the medial



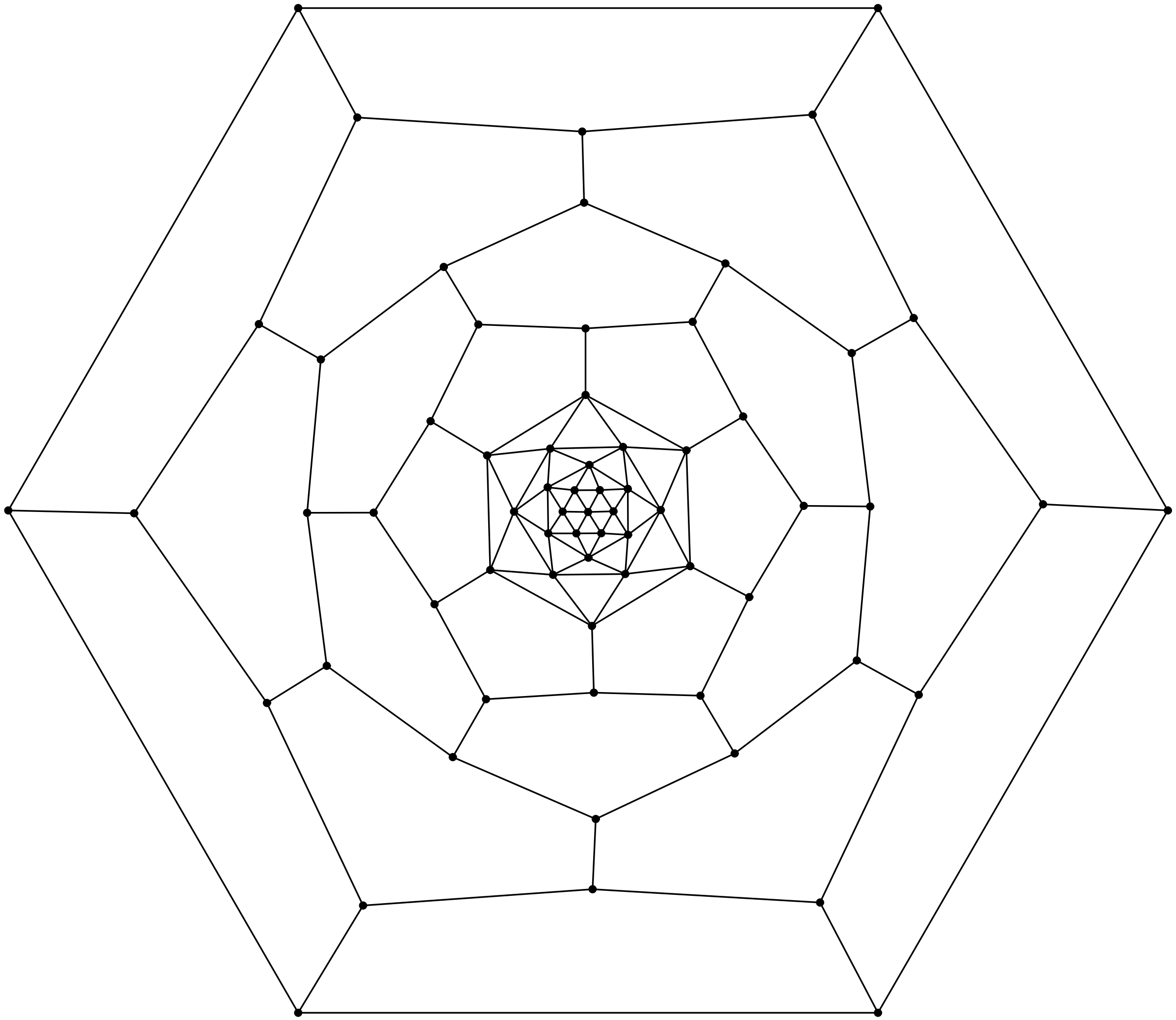
Removing zigzags

Remove the central circuit



Removing zigzags

Take one (out of two) inverse medial graph



Extremal problem

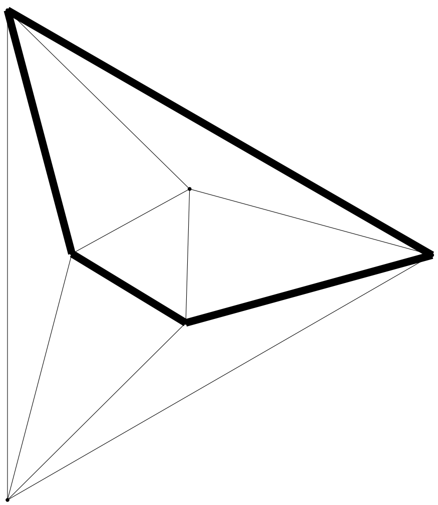
Given a class of tight graphs (octahedrites, graphs q_n), there exist a constant C such that any element of the class has at most C ZC-circuits.

- Every tight octahedrite has at most 6 central circuits.
  Proof method: Local analysis + case by case analysis.
- Every tight 3_n has exactly 3 zigzags.
  Proof method: uses an algebraic formalism on the graphs 3_n.
- Every tight 4_n has at most 9 zigzags.
  Conjecture: The correct upper bound is 8. checked for $n \leq 400$
- Every tight 5_n has at most 15 zigzags.
  Attempted proof: uses a local analysis on zigzags.

Tight with simple central circuits

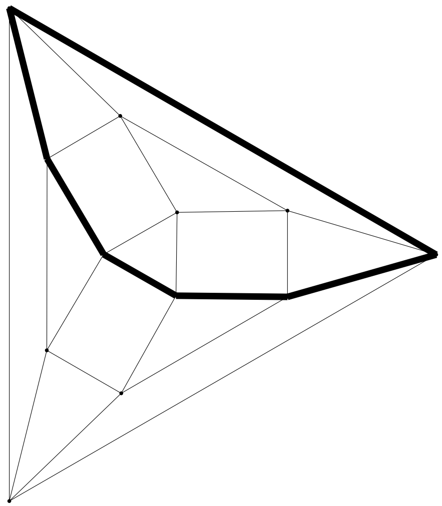
Theorem 1 *There is exactly* 8 *tight octahedrites with simple central circuits.*

Proof method: after removing a central circuit, the obtained graph has faces of gonality at most 4.



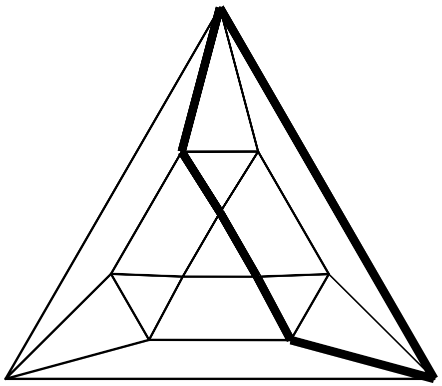
6 O_h

4^3



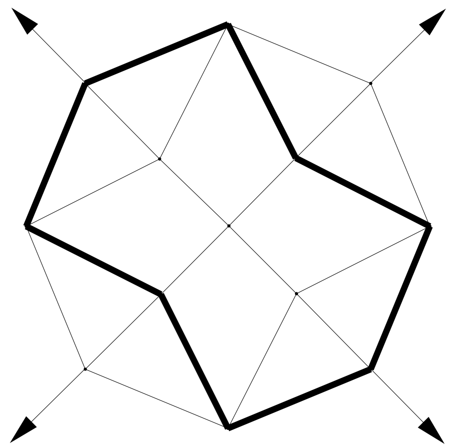
12 O_h

6^4



12 D_{3h}

6^4



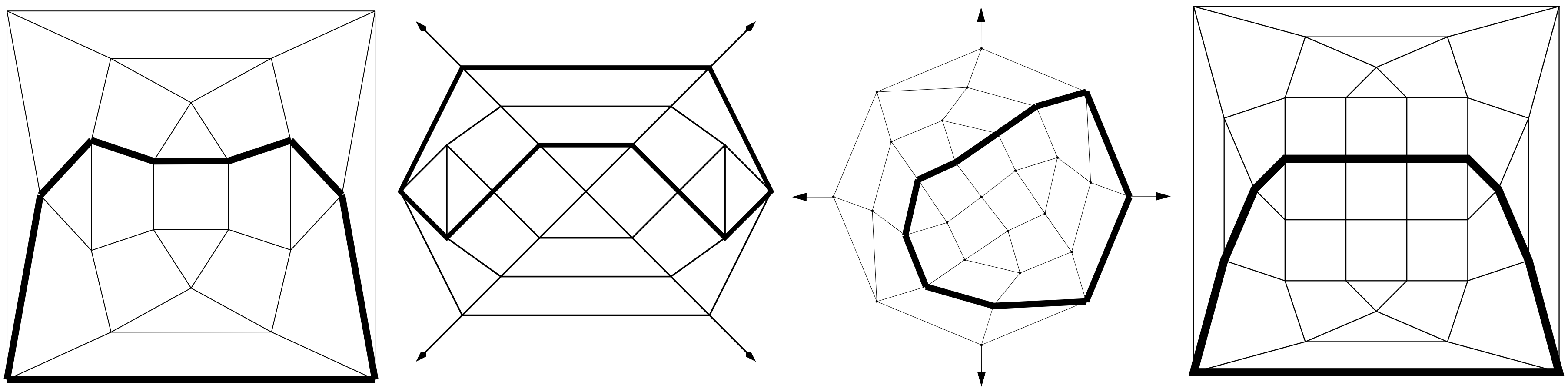
14 D_{4h}

$6^2, 8^2$

Tight with simple central circuits

Theorem 1 *There is exactly* 8 *tight octahedrites with simple central circuits.*

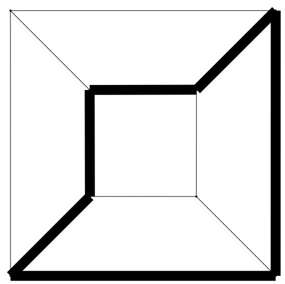
Proof method: after removing a central circuit, the obtained graph has faces of gonality at most 4.



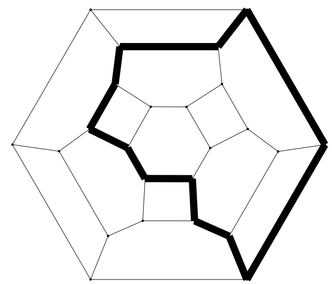
20 D_{2d}	**22** D_{2h}	**30** O	**32** D_{4h}
8^5	$8^3, 10^2$	10^6	$10^4, 12^2$

Tight with simple zigzags

- All tight 3_n have simple zigzags
  ➠Infinity of such graphs
- There are exactly 2 tight graph 4_n with simple zigzags: Cube and Truncated Octahedron=$GC_{1,1}(Cube)$.
  Proof method: the size of intersection of two simple zigzags is at most 6. There is at most 9 zigzags.
  ➠Upper bound on n.



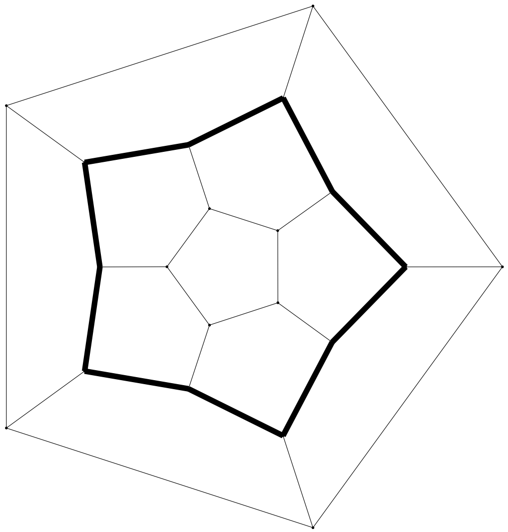
6 O_h, 6^4



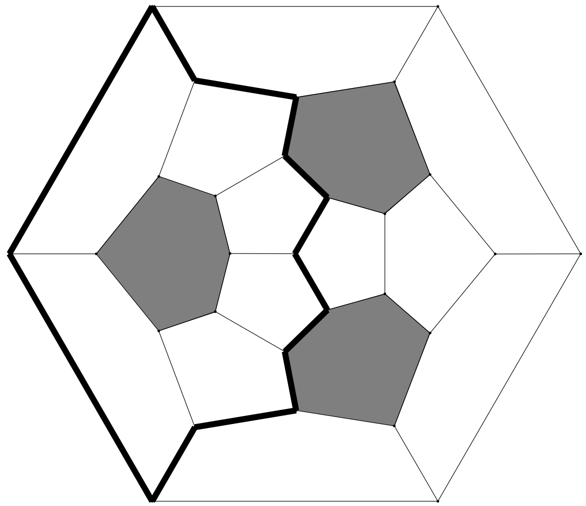
24 O_h, 10^6

- There is at least 9 tight graphs 5_n with simple zigzags. G. Brinkmann and T. Harmuth computation of fullerenes with simple zigzags up to 200 vertices.

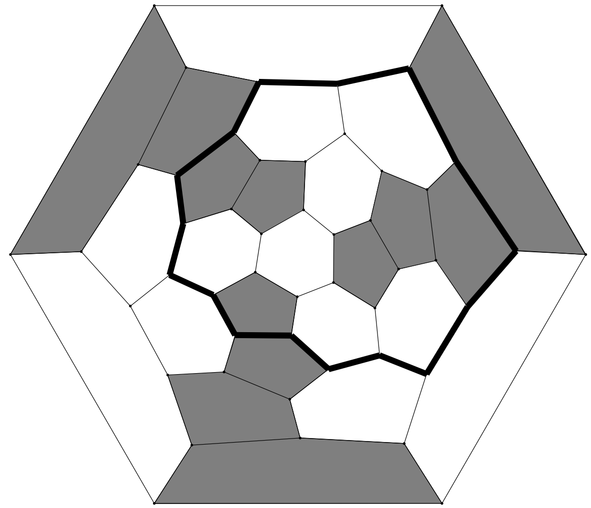
Tight 5_n with simple zigzags



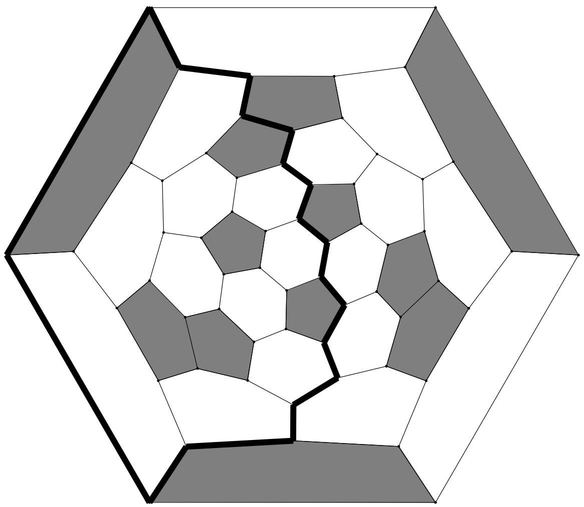
20 $I_h, 20^6$



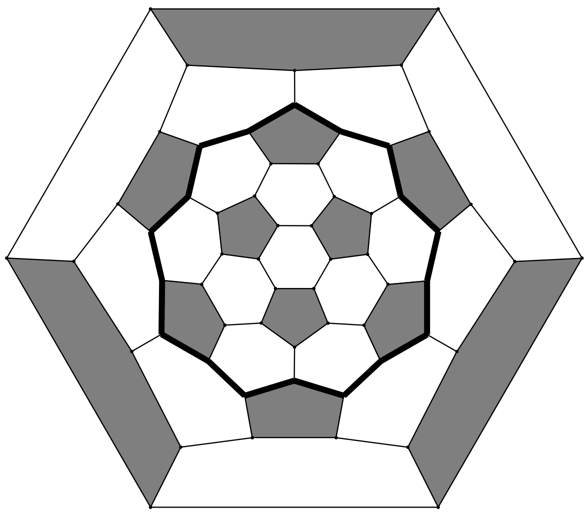
28 $T_d, 12^7$



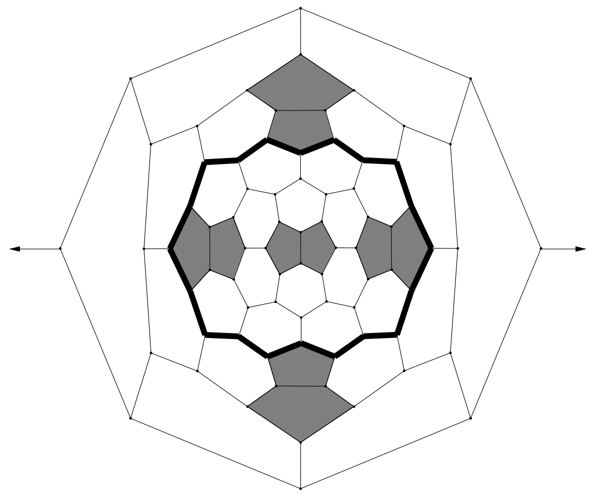
48 $D_3, 16^9$



60 $D_3, 18^{10}$

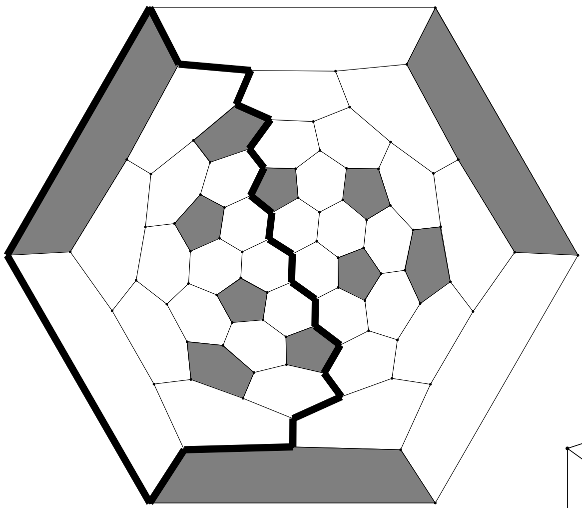


60 $I_h, 18^{10}$

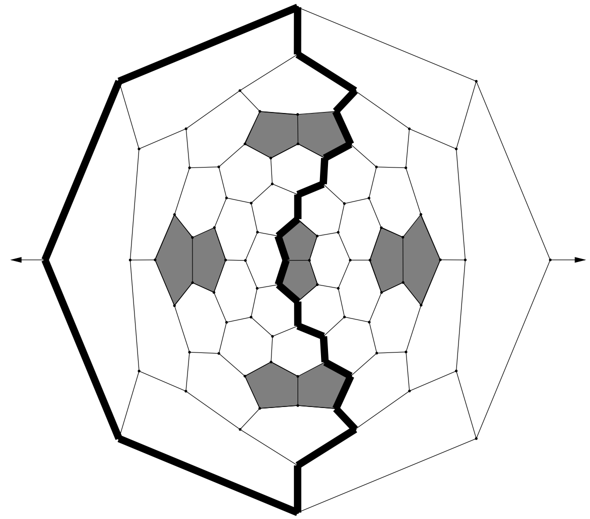


76 $D_{2d}, 22^4, 20^7$

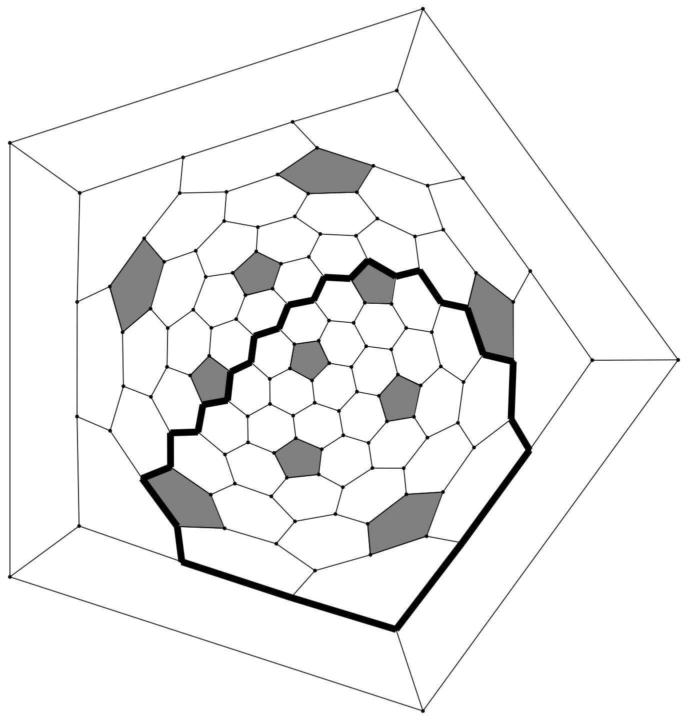
Tight 5_n with simple zigzags



88 $T, 22^{12}$



92 $T_h, 24^6, 22^6$

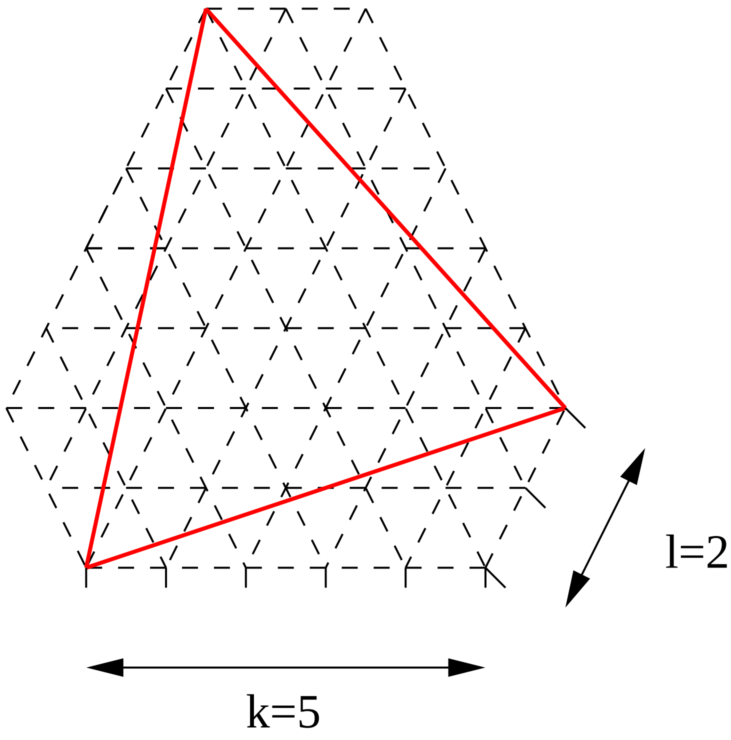


140 $I, 28^{15}$

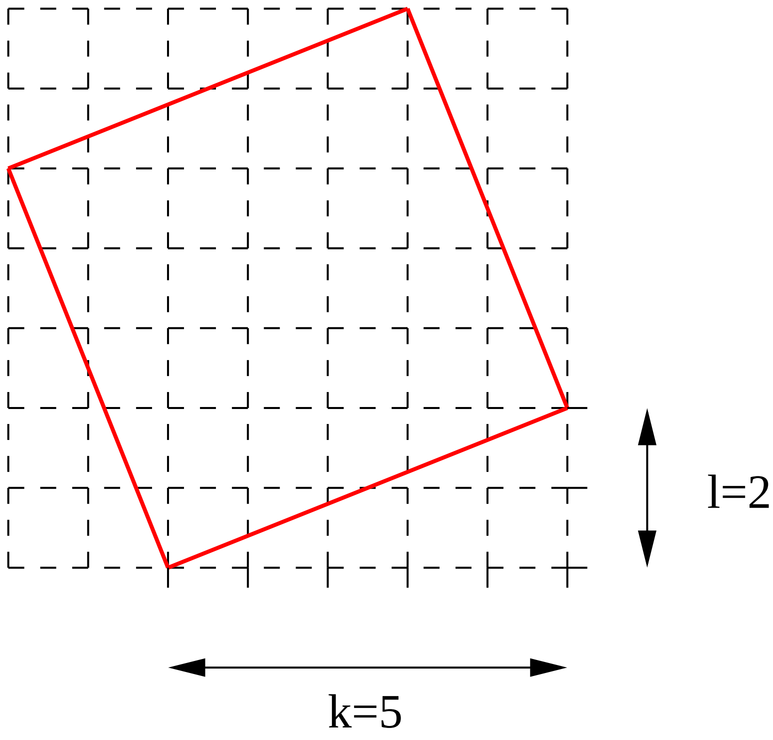
IV. Goldberg-Coxeter construction

The construction

- Take a 3- or 4-valent plane graph G_0. The graph G_0^* is formed of triangles or squares.
- Break the triangles or squares into pieces according to parameter (k, l).



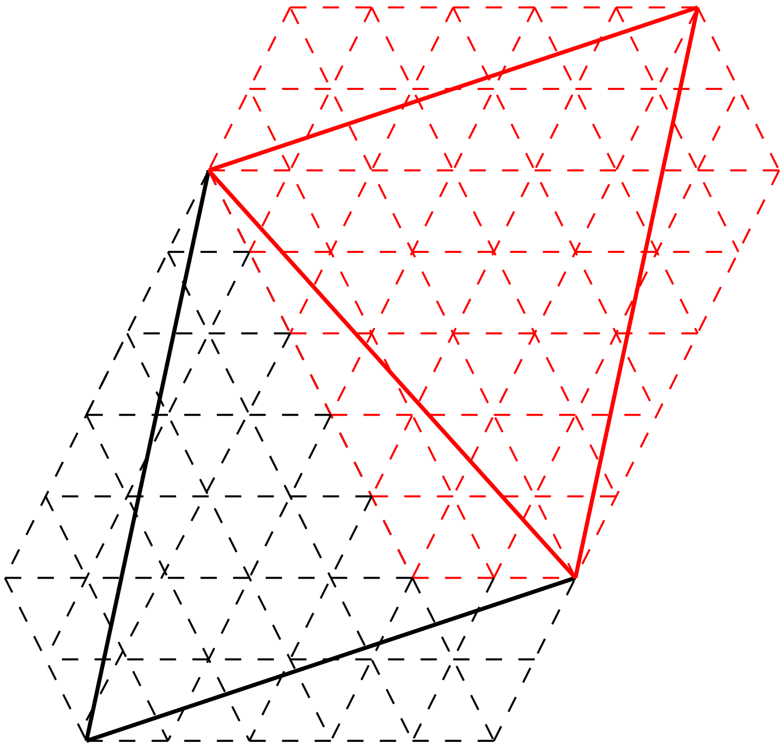
3−valent case



4−valent case

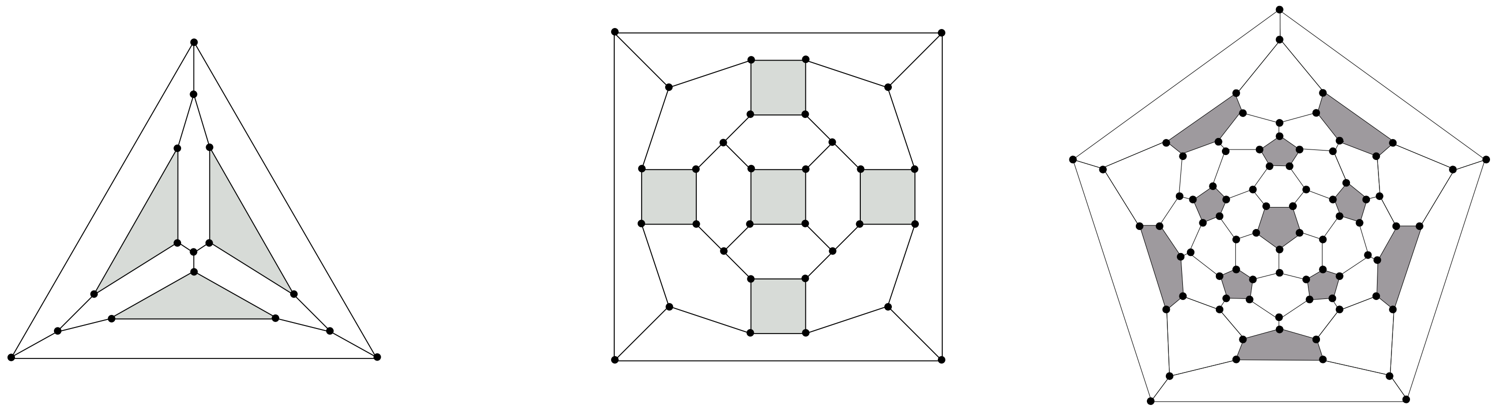
Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



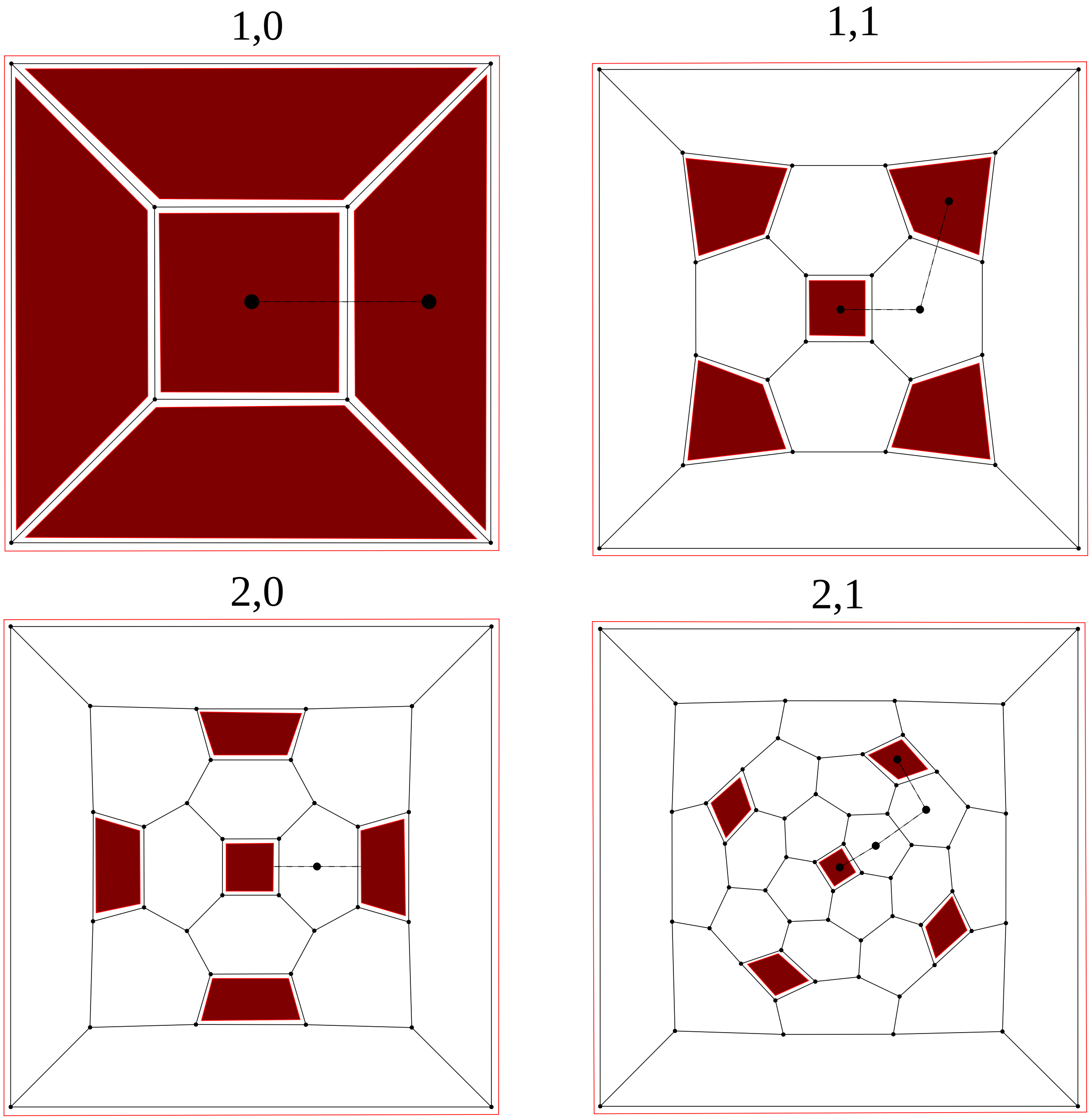
Final steps

- Go to the dual and obtain a 3- or 4-valent plane graph, which is denoted $GC_{k,l}(G_0)$ and called "Goldberg-Coxeter construction".
- The construction works for any 3- or 4-valent map on oriented surface.

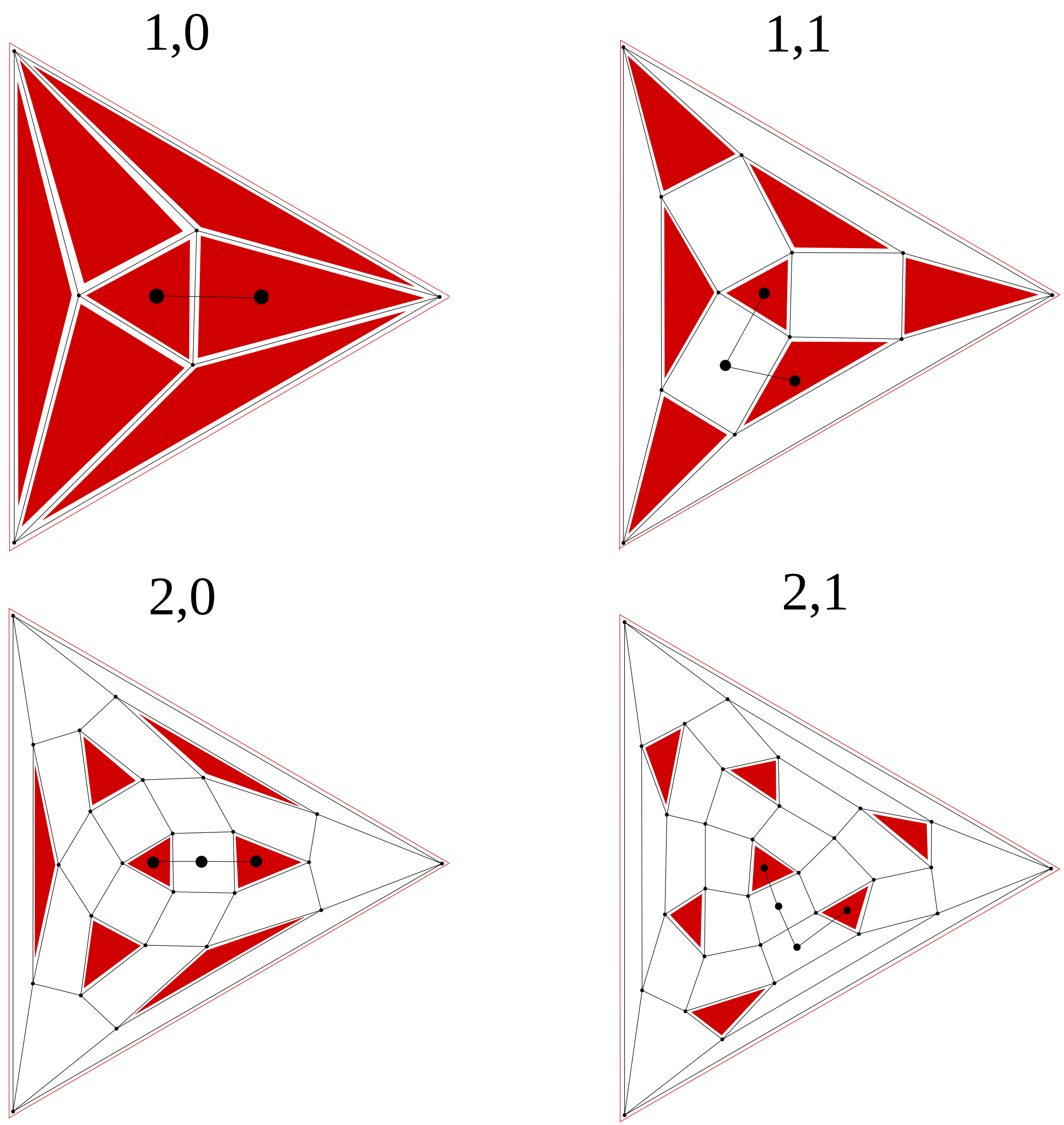


Operation $GC_{2,0}$ on Tetrahedron, Cube and Dodecahedron

Goldberg-Coxeter for Cube



Goldberg-Coxeter for Octahedron



Properties

- One associates $z = k + le^{i\frac{\pi}{3}}$(Eisenstein integer) or $z = k + li$(Gaussian integer) to the pair (k, l) in 3- or 4-valent case.
- If one writes $GC_z(G_0)$ instead of $GC_{k,l}(G_0)$, then one has:

$$GC_z(GC_{z'}(G_0)) \quad = \quad GC_{zz'}(G_0)$$

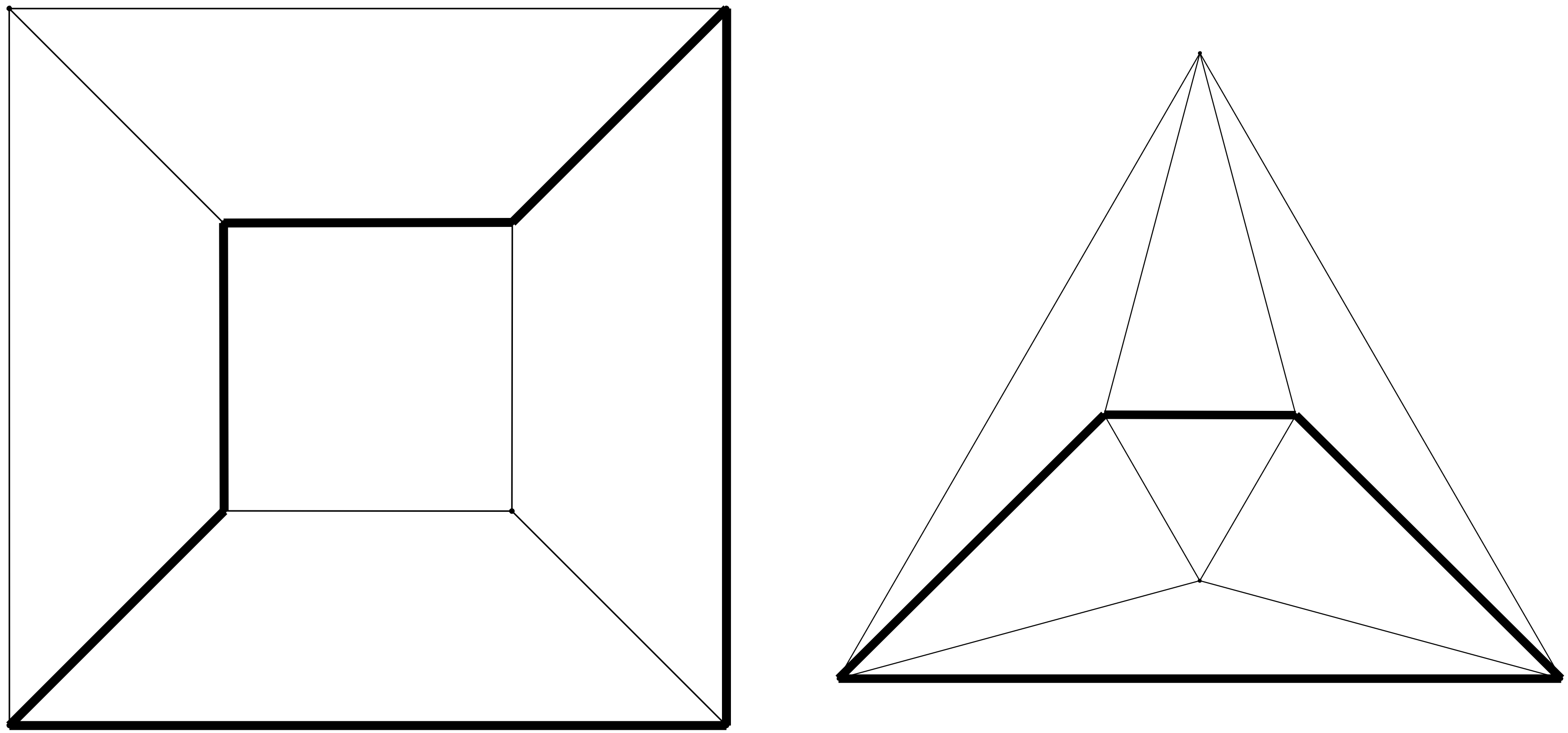
- If G_0 has n vertices, then $GC_{k,l}(G_0)$ has

$$\begin{aligned} n(k^2+kl+l^2) = n|z|^2 \text{ vertices} &\quad \text{if} \quad G_0 \text{ is } 3\text{-valent},\\ n(k^2+l^2) = n|z|^2 \text{ vertices} &\quad \text{if} \quad G_0 \text{ is } 4\text{-valent}. \end{aligned}$$

- If G_0 has a plane of symmetry, we reduce to $0 \le l \le k$.
- $GC_{k,l}(G_0)$ has all rotational symmetries of G_0 and all symmetries if $l = 0$ or $l = k$.

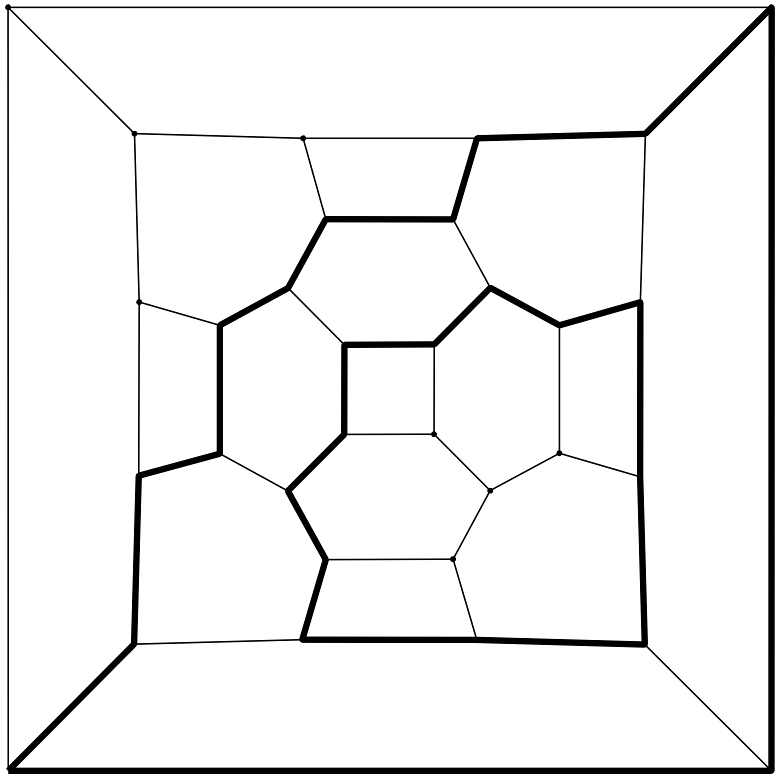
The special case $GC_{k,0}$

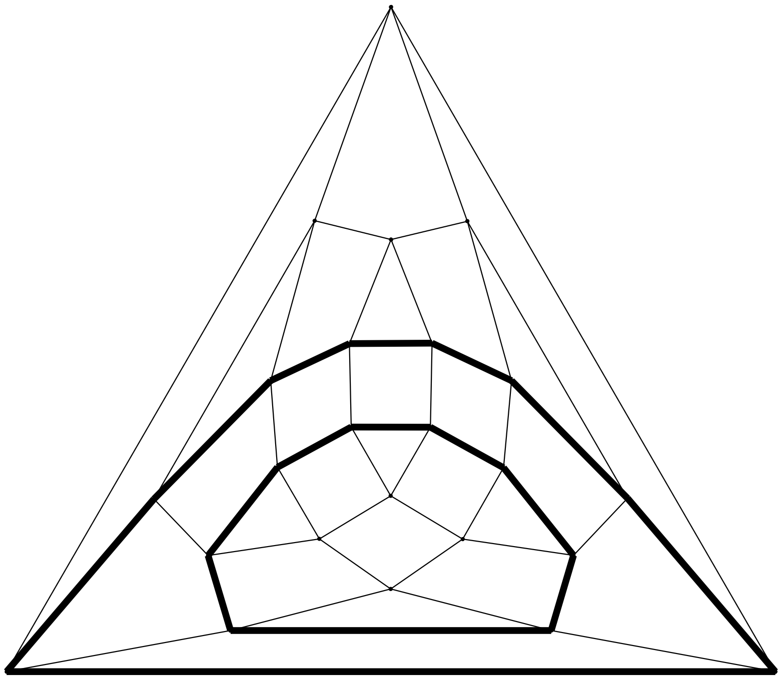
- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.



The special case $GC_{k,0}$

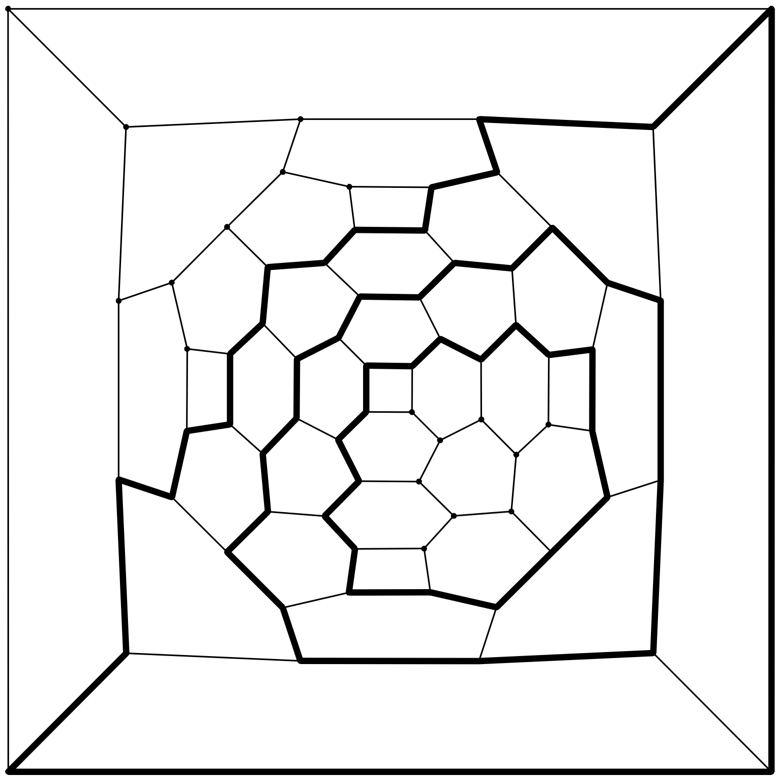
- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.

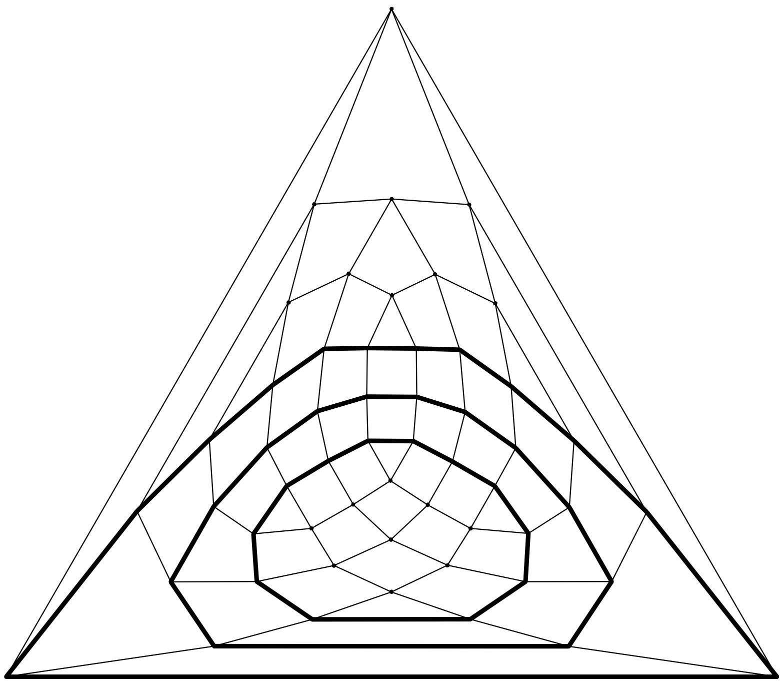




The special case $GC_{k,0}$

- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.





(k,l)-product formalism

Given a 3-valent plane graph G, the zigzags of the Goldberg-Coxeter construction of $GC_{k,l}(G)$ are obtained by:

- Associating to G two elements L and R of a group called moving group,
- Computing the value of the the (k,l)-product $L \odot_{k,l} R$,
- The lengths of zigzags are obtained by computing the cycle structure of $L \odot_{k,l} R$.

For $GC_{k,l}(Dodecahedron)$ with $gcd(k,l) = 1$, this gives 6, 10 or 15 zigzags.

M. Dutour and M. Deza, *Goldberg-Coxeter construction for 3- or 4-valent plane graphs*, Electronic Journal of Combinatorics, **11-1** (2004) R20.

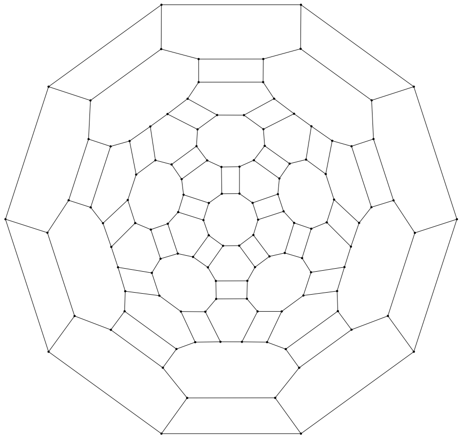
Illustration

- For any ZC-circuit of $GC_{k,l}(G_0)$, there exist $\alpha \geq 1$

  length(ZC)=$2(k^2 + kl + l^2)\alpha$ 3-valent case

  length(ZC)=$(k^2 + l^2)\alpha$ 4-valent case

  The [ZC]-vector of $GC_{k,l}(G_0)$ is the vector $\dots, \alpha_k^{m_k}, \dots$ where m_k is the number of ZC-circuits with order α_k.
- If $gcd(k, l) = 1$, then $GC_{k,l}(Cube)$ has 6 zigzags if $k \equiv l \pmod 3$ and 4 otherwise.
- For Truncated Icosidodecahedron, possible [ZC]:



$2^{30}, 3^{40}$	$2^{30}, 5^{24}$	$3^{20}, 5^{24}$
$2^{60}, 3^{20}$	$2^{60}, 5^{12}$	$3^{40}, 5^{12}$
2^{90}	3^{60}	5^{36}
9^{20}	6^{30}	15^{12}

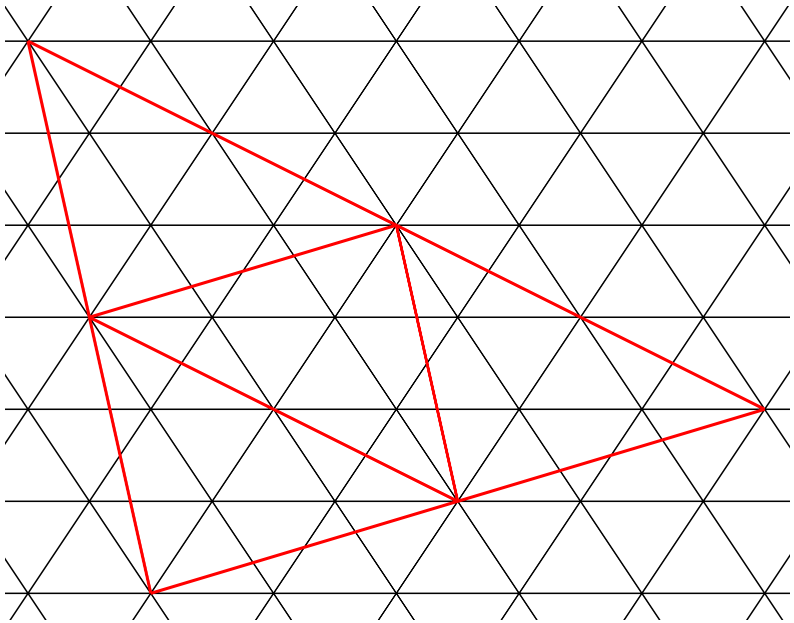
V. Parametrizing graphs

Parametrizing graphs q_n

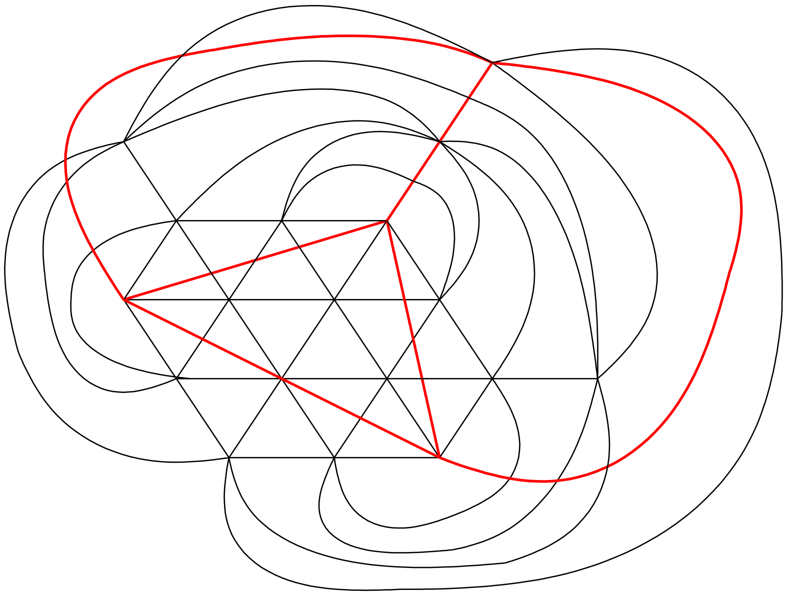
Idea: the hexagons are of zero curvature, it suffices to give relative positions of faces of non-zero curvature.

- Goldberg (1937) All 3_n, 4_n or 5_n of symmetry (T, T_d), (O, O_h) or (I, I_h) are given by Goldberg-Coxeter construction $GC_{k,l}$.
- Fowler and al. (1988) All 5_n of symmetry D_5, D_6 or T are described in terms of 4 parameters.
- Graver (1999) All 5_n can be encoded by 20 integer parameters.
- Thurston (1998) The 5_n are parametrized by 10 complex parameters.
- Sah (1994) Thurston's result implies that the Nrs of 3_n, 4_n, $5_n \sim n, n^3, n^9$.

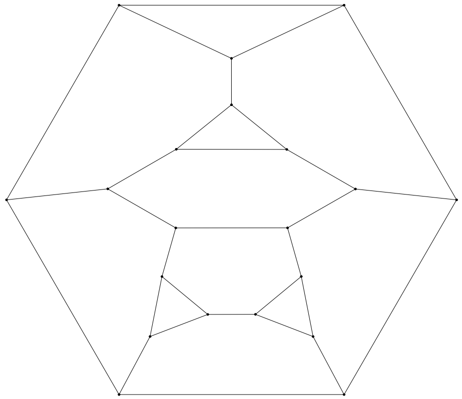
The structure of graphs 3_n



4 triangles in $Z[\omega]$



The corresponding triangulation



The graph $3_{20}(D_{2d})$

z- and railroad-structure of graphs 3_n

All zigzags and railroads are simple.

- The z-vector is of the form

$$(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3} \quad \text{with} \quad s_i m_i = \frac{n}{4};$$

  the number of railroads is $m_1 + m_2 + m_3 - 3$.
- G has ≥ 3 zigzags with equality if and only if it is tight.
- If G is tight, then $z(G) = n^3$ (so, each zigzag is a Hamiltonian circuit).
- All 3_n are tight if and only if $\frac{n}{4}$ is prime.
- There exists a tight 3_n if and only if $\frac{n}{4}$ is odd.

General theory

Extensions:

- 3-valent or 4-valent graphs.
- Classes of graphs with fixed p_i, $i \neq 6$.
- Classes with a fixed symetry.
- Maps on surfaces.

Dictionnary

	3-valent graph G_0	4-valent graph G_0
ring	Eisenstein integers $\mathbb{Z}[\omega]$	Gaussian integers $\mathbb{Z}[i]$
Euler formula	$\sum_i (6-i)p_i = 12$	$\sum_i (4-i)p_i = 8$
zero-curvature	hexagons	squares
ZC-circuits	zigzags	central circuits
Operation	leapfrog graph	medial graph

Number of parameters

Octahedrites:

Group	$\#param.$
C_1	6
C_2	4
D_2	3
D_3	2
D_4	2
O	1

Graphs 3_n:

Groups	$\#param.$
D_2	2
T	1

Graphs 4_n:

Group	$\#param.$
C_1	4
C_2	3
D_2	2
D_3	2
O	1

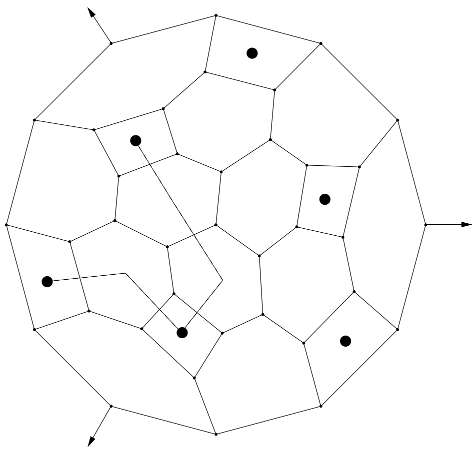
Graph 5_n:

Group	$\#param.$
C_1	10
C_2	6
C_3	4
D_2	4
D_3	3
D_5	2
D_6	2
T	2
I	1

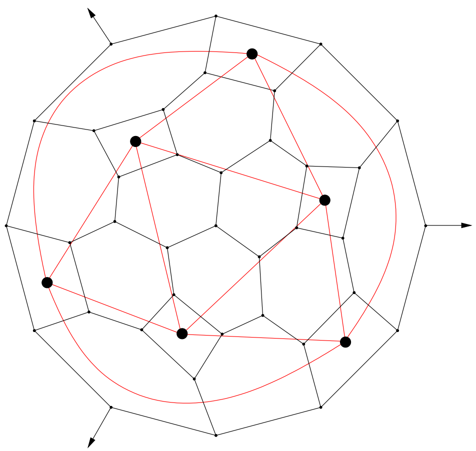
If there is just one parameter, then this is Goldberg-Coxeter construction (of Octahedron, Tetrahedron, Cube, Dodecahedron for octahedrite, 3_n, 4_n, 5_n, respectively).

Conjecture on $4_n(D_{3h}, D_{3d}\ or\ D_3)$

- $4_n(D_3 \subset D_{3h}, D_{3d}, D_6, D_{6h}, O, O_h)$ are described by two complex parameters. They exists if and only if $n \equiv 0, 2 \pmod 6$ and $n \geq 8$.



$4_n(D_3)$ with one zigzag



The defining triangles

- $4_n(D_{3d} \subset O_h, D_{6h})$ exists if and only if $n \equiv 0, 8 \pmod{12}$, $n \geq 8$.
- If n increases, then part of $4_n(D_3)$ amongst $4_n(D_{3h}, D_{3d}, D_3)$ goes to 100%

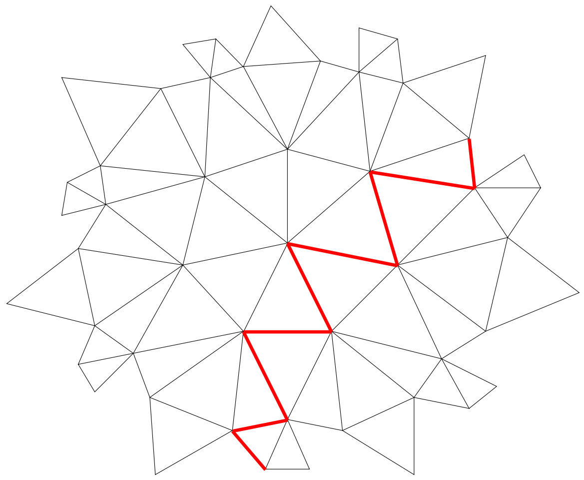
More conjectures

- All 4_n with only simple zigzags are:
  - $GC_{k,0}(Cube)$, $GC_{k,k}(Cube)$ and
  - the family of $4_n(D_3 \subset \dots)$ with parameters $(m,0)$ and $(i, m-2i)$ with $n = 4m(2m-3i)$ and $z = (6m-6i)^{3m-3i}, (6m)^{m-2i}, (12m-18i)^i$

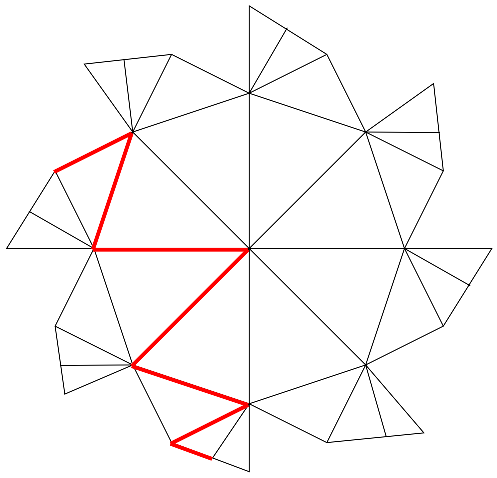
  They have symmetry D_{3d} or O_h or D_{6h}
- Any $4_n(D_3 \subset \dots)$ with one zigzag is a $4_n(D_3)$.
- For tight graphs $4_n(D_3 \subset \dots)$ the z-vector is of the form a^k with $k \in \{1,2,3,6\}$ or a^k, b^l with $k, l \in \{1,3\}$
- Tight $4_n(D_{3d})$ exist if and only if $n \equiv 0 \pmod{12}$, they are z-transitive with
  - $z = (n/2)^6_{n/36,0}$ iff $n \equiv 24 \pmod{36}$ and, otherwise,
  - $z = (3n/2)^2_{n/4,0}$ iff $n \equiv 0, 12 \pmod{36}$

VI. Zigzags on surfaces

Klein and Dyck map



Klein map: $z = 8^{21}$



Dyck map: $z = 6^{16}$

- Zigzags (and central circuits), being local notions, are defined on any surface, even on non-orientable ones.
- Goldberg-Coxeter and parameter constructions are defined only on oriented surfaces.

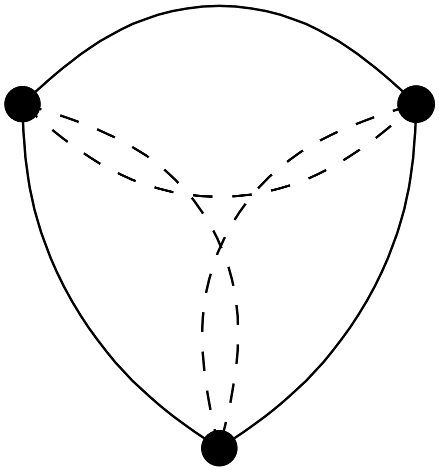
Lins trialities

$(v, f, z) \to$	our notation	notation in [1]	notation in [2]
(v, f, z)	$\mathcal{M}$	gem	$\mathcal{M}$
(f, v, z)	$\mathcal{M}^*$	dual gem	$\mathcal{M}^*$
(z, f, v)	$phial(\mathcal{M})$	phial gem	$p((p(\mathcal{M}))^*)$
(f, z, v)	$(phial(\mathcal{M}))^*$	skew-dual gem	$(p(\mathcal{M}))^*$
(v, z, f)	$skew(\mathcal{M})$	skew gem	$p(M)$
(z, v, f)	$(skew(\mathcal{M}))^*$	skew-phial gem	$p(\mathcal{M}^*)$

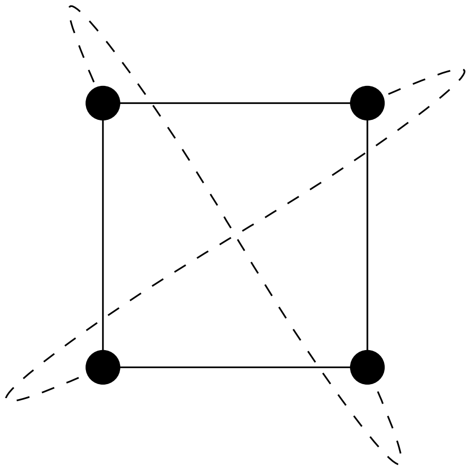
Jones, Thornton (1987): those are only “good” dualities.

1. S. Lins, *Graph-Encoded Maps*, J. Combinatorial Theory Ser. B **32** (1982) 171–181.
2. K. Anderson and D.B. Surowski, *Coxeter-Petrie Complexes of Regular Maps*, European J. of Combinatorics **23-8** (2002) 861–880.

Example: Tetrahedron



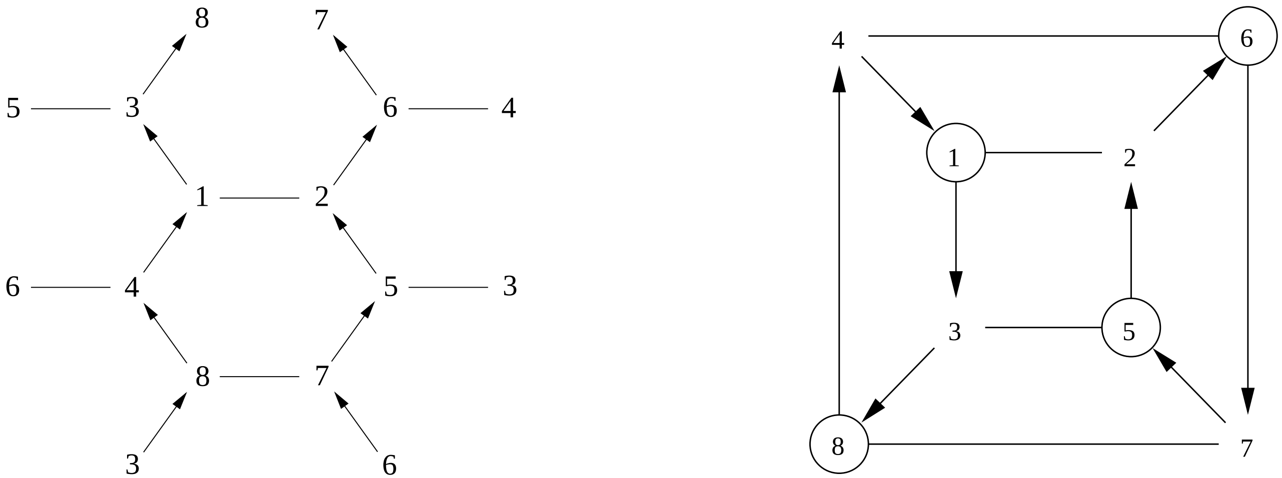
$phial(Tetrahedron)$



$skew(Tetrahedron)$

two Lins maps on projective plane.

Bipartite skeleton case



Two representation of $\textcolor{red}{skew}(Cube)$: on Torus and as a Cube with cyclic orientation of vertices (marked by ○) reversed.

Theorem

For bipartite graph embedded in oriented surface, the skew operation is, in fact, reversing orientation of one of the part of the bipartition.

Prisms and antiprisms

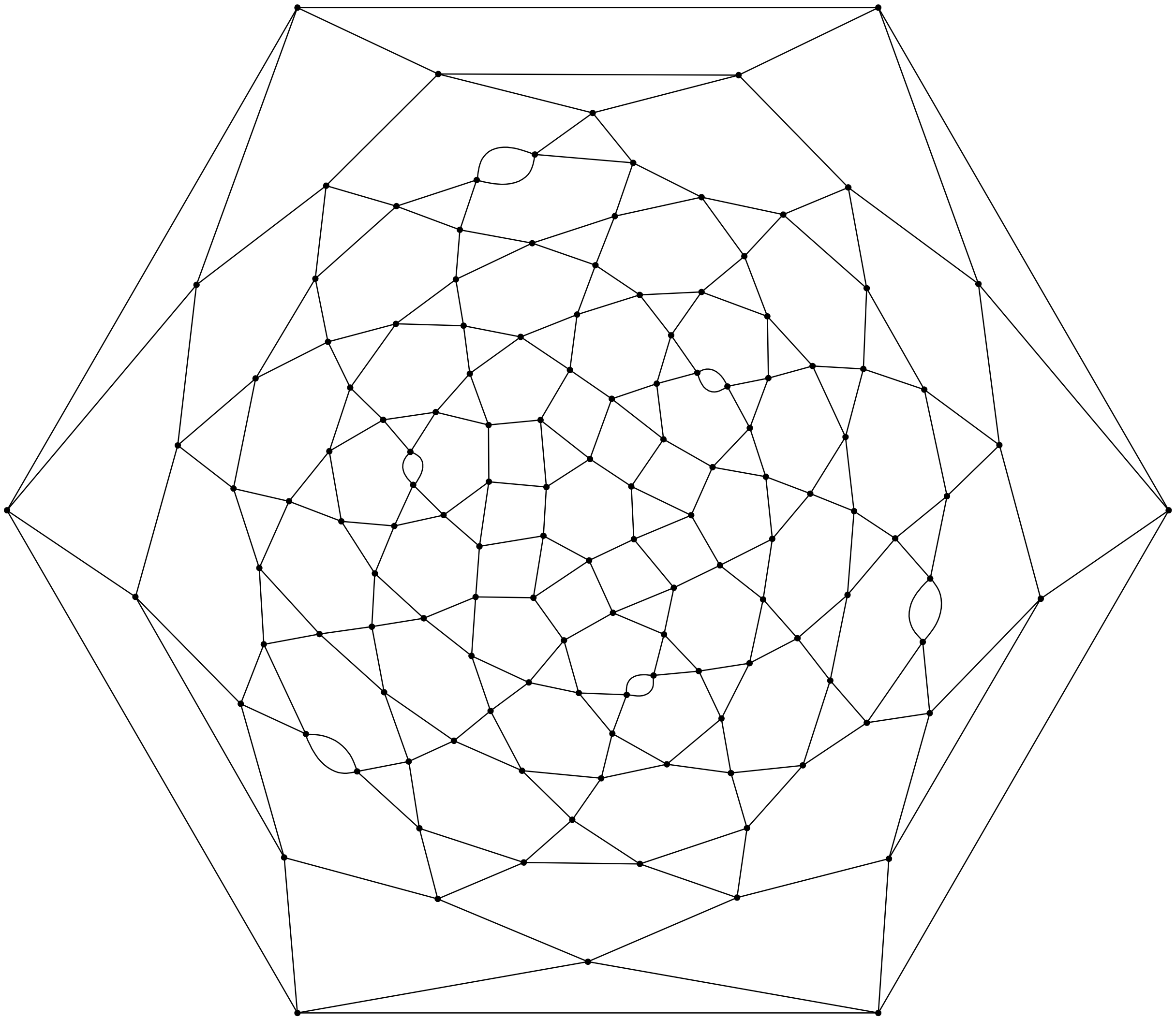
Let χ denotes the Euler characteristic.

Conjecture

- *$skew(Prism_m)$ has $\chi = gcd(m,4) - m$ and is oriented iff m is even;*
- *$phial(Prism_m)$ has $\chi = 2 + gcd(m,4) - 2m$ and is non-oriented.*

- *$skew(APrism_m)$ has $\chi = 1 + gcd(m,3) - 2m$ and is non-oriented;*
- *$phial(APrism_m)$ has $\chi = 3 + gcd(m,3) - 2m$ and is oriented.*

The End



Removing 3 central circuits of $Med(GC_{11,4}(Cube))$.