T-spaces for algebraic rings

Mathieu Dutour Sikirić

Institut Rudjer Bošković

January 12, 2026

I. Problem setting

Group Homology

- Take G a group, suppose that:
  - X is a contractible space.
  - G act fixed point free on X.

  Then we define the group homologies of G to be $H_p(G) = H_p(X/G)$.
- The space X is then a classifying space.
- Examples:
  - The bar construction gives a classifying space which can be used to compute with general groups.
  - If G is a Bieberbach group (acts fixed point free on $\mathbb{R}^n$) then $\mathbb{R}^n$ is the classifying space and the homology is the one of a flat manifold.
- Getting workable classifying space for a group is not easy:
  - If G is finite then $H_i(G) \neq 0$ for an infinity of i and thus X is infinite dimensional.
  - Thus one hopes to work out some "approximate classifying space" and obtain the homology by perturbation arguments.

II. The case of $\mathrm{GL}_n(\mathbb{Z})$

The case of $\mathrm{GL}_n(\mathbb{Z})$

- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{R}^n$.
- So a priori, it would seem that the approximate classifying space would be $\mathbb{R}^n$. But the stabilizer of a point $x \in \mathbb{R}^n$ can be infinite or $\mathrm{GL}_n(\mathbb{Z})$ itself.
- So, we would like another space X on which $\mathrm{GL}_n(\mathbb{Z})$ could act. Our wishes are for:
  - X to be contractible.
  - X to admit a cell decomposition (polyhedral tesselation) invariant under $\mathrm{GL}_n(\mathbb{Z})$.
  - That every face F of the tesselation has finite stabilizer under $\mathrm{GL}_n(\mathbb{Z})$.

Positive definite quadratic forms

- A matrix Q is called positive definite, respectively positive semidefinite, if for every $x \in \mathbb{R}^n - \{0\}$ we have

$$x^t Qx > 0, \text{ respectively } x^t Qx \geq 0.$$

- Denote by $S^n_{>0}$, respectively $S^n_{\geq 0}$ the cones of positive definite, respectively positive semidefinite $n \times n$-matrices.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by the relation

$$(P, Q) \mapsto P^t QP$$

- For any $Q \in S^n_{>0}$ the automorphism group

$$\mathrm{Aut}(Q) = \{P \in \mathrm{GL}_n(\mathbb{Z}) \text{ such that } P^t QP = Q\}$$

is finite.

Perfect form

- If $A \in S^n_{>0}$ then define $\min(A) = \min_{v \in \mathbb{Z}^n \neq 0} A[v]$ and
$$\mathrm{Min}(A) = \{x \in \mathbb{Z}^n \text{ such that } A[x] = \min(A)\}$$
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:
$$Q \mapsto P^t QP$$
and we have $\mathrm{Min}(P^t QP) = P^{-1}\,\mathrm{Min}(Q)$.
- A form is called perfect (Korkine & Zolotarev) if the equation in B
$$B[v] = \min(A) \text{ for all } v \in \mathrm{Min}(A)$$
implies $B = A$.
- A perfect form is necessarily rational and thus up to a multiple integral.

Perfect domains and arithmetic closure

- If $v \in \mathbb{Z}^n$ then the corresponding rank 1 form is $p(v) = vv^T$.
- If A is a perfect form, its perfect domain is

$$\mathrm{Dom}(A) = \sum_{v \in \mathrm{Min}(A)} \mathbb{R}_+ p(v)$$

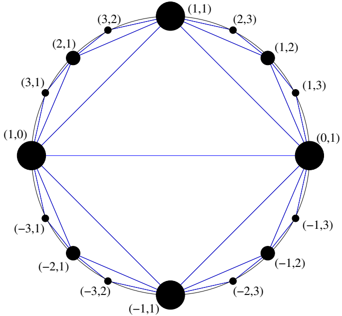
- If A has m shortest vectors then $\mathrm{Dom}(A)$ has $\frac{m}{2}$ extreme rays.
- So actually, the perfect domains realize a tessellation not of $S^n_{>0}$, nor $S^n_{\geq 0}$ but of the rational closure $S^n_{rat,\geq 0}$.
- The rational closure $S^n_{rat,\geq 0}$ has a number of descriptions:
  - $S^n_{rat,\geq 0} = \sum_{v \in \mathbb{Z}^n} \mathbb{R}_+ p(v)$
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if *Ker A* is defined by rational equations.
- So, actually, the stabilizers of some faces of the polyhedral complex are infinite.

Finiteness

- Theorem:(Voronoi) Up to arithmetic equivalence there is only finitely many perfect forms.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:

$$Q \mapsto P^t Q P$$

  and we have $\mathrm{Min}(P^t Q P) = P^{-1}\,\mathrm{Min}(Q)$
- $\mathrm{Dom}(P^T Q P) = c(P)^T\,\mathrm{Dom}(Q) c(P)$ with $c(P) = \left(P^{-1}\right)^T$
- For $n = 2$, we get the classical picture:



Enumeration of Perfect forms (and domains)

dim	Nr of forms	forms	Authors
1	1	A_1	
2	1	A_2	Lagrange
3	1	A_3	Gauss
4	2	D_4, A_4	Korkine & Zolotareff
5	3	D_5, A_5, ...	Korkine & Zolotareff
6	7	E_6, E_6^*, ...	Barnes
7	33	E_7, ...	Jaquet
8	10916	E_8, ...	Dutour Sikirić, Schürmann & Vallentin
9	2237251040	Λ_9, ...	Dutour Sikirić & van Woerden

- The enumeration of perfect forms is done with the Voronoi algorithm.
- The number of orbits of faces of the perfect domain tessellation is much higher but finite (Known for $n \leq 7$)

Well rounded forms and retract

- A form Q is said to be well rounded if it admits vectors v_1, …, v_n such that
  - $(v_1, \ldots, v_n)$ form a $\mathbb{R}$-basis of $\mathbb{R}^n$ (not necessarily a $\mathbb{Z}$-basis)
  - $v_1, \ldots, v_n$ are shortest vectors of Q.
- Well rounded forms correspond to bounded faces of R_n.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of R_n onto a polyhedral complex WR_n of dimension $\frac{n(n-1)}{2}$.
- Every face of WR_n has finite stabilizer.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.
- We also cannot reduce ourselves to lattices whose shortest vectors define a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for $n \geq 5$.

Topological applications

- The fact that we have finite stabilizers for all faces means that we can compute rational homology/cohomology of $\mathrm{GL}_n(\mathbb{Z})$ efficiently.
- This has been done for $n \leq 7$
  - P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math 245 (2013) 587–624.
- As an application, we can compute $K_n(\mathbb{Z})$ for $n \leq 8$.
- By using perfect domains, we can compute the action of Hecke operators on the cohomology.
- This has been done for $n \leq 4$:
  - P.E. Gunnells, *Computing Hecke Eigenvalues Below the Cohomological Dimension*, Experimental Mathematics 9-3 (2000) 351–367.
- The above can, in principle, be extended to the case of $\mathrm{GL}_n(R)$ with R a ring of algebraic integers.

III. Related tesselations and groups

Linear Reduction theories for $S^n_{rat,\geq 0}$

Decompositions related to perfect forms:

- The perfect form theory (Voronoi I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$)
- The central cone compactification (Igusa & Namikawa) (Known for $n \leq 6$)

Decompositions related to Delaunay polytopes:

- The L-type reduction theory (Voronoi II) for Delaunay tessellations (Known for $n \leq 5$)
- The C-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (Known for $n \leq 5$)

Fundamental domain constructions:

- The Minkowski reduction theory (Minkowski) it uses the successive minima of a lattice to reduce it (Known for $n \leq 7$) not face-to-face
- Venkov's reduction theory also known as Igusa's fundamental cone (finiteness proved by Venkov and Crisalli)

T-space of forms

- A T-space $\mathcal{F}$ is a vector space in S^n with $\mathcal{F}_{>0} = \mathcal{F} \cap S^n_{>0}$ being non-empty.
- Relevant group is $\mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \text{ s.t. } g\mathcal{F}g^T = \mathcal{F}\}$.
- For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ of space

$$\mathcal{F}(G) = \left\{ A \in S^n \text{ s.t. } gAg^T = A \text{ for } g \in G \right\}$$

  we have $\mathrm{Aut}(\mathcal{F}(G)) = Norm(G, \mathrm{GL}_n(\mathbb{Z}))$ (Zassenhaus) and a finite number of $\mathcal{F}$-perfect forms.
- For most of the reduction theories that exist for $S^n_{>0}$, there exist an analog in T-spaces.
- The preference is for the perfect form theory. It is reasonably simple, and while it explodes in complexity like others it explode less fast than other reduction theories.

Perfect forms on T-spaces

- The definition of perfect forms is straightforward: The linear equations defining
- Voronoi algorithm works and gets all the perfect forms.
- The well rounded retract can be defined and its cells have finite stabilizer
- Finiteness questions:
  - There exist some T-spaces having a rational basis and an infinity of perfect forms.
  - For a finite subgroup G of $GL_n(\mathbb{Z})$, the space of invariant forms has a finite number of perfect forms.
  - Another finiteness case is for spaces obtained from $\mathrm{GL}_n(R)$ with R number ring.

Case of $\mathrm{GL}_n(R)$

- (Ash) If R is a ring of algebraic integers with r real embedding and s complex embeddings then we can make $\mathrm{GL}_n(R)$ act on $(S^n_{>0})^r \times (H^n_{>0})^s$ with $H^n_{>0}$ the cone of positive definite Hermitian forms.
- Due to the finiteness and the interest for algebraic groups, there is a lot of study for those groups.
- Example papers:
  - Dutour Sikirić M., Gangl H., Gunnells P., Hanke J., Schürmann A., Yasaki D., *On the cohomology of linear groups over imaginary quadratic fields*, J. Pure Appl. Algebra 220 (2016) 2564–2589.
  - Yasaki D., *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic Number theory, 2010, 385–396.
- Among those examples, the real quadratic, imaginary quadratic and totally real have the advantage of being rational.

Embedding $\mathrm{GL}_n(R)$ in $\mathrm{GL}_{nd}(\mathbb{Z})$

- For simplicity assume that $R = \mathbb{Z}[\alpha]$ for α a generating element of the ring.
- So, if we have a basis $(e_i)_{1\leq i\leq n}$ of R^n then the basis of $\mathbb{Z}^{nd}$ is

$$e_i\alpha^{j-1} \text{ for } 1 \leq i \leq n, 1 \leq j \leq d$$

- From this we get an injective homomorphism

$$\phi : \mathrm{GL}_n(R) \mapsto \mathrm{GL}_{nd}(\mathbb{Z})$$

- The multiplication by α gives an element A of $\mathrm{GL}_{nd}(\mathbb{Z})$
- We then have the characterization

$$Im(\phi) = \{M \in \mathrm{GL}_{nd}(\mathbb{Z}) \text{ s.t. } AM = MA\}$$

Real quadratic rings I

- Let us take a ring $R = \mathbb{Z}[\alpha]$ with $\alpha^2 - S\alpha + P = 0$. We define σ the conjugation of the ring, which gets us $S = \alpha + \alpha^\sigma$ and $P = \alpha\alpha^\sigma$.
- The quadratic form that we have on R^n for $v = x + \alpha y$ with $x, y \in \mathbb{Z}^n$.

$$Tr(v) = A_1[x + \alpha y] + A_2[x + \alpha^\sigma y]$$

- We can write $A_1 = A + \alpha^\sigma B$ and $A_2 = A + \alpha B$ with $A, B \in S^n$.
- After expanding we get

$$\begin{array}{rcl} Tr(v) & = & x^T(2A + SB)x \\ & + & y^T\left((S^2 - P)A + PSB\right)y \\ & + & x^T(2SA + 4PB)y \end{array}$$

Real quadratic rings II

- So, this defines the following space of quadratic forms $SP(R)$

$$\begin{pmatrix} 2A + SB & SA + 2PB \\ SA + 2PB & (S^2 - P)A + PSB \end{pmatrix}$$

- For $A, B \in \mathcal{S}^n$.
- The dimension of the T-space in $\mathcal{S}^{2n}$ is $n(n+1)$.

Imaginary quadratic rings I

- If the ring R is imaginary quadratic then we take an Hermitian matrix $A \in \mathcal{H}^n$ and write $A = U + V$ with U a symmetric matrix and V an antisymmetric matrix and get

$$\begin{array}{rcl} Tr(v) & = & A[x+\alpha y] = (x+\alpha y)^{\sigma t} A (x+\alpha y) \\ & = & x^T U x + P y^t U y + x^t S U y \\ & + & \alpha x^T V y + \alpha^\sigma y^T V x \end{array}$$

- The last line is simplified with $y^T V x = -x^T V y$.
- So, we write $W = (\alpha - \alpha^\sigma) V/2$ and the last line becomes

$$x^T 2 W y$$

Imaginary quadratic rings II

- The space in question becomes $SP(R)$ with the matrices

$$\begin{pmatrix} U & (S/2)U + W \\ (S/2)U + W & PU \end{pmatrix}$$

  with

$$U \in \mathcal{S}^n \text{ and } W \in \mathcal{A}S^n$$

- The dimension of the T-space in $\mathcal{S}^{2n}$ is n^2.
- Define t the dimension of the T-space so defined.

Embedding $\mathrm{GL}_n(R)$ in $\mathrm{GL}_t(\mathbb{Z})$

- The action of $\mathrm{GL}_n(R)$ embeds into $\mathrm{GL}_{nd}(\mathbb{Z})$.
- For the T-space $SP(R)$, the action is

  $$(P, A) \mapsto PAP^T$$

  and so this embeds into $\mathrm{GL}_t(\mathbb{Z})$ for a good basis of $SP(R)$.
- The kernel is non-trivial. At least $\pm I_n$ is part of it.
- For R an imaginary quadratic ring the kernel is the ring of units of the ring.
- So, we get an embedding of $\mathrm{PSL}_n(\mathbb{Z}[i])$ into $\mathrm{GL}_{n^2}(\mathbb{Z})$.

IV. Computational techniques

Isomorphism and Automorphism computation I

- Let us consider first the computation of the automorphism of a quadratic form Q.
- We need to have a family of vectors $(v_i)_{1\leq i\leq N}$ which is invariant under any automorphism of Q and is generating $\mathbb{Z}^n$ as a $\mathbb{Z}$-lattice:
  - Since we are with perfect forms computing the shortest vectors is a good bet.
  - But we are in a T-space, so there are some T-perfect forms for which the set of shortest vectors is not even full-dimensional.
  - Also, we do need to consider forms which are not perfect.
  - One strategy is to take the short vectors, that is vectors v such that $Q[v] \leq \lambda$.

Isomorphism and Automorphism computation II

- We consider the edge-weighted graph G with edge weights $w_{i,j} = v_i Q v_j^T$.
- We can compute the automorphism group of this graph. The graph automorphism map to matrix automorphism and this defines a group G_1 of $\mathrm{GL}_n(\mathbb{Z})$.
- However, there are 3 groups:
  - The group G_1 in question.
  - The subgroup G_2 of G_1 stabilizing the T-space
  - The subgroup G_3 of G_2 that belongs to the image of $\mathrm{GL}_n(R)$.
- For computing the group G_3, the trick is to use the vector-valued edge-weights $w_{i,j} = (v_i Q v_j^T, v_i P Q v_j^T)$ with P the matrix element corresponding to the multiplication by α.
- For the group G_2, there is no good algorithm for doing the computation. What we use is single-coset iteration and plan is to use double coset iteration.