Topological application of Perfect form theory

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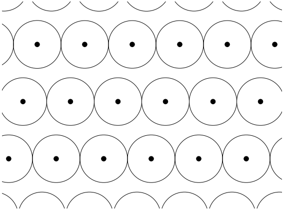
I. Lattices and Gram matrices

Lattice packings

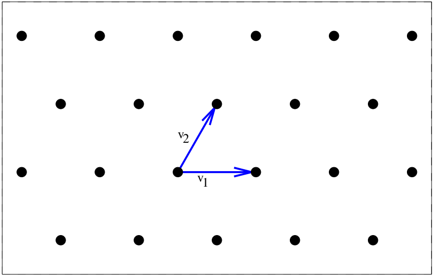
- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ with $(v1, \ldots, v_n)$ independent.
- A packing is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius r and center x_i such that their interiors are disjoint.



- If L is a lattice, the lattice packing is the packing defined by taking the maximal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a packing.
- The maximum α is called $\lambda(L)$ and the determinant of $(v_1, \ldots, v_n)$ is det L.

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices, $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices and $S^n_{\geq 0}$ the convex cone of real symmetric positive semidefinite $n \times n$ matrices.
- Take a basis $(v_1, \dots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$$G_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Isometric lattices

- Take a basis $(v_1, \ldots, v_n)$ of a lattice L with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \ldots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \ldots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T \, V$.
  The matrix $G_{\mathbf{v}}$ is defined by $\frac{n(n+1)}{2}$ variables as opposed to n^2 for the basis V.
- If $M \in S^n_{>0}$, then there exists V such that $M = V^T \, V$ (Gram Schmidt orthonormalization)
- If $M = V_1^T \, V_1 = V_2^T \, V_2$, then $V_1 = OV_2$ with $O^T \, O = I_n$ (i.e. O corresponds to an isometry of $\mathbb{R}^n$).
- Also if L is a lattice of $\mathbb{R}^n$ with basis $\mathbf{v}$ and u an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Arithmetic minimum

- For $A \in S^n$ and $x \in \mathbb{R}^n$ we write $A[x] = x^T A x$.
- The arithmetic minimum of $A \in S^n_{>0}$ is

$$\min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} A[x]$$

- The minimal vector set of $A \in S^n_{>0}$ is

$$\mathrm{Min}(A) = \{x \in \mathbb{Z}^n \mid A[x] = \min(A)\}$$

- Both $\min(A)$ and $\mathrm{Min}(A)$ can be computed using some programs (for example SV by Vallentin)
- The matrix $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}.$$

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice L then $V' = VP$ with $P \in \mathrm{GL}_n(\mathbb{Z})$. This implies

$$G_{\mathbf{v}'} = {V'}^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$A = P^T B P$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken/Souvignier wrote a program ISOM for testing arithmetic equivalence and a program AUTO for computing automorphism group of lattices.
  All such programs take Gram matrices as input.

II. Computational techniques

Dual description problem

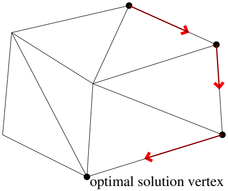
- A vertex of a polytope P is a point $v \in P$, which cannot be expressed as $v = \lambda v^1 + (1-\lambda)v^2$ with $0 < \lambda < 1$ and $v^1 \neq v^2 \in P$.
- A polytope is the convex hull of its vertices and this is the minimal set defining it.
- A facet of a polytope is an inequality $f(x) - b \geq 0$, which cannot be expressed as $f(x) - b = \lambda(f_1(x) - b_1) + (1-\lambda)(f_2(x) - b_2)$ with $f_i(x) - b_i \geq 0$ on P.
- A polytope is defined by its facet inequalities. and this is the minimal set of linear inequalities defining it.
- The dual-description problem is the problem of passing from one description to another.
- There are several programs CDD, LRS for computing dual-description computations.
- In case of large problems, we can use the symmetries for faster computation.

Linear programs

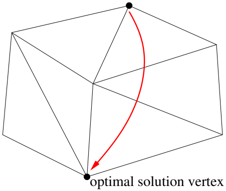
- A linear program is the problem of maximizing a linear function $f(x)$ over a set $\mathcal{P}$ defined by linear inequalities.

$$\mathcal{P} = \{x \in \mathbb{R}^d \quad \text{such that} \quad f_i(x) \geq b_i\}$$

  with f_i linear and $b_i \in \mathbb{R}$.
- The solution of linear programs is attained at vertices of $\mathcal{P}$.
- There are two classes of solution methods:



Simplex method



Interior point method

- Simplex methods use exact arithmetic but have bad theoretical complexity
- Interior point methods have good theoretical complexity but only gives an approximate vertex.

III. Perfect forms and domains

Perfect forms

- A form A is extreme if it is a local maximum of the packing density.
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev) if the equation

$$B \in S^n \text{ and } B[x] = \min(A) \text{ for all } x \in \mathrm{Min}(A)$$

  implies $B = A$.
- Theorem: (Korkine & Zolotarev) If a form is extreme then it is perfect.
- Up to a scalar multiple, perfect forms are rational.
- All root lattices are perfect, many other families are known.

Perfect domains and arithmetic closure

- If $v \in \mathbb{Z}^n$ then the corresponding rank 1 form is $p(v) = vv^T$.
- If A is a perfect form, its perfect domain is

$$\mathrm{Dom}(A) = \sum_{v \in \mathrm{Min}(A)} \mathbb{R}_+ p(v)$$

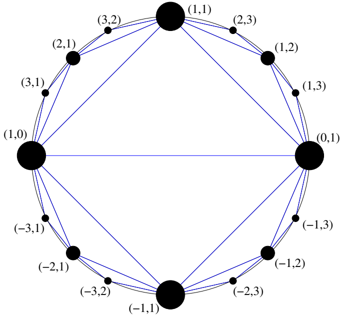
- If A has m shortest vectors then $\mathrm{Dom}(A)$ has $\frac{m}{2}$ extreme rays.
- So actually, the perfect domains realize a tessellation not of $S^n_{>0}$, nor $S^n_{\geq 0}$ but of the rational closure $S^n_{rat,\geq 0}$.
- The rational closure $S^n_{rat,\geq 0}$ has a number of descriptions:
  - $S^n_{rat,\geq 0} = \sum_{v \in \mathbb{Z}^n} \mathbb{R}_+ p(v)$
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if *Ker A* is defined by rational equations.
- So, actually, the stabilizers of some faces of the polyhedral complex are infinite.

Finiteness

- Theorem:(Voronoi) Up to arithmetic equivalence there is only finitely many perfect forms.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:

$$Q \mapsto P^t Q P$$

  and we have $\mathrm{Min}(P^t Q P) = P^{-1}\,\mathrm{Min}(Q)$
- $\mathrm{Dom}(P^T Q P) = c(P)^T\,\mathrm{Dom}(Q) c(P)$ with $c(P) = \left(P^{-1}\right)^T$
- For $n = 2$, we get the classical picture:



Known results on lattice packing density maximization

dim.	Nr. of perfect forms	Best lattice packing
2	1 (Lagrange)	A_2
3	1 (Gauss)	A_3
4	2 (Korkine & Zolotarev)	D_4
5	3 (Korkine & Zolotarev)	D_5
6	7 (Barnes)	E_6 (Blichfeldt & Watson)
7	33 (Jaquet)	E_7 (Blichfeldt & Watson)
8	10916 (DSV)	E_8 (Blichfeldt & Watson)
9	$\geq$500000	Λ_9?
24	?	Leech (Cohn & Kumar)

- The enumeration of perfect forms is done with the Voronoi algorithm.
- The number of orbits of faces of the perfect domain tessellation is much higher but finite (Known for $n \leq 7$)
- Blichfeldt used Korkine-Zolotarev reduction theory.
- Cohn & Kumar used Fourier analysis and Linear programming.

IV. Ryshkov polyhedron and the Voronoi algorithm

The Ryshkov polyhedron

- The Ryshkov polyhedron R_n is defined as

$$R_n = \{A \in S^n \text{ s.t. } A[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n - \{0\}\}$$

- The cone is invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$.
- The cone is locally polyhedral, i.e. for a given $A \in R_n$

$$\{x \in \mathbb{Z}^n \text{ s.t. } A[x] = 1\}$$

  is finite
- Vertices of R_n correspond to perfect forms.
- For a form $A \in R_n$ we define the local cone

$$Loc(A) = \{Q \in S^n \text{ s.t. } Q[x] \geq 0 \text{ if } x \in \mathrm{Min}(A)\}$$

The Voronoi algorithm

- Find a perfect form (say A_n), insert it to the list $\mathcal{L}$ as undone.
- Iterate
  - For every undone perfect form A in $\mathcal{L}$, compute the local cone $Loc(A)$ and then its extreme rays.
  - For every extreme ray r of $Loc(A)$ realize the flipping, i.e. compute the adjacent perfect form $A' = A + \alpha r$.
  - If A' is not equivalent to a form in $\mathcal{L}$, then we insert it into $\mathcal{L}$ as undone.
- Finish when all perfect forms have been treated.

The sub-algorithms are:

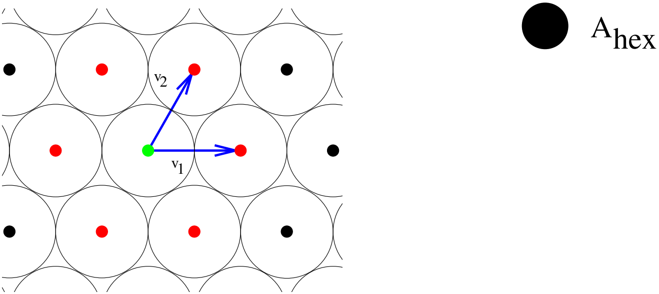
- Find the extreme rays of the local cone $Loc(A)$ (use CDD or LRS or any other program)
- For any extreme ray r of $Loc(A)$ find the adjacent perfect form A' in the Ryshkov polyhedron R_n
- Test equivalence of perfect forms using ISOM

Flipping on an edge I

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$$

with

$$A_{hex} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

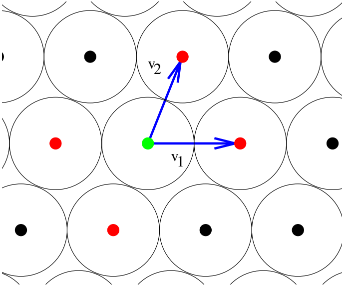


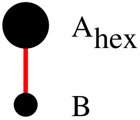
Flipping on an edge II

$$\mathrm{Min}(B) = \{\pm(1,0), \pm(0,1)\}$$

with

$$B = \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix} = A_{hex} + D/4$$



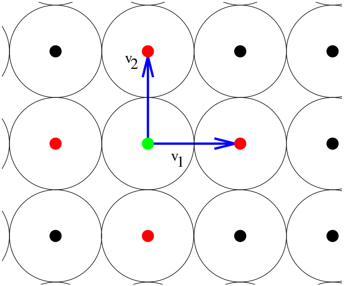


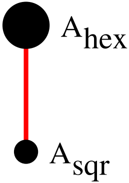
Flipping on an edge III

$$\mathrm{Min}(A_{sqr}) = \{\pm(1,0), \pm(0,1)\}$$

with

$$A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_{hex} + D/2$$



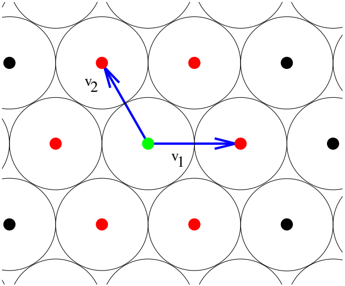


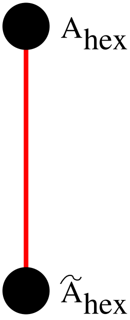
Flipping on an edge IV

$$\mathrm{Min}(\tilde{A}_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$$

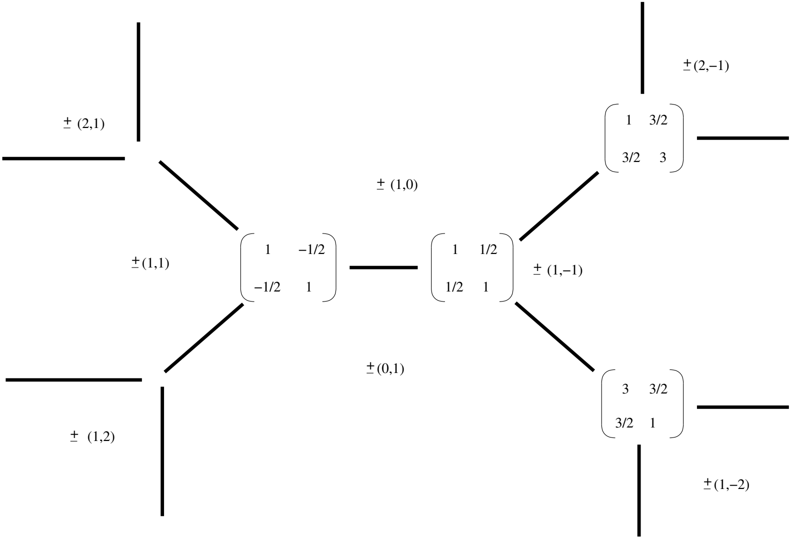
with

$$\tilde{A}_{hex} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = A_{hex} + D$$





The Ryshkov polyhedron R_2



Well rounded forms and retract

- A form Q is said to be well rounded if it admits vectors v_1, ..., v_n such that
  - $(v_1, \ldots, v_n)$ form a $\mathbb{R}$-basis of $\mathbb{R}^n$ (not necessarily a $\mathbb{Z}$-basis)
  - $v_1, \ldots, v_n$ are shortest vectors of Q.
- Well rounded forms correspond to bounded faces of R_n.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of R_n onto a polyhedral complex WR_n of dimension $\frac{n(n-1)}{2}$.
- Every face of WR_n has finite stabilizer.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.
- We also cannot reduce ourselves to lattices whose shortest vectors define a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for $n \geq 5$.

Topological applications

- The fact that we have finite stabilizers for all faces means that we can compute rational homology/cohomology of $\mathrm{GL}_n(\mathbb{Z})$ efficiently.
- This has been done for $n \leq 7$
  - P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math 245 (2013) 587–624.
- As an application, we can compute $K_n(\mathbb{Z})$ for $n \leq 8$.
- By using perfect domains, we can compute the action of Hecke operators on the cohomology.
- This has been done for $n \leq 4$:
  - P.E. Gunnells, *Computing Hecke Eigenvalues Below the Cohomological Dimension*, Experimental Mathematics 9-3 (2000) 351–367.
- The above can, in principle, be extended to the case of $\mathrm{GL}_n(R)$ with R a ring of algebraic integers.

V. Tessellations

Linear Reduction theories for $S^n_{rat,\geq 0}$

Decompositions related to perfect forms:

- The perfect form theory (Voronoi I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$)
- The central cone compactification (Igusa & Namikawa) (Known for $n \leq 6$)

Decompositions related to Delaunay polytopes:

- The L-type reduction theory (Voronoi II) for Delaunay tessellations (Known for $n \leq 5$)
- The C-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (Known for $n \leq 5$)

Fundamental domain constructions:

- The Minkowski reduction theory (Minkowski) it uses the successive minima of a lattice to reduce it (Known for $n \leq 7$) not face-to-face
- Venkov's reduction theory also known as Igusa's fundamental cone (finiteness proved by Venkov and Crisalli)

Self-dual cones

- For an open cone C in $\mathbb{R}^n$ the dual cone is

$$C^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y \rangle > 0 \text{ for } y \in C\}$$

- Such cones are classified by Euclidean Jordan algebras and the classification gives:
  - S^n: The cone of positive definite real quadratic forms
  - H^n: The cone of positive definite Hermitian quadratic forms
  - Q^n: The cone of positive definite quaternionic quadratic forms
  - The cone of 3×3 positive definite octonion matrices.
  - The hyperbolic cone H_n

$$H_n = \left\{(x_1, \ldots, x_n) \text{ s.t. } x_1 > 0 \text{ and } x_1^2 - x_2^2 - \cdots - x_n^2 > 0\right\}$$

- References

  - A. Ash, D. Mumford, M. Rapoport, Y. Tai *Smooth compactifications of locally symmetric varieties*, Cambridge University Press
  - M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivtätsbereichan I/II*, Math. Annalen 141, 384–432, 144, 175–182

T-space theory

- A T-space $\mathcal{F}$ is a vector space in S^n with $\mathcal{F}_{>0} = \mathcal{F} \cap S^n_{>0}$ being non-empty.
- All above reduction theories apply to that case.
- But some dead ends exist to the polyhedral tessellations.
- Relevant group is $\mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \text{ s.t. } g\mathcal{F}g^T = \mathcal{F}\}$.
- For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ of space

  $$\mathcal{F}(G) = \left\{A \in S^n \text{ s.t. } gAg^T = A \text{ for } g \in G\right\}$$

  we have $\mathrm{Aut}(\mathcal{F}(G)) = \mathit{Norm}(G, \mathrm{GL}_n(\mathbb{Z}))$ (Zassenhaus) and a finite number of $\mathcal{F}$-perfect forms.
- There exist some T-spaces having a rational basis and an infinity of perfect forms.
- Another finiteness case is for spaces obtained from $\mathrm{GL}_n(R)$ with R number ring.

Non-polyhedral reduction theories

- Some works with non-polyhedral, but still manifold domains:
  - R. MacPherson and M. McConnel, *Explicit reduction theory for Siegel modular threefolds*, Invent. Math. **111** (1993) 575–625.
  - D. Yasaki, *An explicit spine for the Picard modular group over the Gaussian integers*, Journal of Number Theory, **128** (2008) 207–234.
- Other works in complex hyperbolic space using Poincaré polyhedron theorem:
  - M. Deraux, *Deforming the* $\mathbb{R}$*-fuchsian* $(4,4,4)$*-lattice group into a lattice*.
  - E. Falbel and P.-V. Koseleff, *Flexibility of ideal triangle groups in complex hyperbolic geometry*, Topology **39** (2000) 1209–1223.
- Other works for non-manifold setting would be:
  - T. Brady, *The integral cohomology of* $Out_{+}(F_3)$, Journal of Pure and Applied Algebra **87** (1993) 123–167.
  - K.N. Moss, *Cohomology of* $\mathrm{SL}(n, \mathbb{Z}[1/p])$, Duke Mathematical Journal **47-4** (1980) 803–818.

VI. Central cone compactification

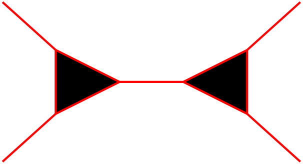
Central cone compactification

- We consider the space of integral valued quadratic forms:

$$I_n = \{A \in S^n_{>0} \text{ s.t. } A[x] \in \mathbb{Z} \text{ for all } x \in \mathbb{Z}^n\}$$

  All the forms in I_n have integral coefficients on the diagonal and half integral outside of it.
- The centrally perfect forms are the elements of I_n that are vertices of conv I_n.
- For $A \in I_n$ we have $A[x] \geq 1$. So, $I_n \subset R_n$
- Any root lattice gives a vertex both of R_n and conv I_n.
- The centrally perfect forms are known for $n \leq 6$:

dim.	Centrally perfect forms
2	A_2 (Igusa)
3	A_3 (Igusa)
4	A_4, D_4 (Igusa)
5	A_5, D_5 (Namikawa)
6	A_6, D_6, E_6 (Dutour Sikirić)



- By taking the dual we get tessellations of $S^n_{rat,\geq 0}$.

Enumeration of centrally perfect forms

- Suppose that we have a conjecturally correct list of centrally perfect forms $A_1, \ldots, A_m$. Suppose further that for each form A_i we have a conjectural list of neighbors $N(A_i)$.
- We form the cone

$$C(A_i) = \{X - A_i \text{ for } X \in N(A_i)\}$$

  and we compute the orbits of facets of $C(A_i)$.
- For each orbit of facet of representative f we form the corresponding linear form f and solve the Integer Linear Problem

$$f_{opt} = \min_{X \in I_n} f(X)$$

  We have to use GLPK program for that. It is done iteratively since I_n is defined by an infinity of inequalities.
- If $f_{opt} = f(A_i)$ always then the list is correct. If not then the X realizing $f(X) < f(A_i)$ need to be added to the full list.

VII. Perfect form complex

Known number of orbits of faces for $n \leq 9$

- Each orbit of face corresponds to a vector configuration.
- The rank $rk(\mathcal{V})$ of a vector configuration $\mathcal{V} = \{v_1, \ldots, v_m\}$ is the rank of the matrix family $\{p(v_i) = v_i^T v_i\}$.
- The complex is fully known for $n \leq 7$. Number of orbits by rank:
  - $n = 4$: 1, 3, 4, 4, 2, 2, 2.
  - $n = 5$: 2, 5, 10, 16, 23, 25, 23, 16, 9, 4, 3.
  - $n = 6$: 3, 10, 28, 71, 162, 329, 589, 874, 1066, 1039, 775, 425, 181, 57, 18, 7
  - $n = 7$: 6, 28, 115, 467, 1882, 7375, 26885, 87400, 244029, 569568, 1089356, 1683368, 2075982, 2017914, 1523376, 876385, 374826, 115411, 24623, 3518, 352, 33
- For $n = 8$ the following is known:
  - Number of perfect forms is 10916
  - Number of orbits of low rank faces is: 13 (Zahareva & Martinet), 106, 783, 6167, 50645
- For $n = 9$ the following is known:
  - Number of orbits of low rank faces is: 44 (Keller, Martinet & Schürmann), 759, 13437

Testing realizability of vector families

- Problem: Suppose we have a configuration of vector $\mathcal{V}$. Does there exist a matrix $A \in S^n_{>0}$ such that $\mathrm{Min}(A) = \mathcal{V}$?
- Consider the linear program

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{with} \quad & \lambda = A[v] \text{ for } v \in \mathcal{V} \\ & A[v] \geq 1 \text{ for } v \in \mathbb{Z}^n - \{0\} - \mathcal{V} \end{aligned}$$

  If $\lambda_{opt} < 1$ then $\mathcal{V}$ is realizable, otherwise no.
- In practice one replaces $\mathbb{Z}^n$ by a finite set and iteratively increases it until a conclusion is reached.
- The number of iterations can unfortunately be very high.
- We use integral symmetries of the configuration of vectors in order to make the linear program simpler.
- A related problem is to find the smallest configuration $\mathcal{W}$ such that there exist a $A \in S^n_{>0}$ with $\mathcal{V} \subseteq \mathcal{W} = \mathrm{Min}(A)$ and $rk(\mathcal{V}) = rk(\mathcal{W})$.

Simpliciality results

- Theorem: If $\mathcal{V} = \{v_1, \dots, v_m\}$ is a configuration of shortest vectors in dimension n such that $rk(\mathcal{V}) = r$ with $r \in \{n, n+1, n+2\}$. Then $m = r$.
- The proof of this is relatively elementary and use simple matrix arguments. See:
  - M. Dutour Sikirić, K. Hulek, A. Schürmann, *Smoothness and singularities of the perfect form compactification of Ag*, Algebraic Geometry 2(5) (2015) 642-653.
- Conjecture: The equality $m = r$ also holds if $r \in \{n+3, n+4\}$.
  It is true for $n \leq 8$.
- For Eisenstein or Gaussian integers similar results hold only for $r = n$.

Enumeration of vector configurations for $r = n + 1$, $r = n + 2$

Suppose we know the configuration of shortest vectors in dimension n of rank $r = n$.

- Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be a short vector configuration with n vectors.
- We search for the vectors v such that $\mathcal{W} = \mathcal{V} \cup \{v\}$ is a vector configuration.
- We can assume that $\mathcal{V}$ has maximum determinant in the $n + 1$ subvector configurations with n vectors of $\mathcal{W}$. Thus

  $$|\det(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n, v)| \le |\det(v_1, \dots, v_n)|$$

  for $1 \le i \le n$.
- The above inequalities determine a n-dim. polytope.
- We enumerate all the integer points by exhaustive enumeration.
- We then check for realizability of the vector families.

For rank $r = n + 2$, we proceed similarly.

Enumeration of vector configurations for $r > n+2$

We assume that we know all the realizable vector configurations of rank $r-1$ and $r-2$.

- We enumerate all pairs $(\mathcal{V}, \mathcal{W})$ with $\mathcal{V} \subset \mathcal{W}$, $rk(\mathcal{V}) = r-2$ and $rk(\mathcal{W}) = r-1$.
- If we have a configuration of rank r, then it contains a configuration $\mathcal{V}$ of rank $r-2$ and dimension n which is contained in two configurations $\mathcal{W}_1$ and $\mathcal{W}_2$ of rank $r-2$ such that $\mathcal{V} \subset \mathcal{W}_1$ and $\mathcal{V} \subset \mathcal{W}_2$.
- So, we combine previous enumeration and obtain a set of configurations $\mathcal{W}_1 \cup \mathcal{W}_2$
- We check for each of them if there exist a realizable vector configuration $\mathcal{W}$ such that $\mathcal{W}_1 \cup \mathcal{W}_2 \subset \mathcal{W}$ and $rk(\mathcal{W}) = r$.

Enumerating the configurations of rank $r = n$

- This is in general a very hard problem with no satisfying solution.
- One would expect that the number of realizable vector configurations in dimension 10, 11 and 12 not be too high.
- It does not seem possible to use the polyhedral structure in order to enumerate them.
- The only known upper bound on the possible determinant of realizable configurations V is

$$|\det(V)| \leq \left\lfloor \gamma_n^{n/2} \right\rfloor$$

- For dimension 9 and 10 the bound combined with known upper bound on γ_n gives 30 and 59 as upper bound.
- Those are quite large bounds.

The case of cyclic lattices

- For an index $d \in \mathbb{N}$ we consider a lattice L spanned by $e_1 = (1, 0, \dots, 0)$, …, $e_n = (0, \dots, 0, 1)$ and

$$e_{n+1} = \frac{1}{d}(a_1, \dots, a_n), a_i \in \mathbb{Z}$$

  such that $(e_1, \dots, e_n)$ is the configuration of shortest vectors of a lattice.
- By standard reductions, we can assume that
  - $a_1 \leq a_2 \leq \cdots \leq a_n$.
  - $1 \leq a_i \leq \lfloor d/2 \rfloor$.
- For $n = 10$ at present one can only state $d \leq 59$ and d prime.
- This case is important for two reasons:
  - It exemplifies the difficulty of the problem.
  - It shows up when enumerating the minimal configuration of shortest vectors.
- For $n = 9$, the largest feasible prime d is 7.

THANK

YOU