Sphere packings
and
lattice sphere packings

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I. Introduction

Norms, balls and spheres

- On the vector space $\mathbf{R}^n$, we define the Euclidean norm

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2} \quad \text{with} \quad x = (x_1, \ldots, x_n)$$

- The Euclidean distance on $\mathbf{R}^n$ is $d(x, y) = ||x - y||$
- A ball $B(c, r)$ of center c and radius r is defined by

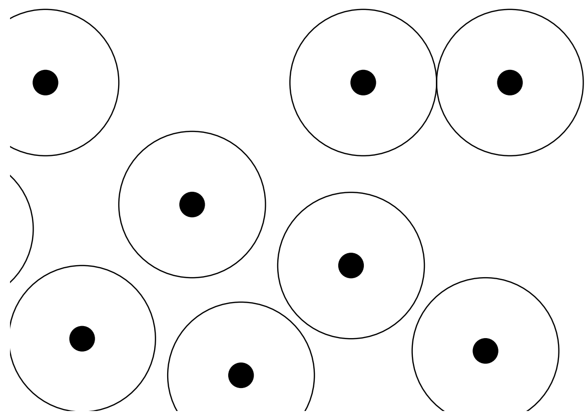
$$B(c, r) = \{x \in \mathbf{R}^n \ \text{ with } \ ||x - c|| \leq r\}$$

- The interior of $B(c, r)$ is $\{x \in \mathbf{R}^n \ \text{ with } \ ||x - c|| < r\}$
- The sphere $S(c, r)$ of center c and radius r is defined as

$$S(c, r) = \{x \in \mathbf{R}^n \ \text{ with } \ ||x - c|| = r\}$$

Packings

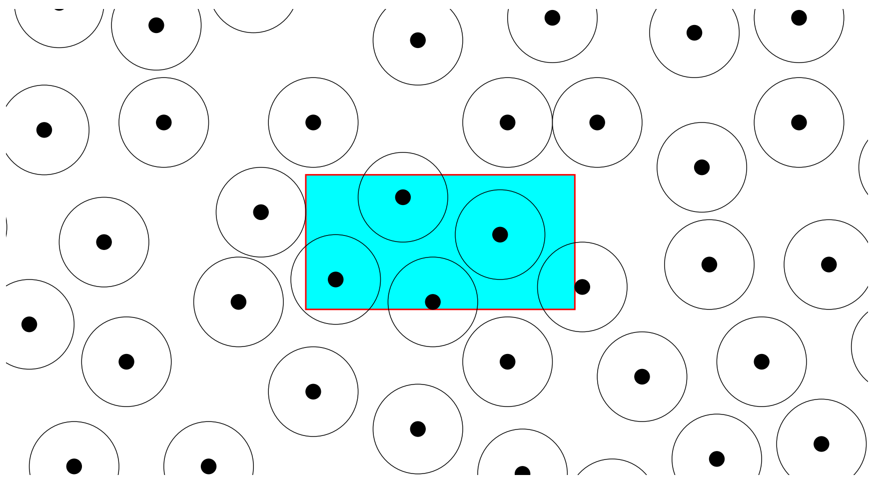
- A packing in $\mathbf{R}^n$ is a set of balls of the same radius, whose interiors are non-overlapping.



- For historical reasons, those packings are called sphere packings, instead of ball packings.
- We will consider only infinite sphere packings.

Density of packing

- The density of a sphere packing $\mathcal{SP}$ is the fraction of space covered. It is defined by
  $\delta(\mathcal{SP}) = \lim_{m\to\infty} \frac{\mathbf{vol}(\mathcal{SP}\cap R_m)}{\mathbf{vol}(R_m)}$ with $R_m \to \mathbf{R}^n$



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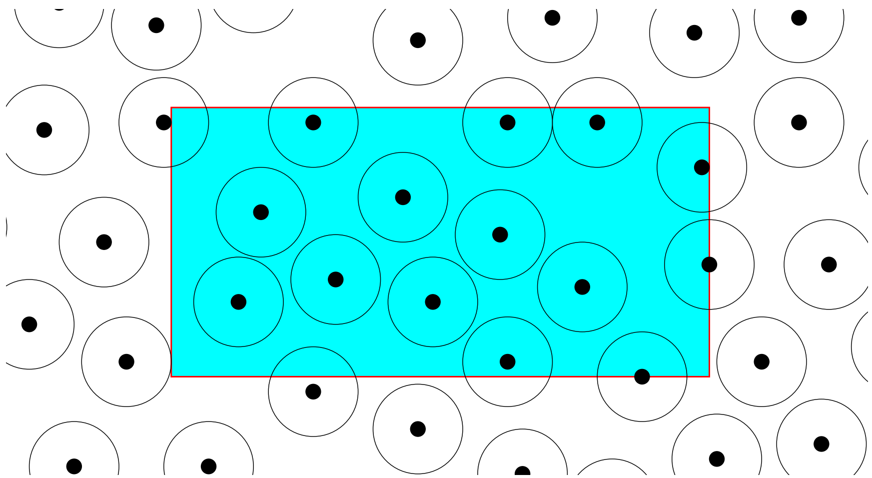

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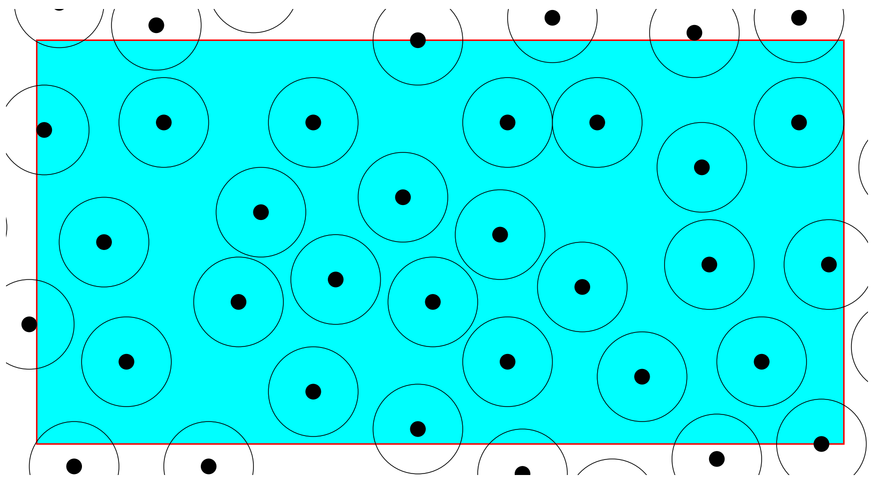

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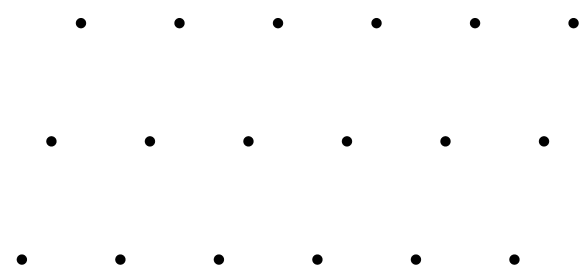
Packing problem

- Denote by δ_n the highest density of sphere packings in $\mathbf{R}^n$.
- The packing problem in dimension n is:
  - Determine the value of δ_n.
  - Describe packings of density δ_n.
- In dimension 3, the problem is sometimes called Kepler problem.
- Removing one sphere in a packing does not change its density!
- A “reasonable” problem is to describe all periodic n-dimensional packings having highest constant.

Lattice packings

- A lattice L is a subgroup of $\mathbb{R}^d$ of the form $L = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_d$.



- If L is a lattice, the lattice packing is the packing defined by taking the maximal value of $\alpha > 0$ such that $L + B(0, \alpha)$ is a packing.

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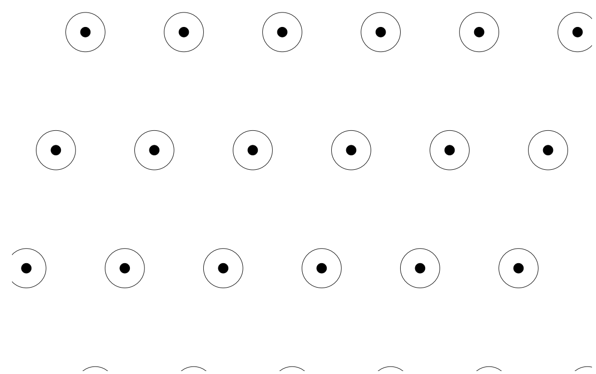

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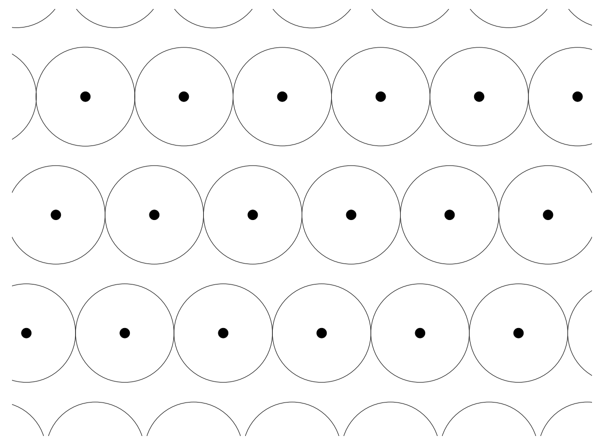

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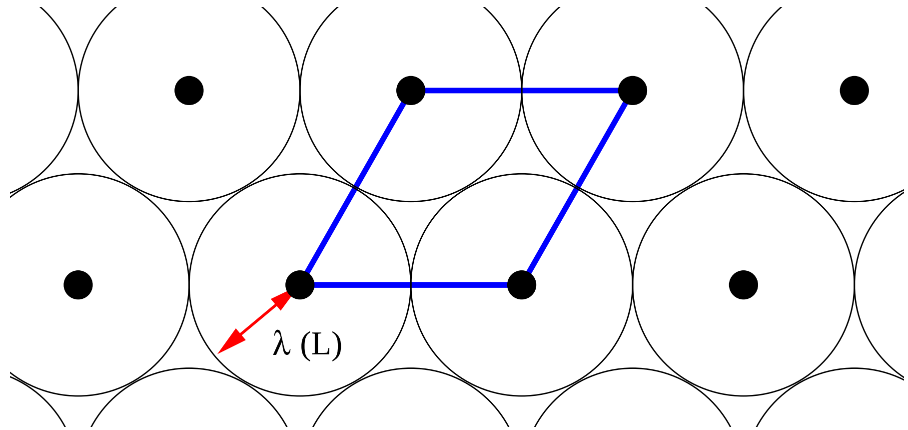
Density of lattice packings

- Take the lattice packing defined by a lattice L:



- The packing density has the expression

$$\delta(L) = \frac{\lambda(L)^n \kappa_n}{det\, L} \text{ with } \lambda(L) = \frac{1}{2} \min_{v \in L - \{0\}} ||v||,$$

κ_n the volume of the unit ball $B(0,1)$ and $det\, L$ the volume of an unit cell.

- Denote by δ_n^* the highest density of lattice packings.

Known results

Dimension	Best lattice packing	Best packing
2	A_{hex} (Lagrange)	A_{hex} (Lagrange)
3	A_3 (Gauss)	A_3 (Hales & Ferguson)
4	D_4 (Korkine & Zolotarev)	?
5	D_5 (Korkine & Zolotarev)	?
6	E_6 (Blichfeldt)	?
7	E_7 (Blichfeldt)	?
8	E_8 (Blichfeldt)	?
24	$Leech$ (Cohn & Kumar)	?

We do not know if $\delta_n > \delta_n^*$ for some n, i.e. an example of a sphere packing of higher density than any lattice sphere packing.

Plan of the presentation

We will present:

- Those results
- Methods of their proof
- Some conjectures
- Some new techniques

Subjects not covered:

- Finite packings
- Asymptotic theory as the dimension goes to ∞.
- Description of remarkable lattices $Leech$, E_8, ...
- Random packing.

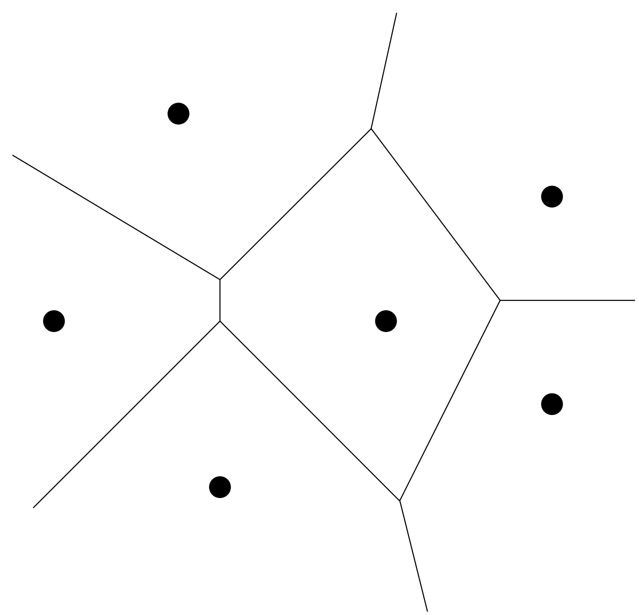
II. Voronoi polytope technique

Voronoi domain

- Suppose X is a locally finite set in $\mathbf{R}^n$, for any $x \in X$, define

  $$N(x) = \{v \in \mathbf{R}^n \mid \ ||v - x|| \leq ||v - y||, \text{ for all } y \in X - \{x\}\}$$

  also known as “Nearest neighbor region”, “Brillouin zone”, “Wigner Seitz cell”.
- They form a face to face tiling of $\mathbf{R}^n$



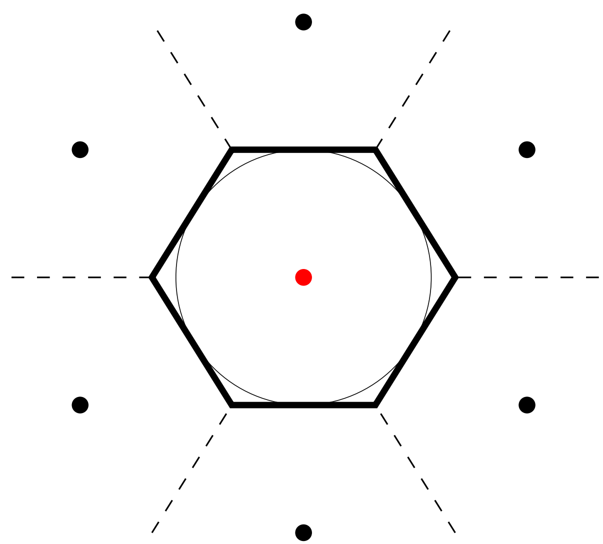
Results

- Take a subset X in $\mathbf{R}^n$ and assume that for every $x, x' \in X$, $x \neq x'$, we have $||x - x'|| \geq 2$. Then one defines
  - the sphere packings with balls $B(x, 1)$ of radius 1 and center in X,
  - the Voronoi domain region $N(x)$

  and gets for any $x \in X$, the inclusion $B(x, 1) \subset N(x)$.
- Denote by α_n the smallest value of α such that for every sphere packing with balls $B(x, 1)$, one has $\alpha \mathbf{vol}(N(x)) \geq \mathbf{vol}(B(x, 1))$.
- One has $\delta_n \leq \alpha_n$.

Dimension 2

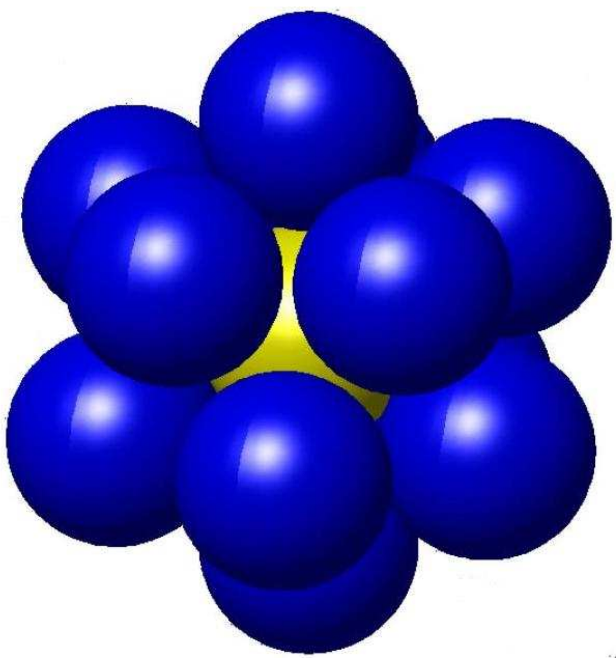
- It is proved that the Voronoi cell of minimal volume in a packing by sphere of radius 1 is regular hexagon. So, $\alpha_2 = \frac{\pi}{\sqrt{12}}$ and $\delta_2 \leq \alpha_2$.



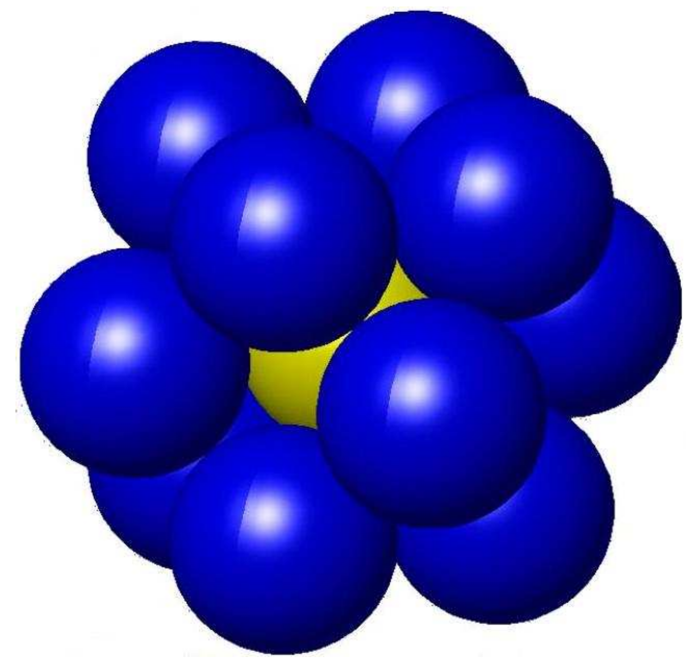
- Regular hexagon realizes a face-to-face packing of $\mathbf{R}^2$, so in fact $\delta_2 = \alpha_2 = \frac{\pi}{\sqrt{12}}$ (Lagrange)
- Hexagonal packing is the unique periodic packing of highest density.

Voronoi polytope in dimension 3

- The Voronoi region of minimal volume is the Dodecahedron (Thomas Hales & Sean McLaughlin, proved by computer computations)
- There is no set X in $\mathbb{R}^3$, whose Voronoi region are Dodecahedron. So, $\delta_3 < \alpha_3$



Configuration of spheres with minimal Voronoi

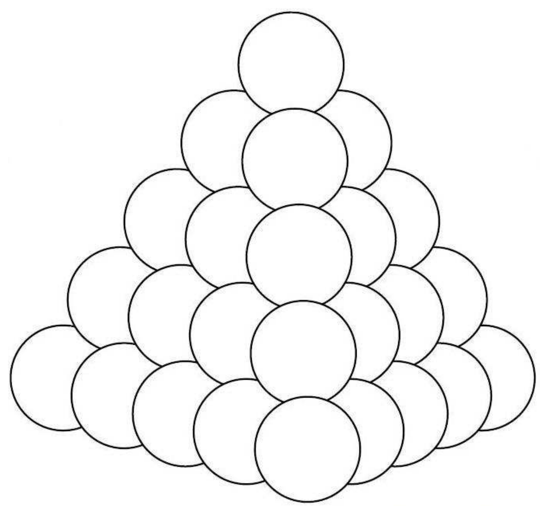


Configuration of spheres with maximal density

Sphere packing in dimension 3

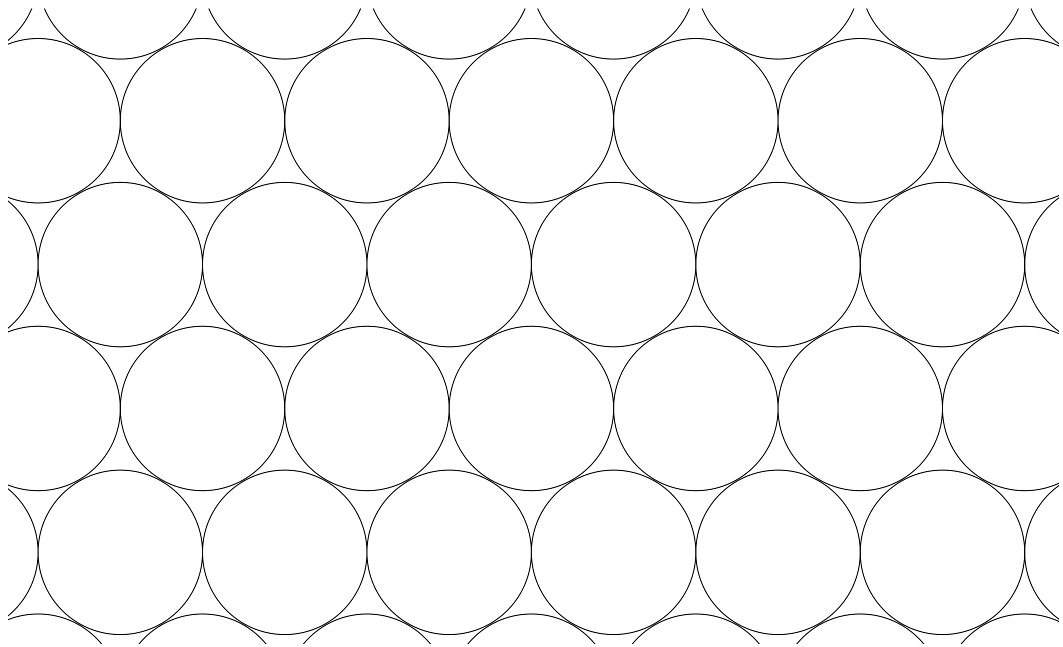
Hales & Ferguson proved that there is no packing of density higher than the one by A_3 lattice (cannon ball packing).

- The method is computer based, extremely long, extraordinarily complicated, unchecked.
- It uses a decomposition of the space intermediate between Voronoi and Delaunay decomposition.
- It uses global optimization, branch & bound and interval arithmetic.



Highest density packings in dimension 3

- Packings with highest density in dimension 3 are formed by lamination on the hexagonal packing:



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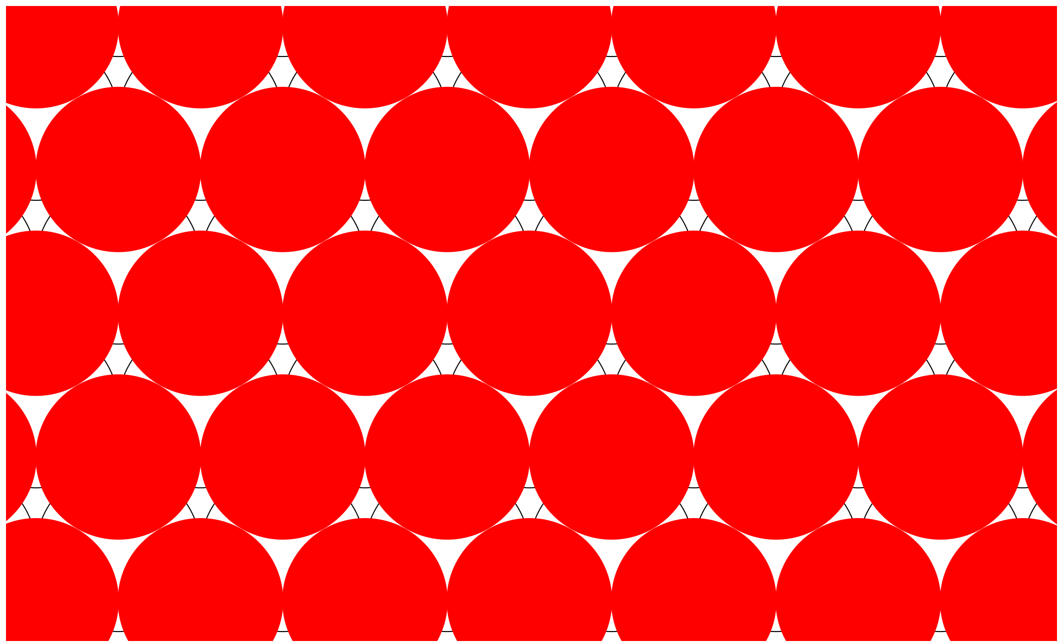

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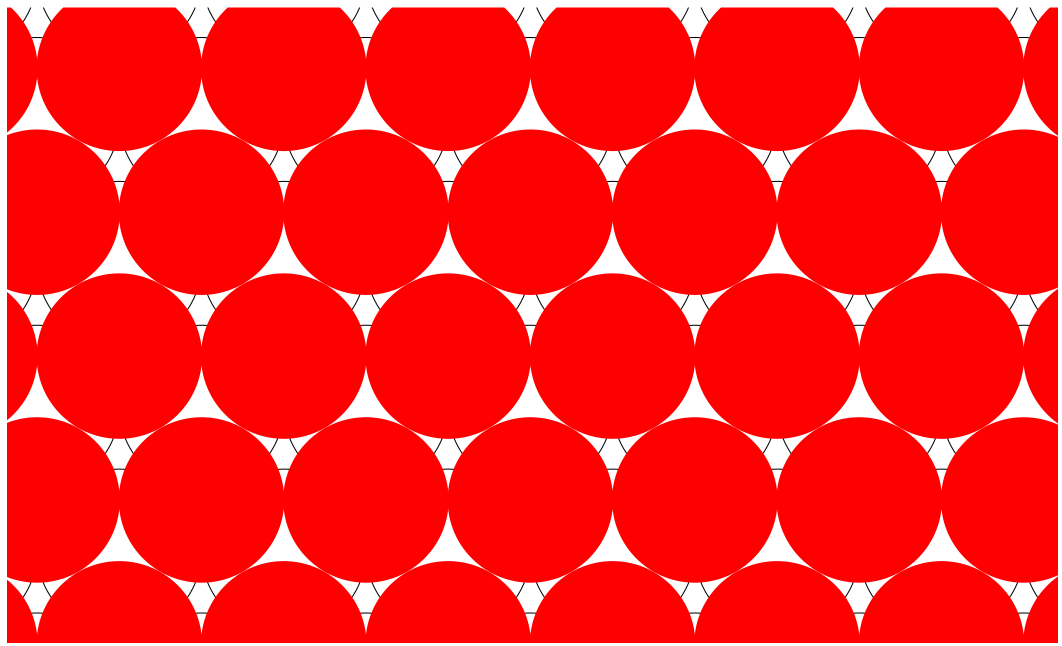

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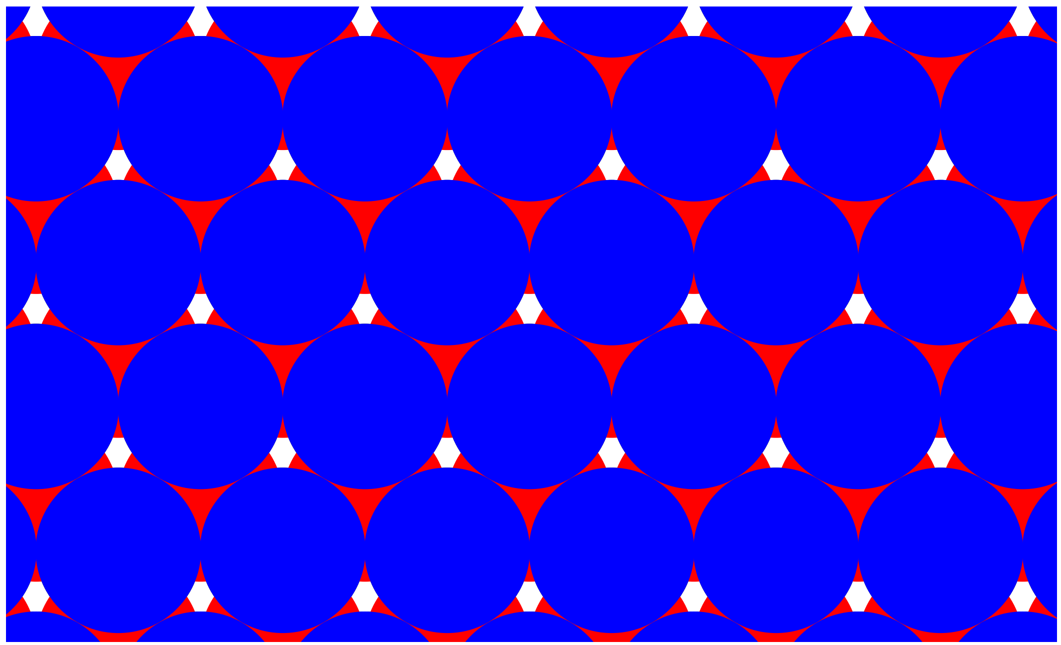

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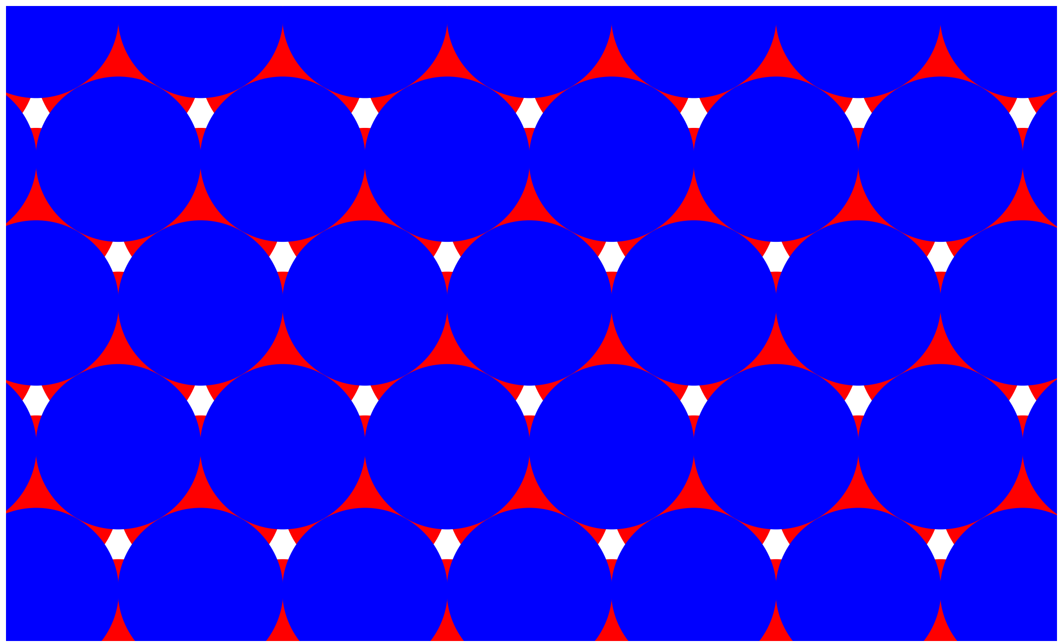

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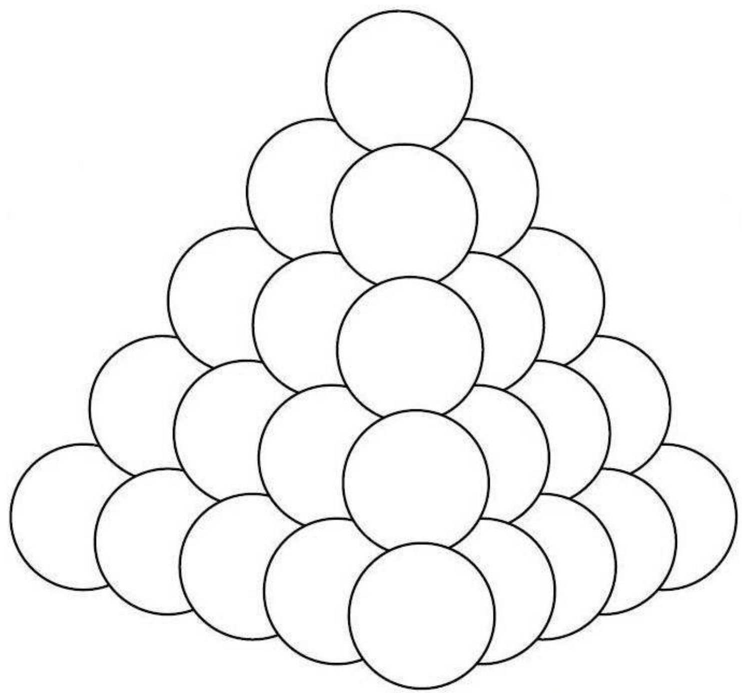


- The above “Cannon ball” packing is when the choice of layer is uniform.

Other dimensions

- Conjecture It seems likely that

$$\delta_4 = \alpha_4, \quad \delta_8 = \alpha_8 \quad \text{and} \quad \delta_{24} = \alpha_{24}$$

  And that the equality is realized only by the Voronoi domain of D_4, E_8 and $Leech$ lattice
- This would prove that they are best packings in dimension 4, 8 and 24
- Problem Find Voronoi cell of “small” volume in dimension 5 to 24.

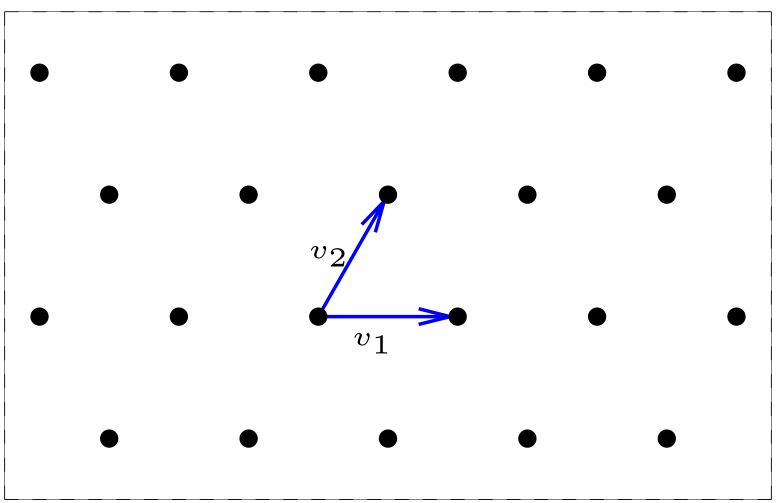
III. Lattices

and

Gram matrices

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices and $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices.
- Take a lattice $L = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_n$ and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$



$$G_{\mathbf{v}} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Isometric lattices

- Take a lattice $L = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_n$ with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbf{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T\, V$
- If $M \in S^n_{>0}$, then there exists V such that $M = V^T\, V$
- If $M = V_1^T\, V_1 = V_2^T\, V_2$, then $V_1 = OV_2$ with $O^T\, O = I_n$ (i.e. O corresponds to an isometry of $\mathbf{R}^n$).
- Also if L is a lattice of $\mathbf{R}^n$ with basis $\mathbf{v}$ and u an isometry of $\mathbf{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Arithmetic minimum

- The arithmetic minimum of $A \in S^n_{>0}$ is

$$min(A) = \min_{x \in \mathbf{Z}^n - \{0\}} x^T A x$$

- The minimal vector set of $A \in S^n_{>0}$ is

$$Min(A) = \{x \in \mathbf{Z}^n \mid x^T A x = min(A)\}$$

- Both $min(A)$ and $Min(A)$ can be computed using some programs (for example sv by Vallentin)
- The matrix $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has $Min(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$.

Reexpression of previous definitions

- Take a lattice $L = \mathbf{Z}v_1 + \cdots + \mathbf{Z}v_n$. If $x \in L$,

$$x = x_1 v_1 + \cdots + x_n v_n \text{ with } x_i \in \mathbf{Z}$$

  we associate to it the column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

- We get $||x||^2 = X^T G_{\mathbf{v}} X$ and

$$det\, L = \sqrt{det\, G_{\mathbf{v}}} \text{ and } \lambda(L) = \frac{1}{2}\sqrt{min(G_{\mathbf{v}})}$$

- For $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $det\, A_{hex} = 3$ and $min(A_{hex}) = 2$

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice L then $V' = VP$ with $P \in GL_n(\mathbf{Z})$. This implies

$$G_{\mathbf{v}'} = V'^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

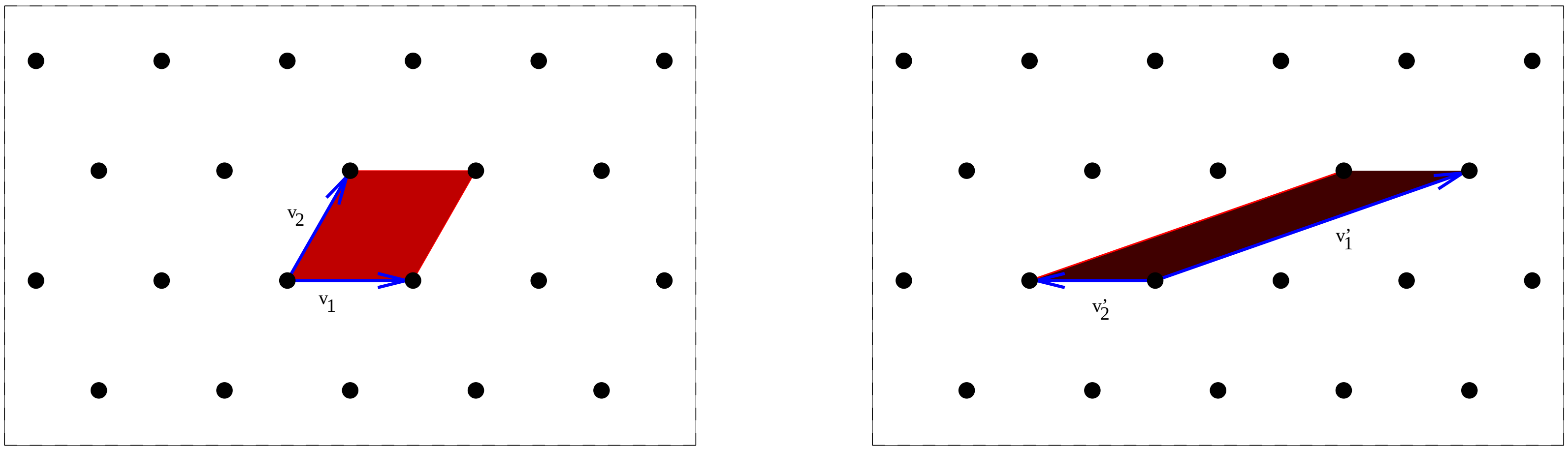
- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in GL_n(\mathbf{Z})$ such that

$$A = P^T B P$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken wrote a program isom for testing arithmetic equivalence.

An example

- Take the hexagonal lattice and two basis in it.



$v_1 = (1,0)$ and $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ $\quad v'_1 = (\frac{5}{2}, \frac{\sqrt{3}}{2})$ and $v'_2 = (-1, 0)$

- One has $v'_1 = 2v_1 + v_2$, $v'_2 = -v_1$ and $P = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

$$G_{\mathbf{v}} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \text{ and } G_{\mathbf{v}'} = \begin{pmatrix} 7 & -\frac{5}{2} \\ -\frac{5}{2} & 1 \end{pmatrix} = P^T G_{\mathbf{v}} P$$

Hermite constant

- The Hermite function is defined on $S^n_{>0}$ as

$$\gamma(A) = \frac{min(A)}{(det\ A)^{1/n}}$$

- The Hermite constant is:

$$\gamma_n = \max_{A \in S^n_{>0}} \gamma(A)$$

- The density of the lattice packing associated to A is

$$\sqrt{\gamma(A)^n}\frac{\kappa_n}{2^n}$$

- Finding lattice packings with highest packing density is the same as maximizing the Hermite function.

Extreme lattices

- The function γ is continuous on $S^n_{>0}$.
- The expression of the lattice packing problem in form of a matrix problem allows to use analytical tools.
- A form $A \in S^n_{>0}$ is extreme if the Hermite function γ attains a local maximum at A.
- If one determines all the extreme lattices, then by computing the value of γ for all of them, one would get the absolute maximum at A.

IV. Lattice packings, perfect lattices and Voronoi algorithm

Perfect lattices

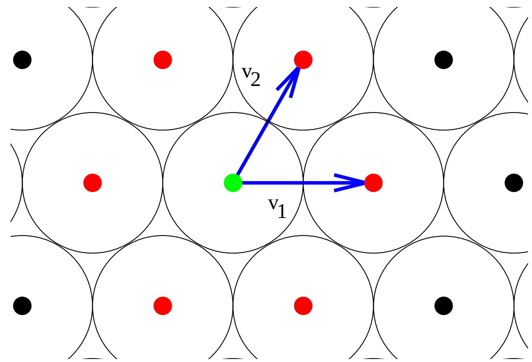
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev) if the equation

$$B \in S^n \text{ and } {}^t x B x = min(A) \text{ for all } x \in Min(A)$$

  implies $B = A$.
- A lattice is perfect if it has a basis $(v_1, \dots, v_n)$ with $G_{\mathbf{v}}$ being perfect.
- Since $x \in \mathbf{Z}^n$, we have a linear system with integral coefficient so perfect matrices are rational.
- $dim(S^n) = \frac{n(n+1)}{2}$ and shortest vector comes into pairs $\{v, -v\}$. So one has $|Min(A)| \geq n(n+1)$.
- A extreme $\Rightarrow$ A perfect. (Korkine & Zolotarev)

A perfect lattice

- $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ corresponds to the lattice:



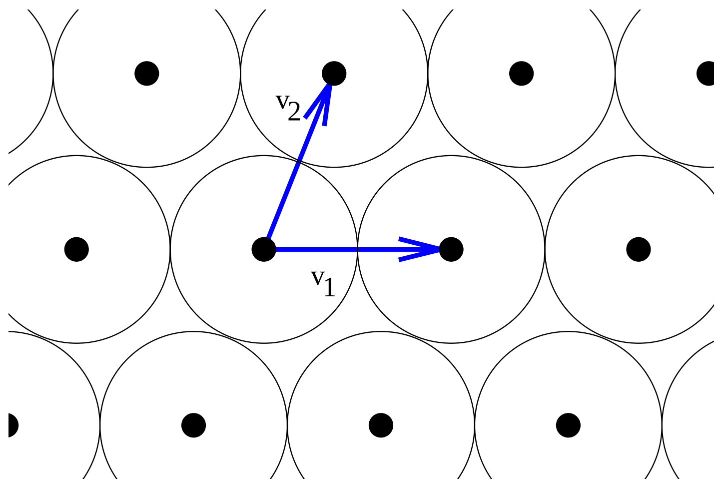
- If $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfies to $x^T B x = min(A_{hex})$ for $x \in Min(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$, then:

$$a = 2, \quad b = 2 \text{ and } a - 2c + b = 2$$

which implies $B = A_{hex}$. A_{hex} is perfect.

A non-perfect lattice

- $A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $Min(A_{sqr}) = \{\pm(0,1), \pm(1,0)\}$.
- See below two lattices L associated to matrices $B \in S^2_{>0}$ with $Min(B) = Min(A_{sqr})$:



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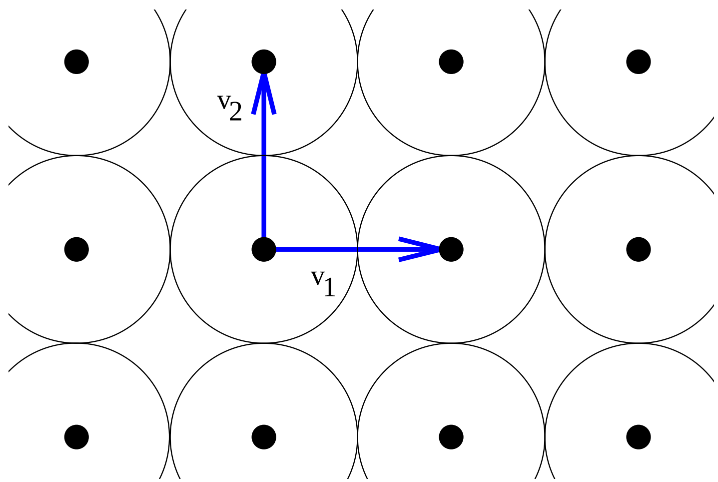


Perfect domains

- If $A \in S^n_{>0}$ is a perfect matrix then the perfect domain is

$$Dom(A) = \{ \sum_{v \in Min(A)} \lambda_v v v^T \text{ with } \lambda_v \geq 0 \}$$

- A perfect implies that $Dom(A)$ is full-dimensional in S^n.
- Thm. (Voronoi): Perfect domains form a face-to-face tesselation of $S^n_{>0}$.
- Thm. (Voronoi): In a fixed dimension n, there exist a finite number of perfect matrices $A_1, \dots, A_r \in S^n_{>0}$ such that for every perfect matrix $A \in S^n_{>0}$, there exists $P \in GL_n(\mathbf{Z})$ and $1 \leq i \leq r$ such that $A = P^T A_i P$

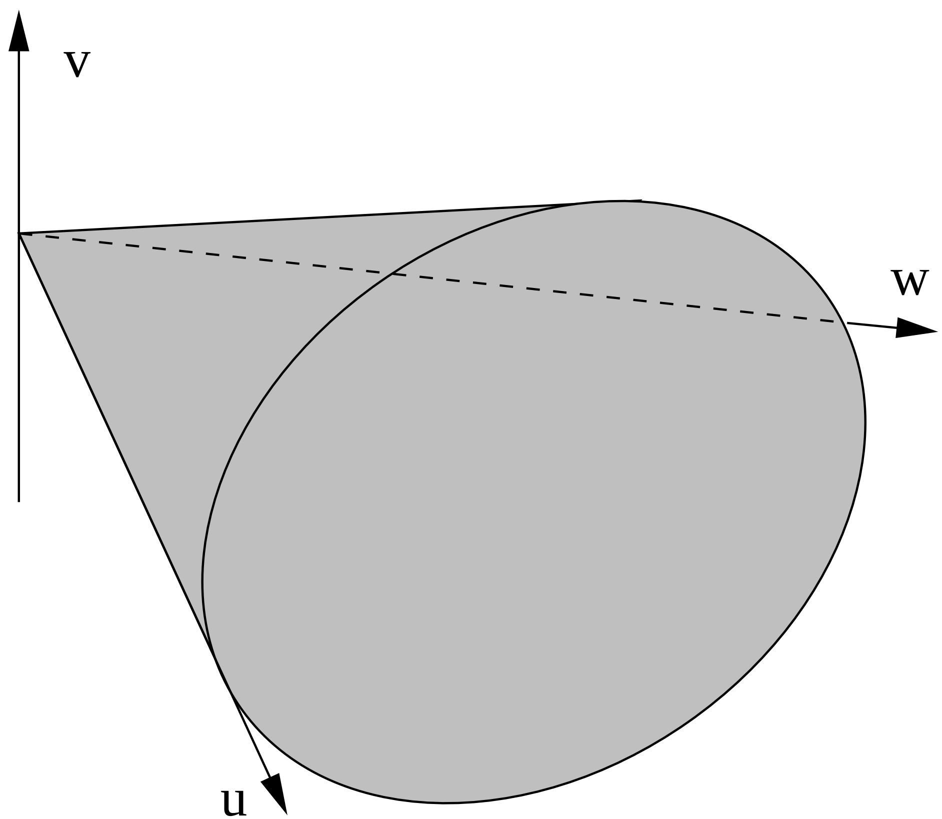
Enumeration of Perfect lattices

dim.	Nr. of perfect lattices	Absolute maximum of γ realized by
2	1 (Lagrange)	A_{hex}
3	1 (Gauss)	A_3
4	2 (Korkine & Zolotareff)	D_4
5	3 (Korkine & Zolotareff)	D_5
6	7 (Barnes)	E_6
7	33 (Jaquet)	E_7
8	10916 (Dutour, Schürmann & Vallentin)	E_8

We will explain Voronoi algorithm that allows the classification up to dimension 8 (but for dimension up to 5 Korkine-Zolotarev methods are sufficient).

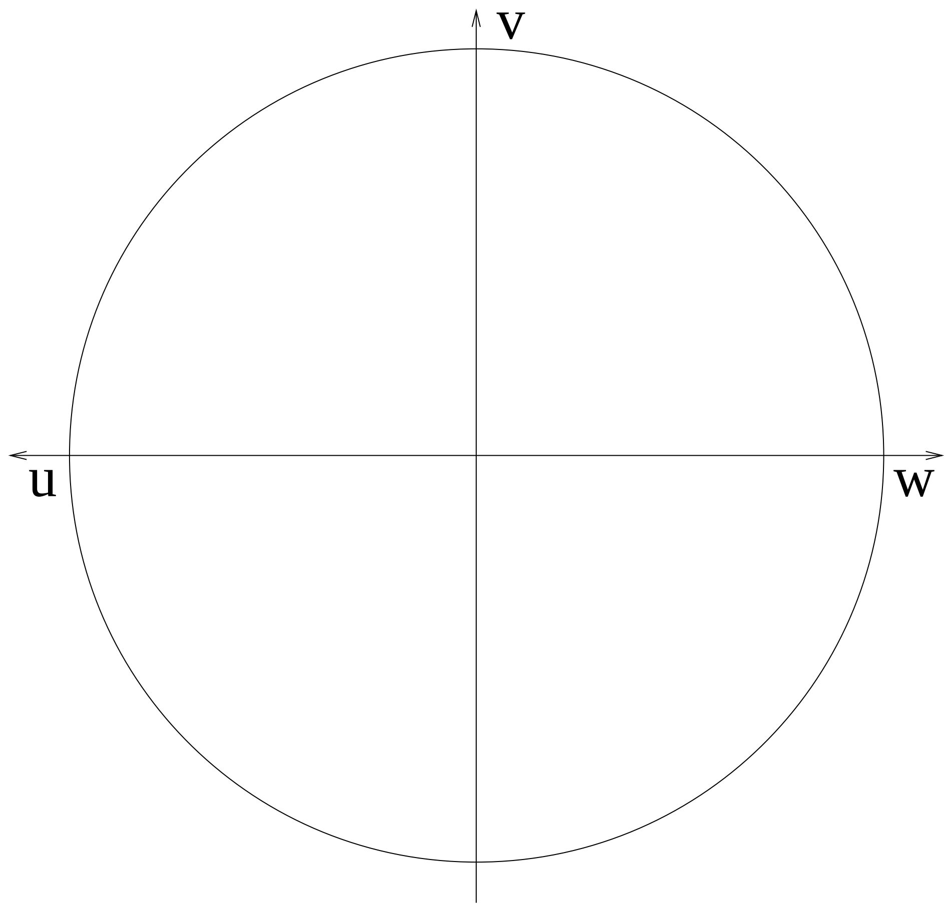
The partition of $S^2_{>0} \subset \mathbf{R}^3$

$\begin{pmatrix} u & v \\ v & w \end{pmatrix} \in S^2_{>0}$ if and only if $v^2 < uw$ and $u > 0$.



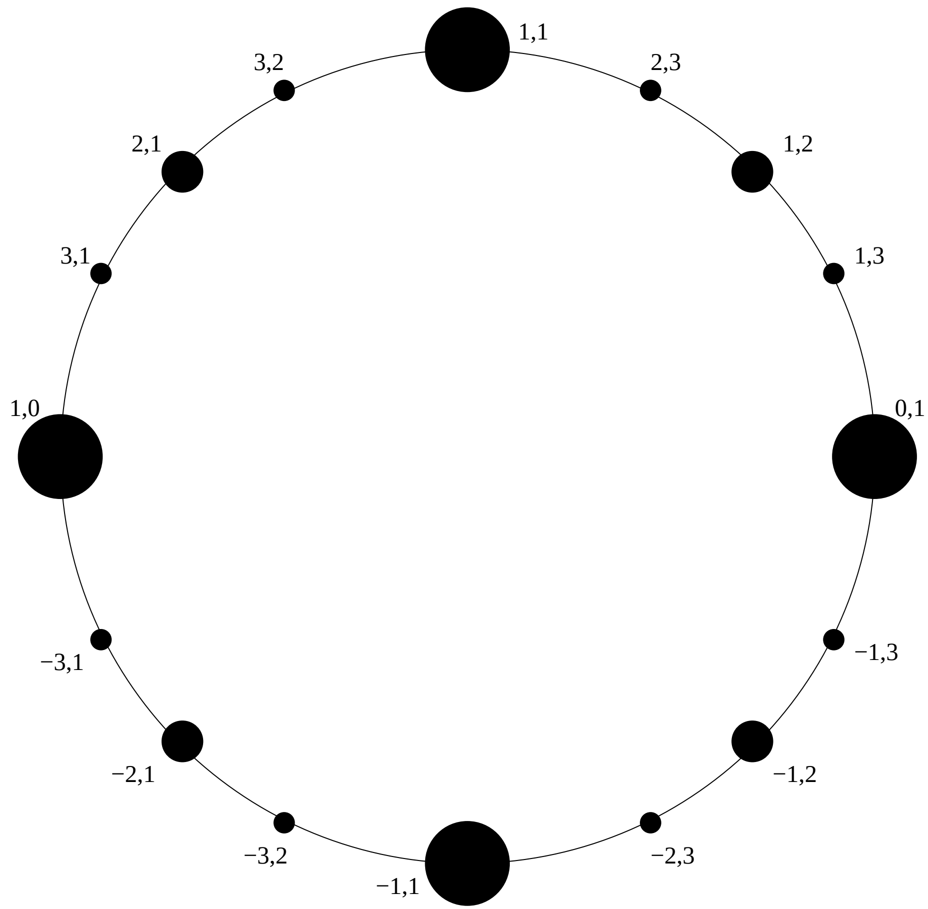
The partition of $S^2_{>0} \subset \mathbf{R}^3$

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



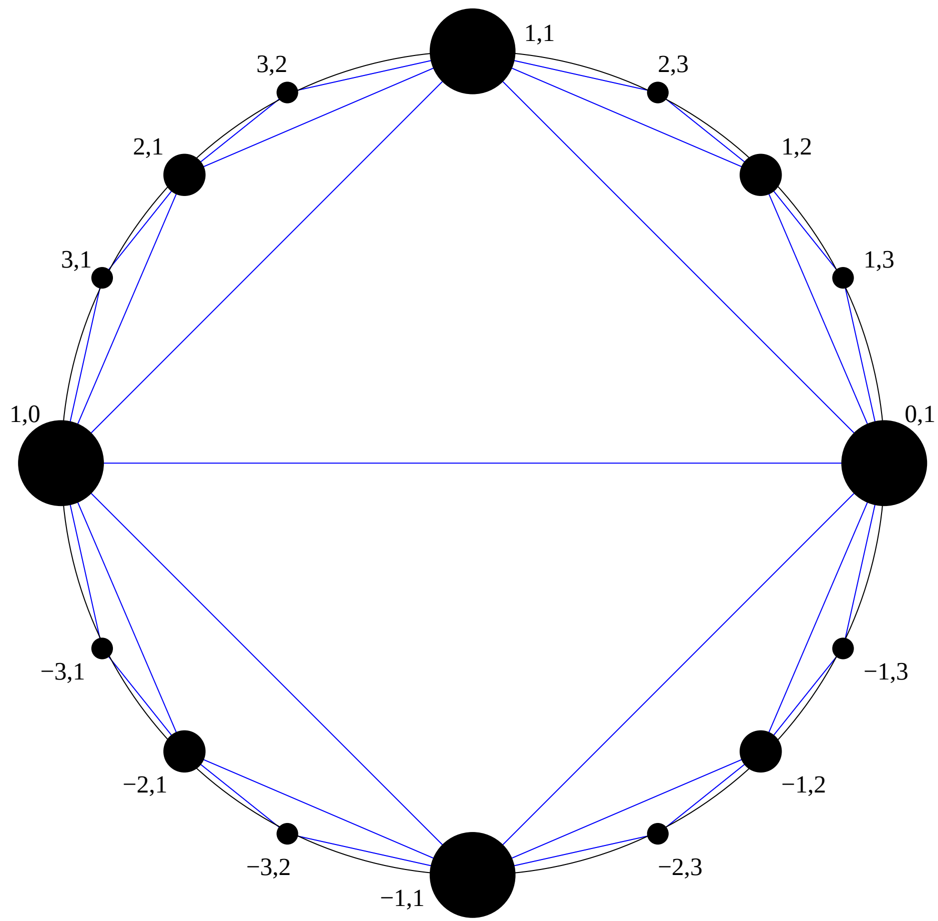
The partition of $S^2_{>0} \subset \mathbf{R}^3$

The rank 1 matrices $(a,b)(a,b)^T$ with $a, b \in \mathbf{Z}$ lie on the boundary of $S^2_{>0}$.



The partition of $S^2_{>0} \subset \mathbf{R}^3$

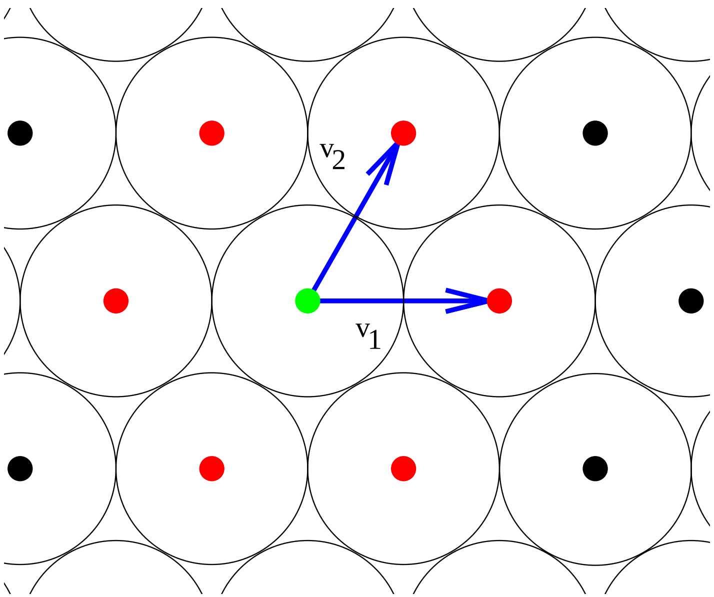
$S^2_{>0}$ partition: every triangle corresponds to a perfect domain $Dom(B)$ with B arithmetically equivalent to A_{hex}



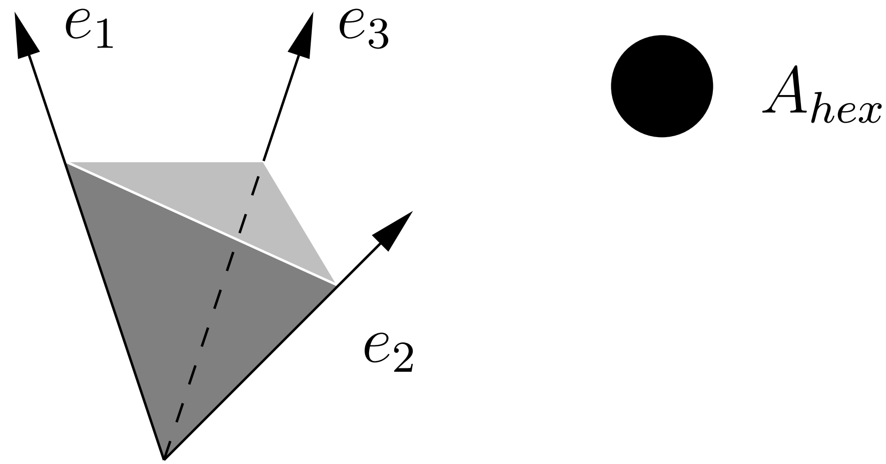
Voronoi algorithm

- Find a perfect matrix, insert it to the list as undone.
- Iterate
  - For every undone perfect matrix, compute the perfect domain and then its facets.
  - For every facet realize the flipping, i.e. compute the adjacent perfect domain (and perfect lattice).
  - If the perfect lattice is new, then we insert it into the list of perfect lattices as undone.
- Finish when all perfect domains have been treated.

Flipping on a facet



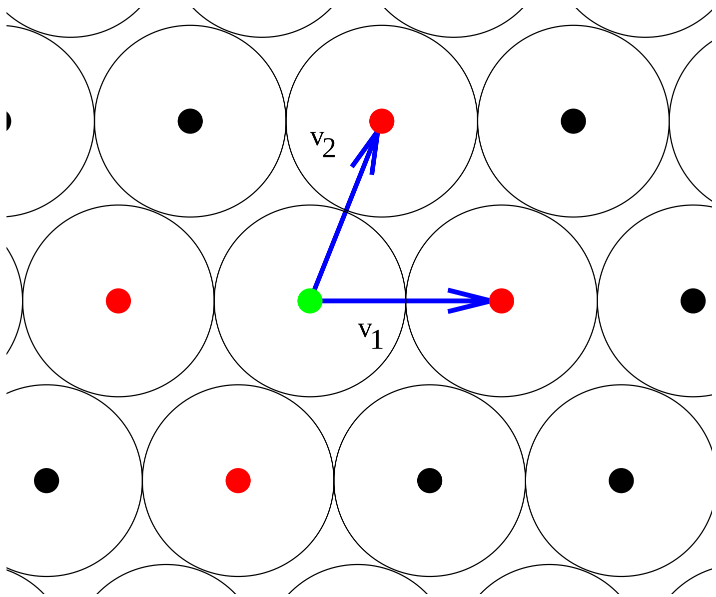
$$Min(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$$

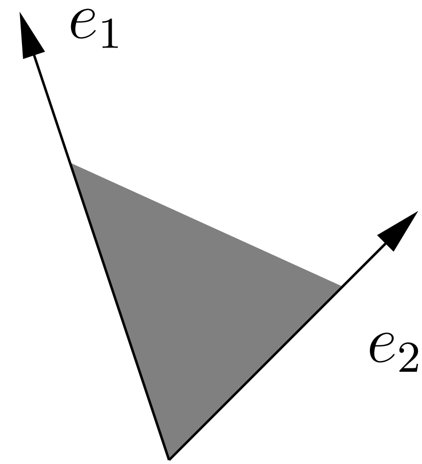


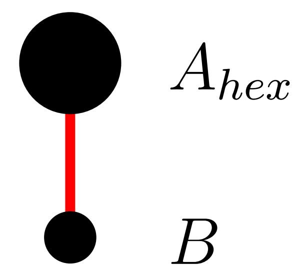
$$e_1 = (1,0)(1,0)^T$$
$$e_2 = (0,1)(0,1)^T$$
$$e_3 = (1,-1)(1,-1)^T$$

Flipping on a facet

$$Min(B) = \{\pm(1,0), \pm(0,1)\}$$



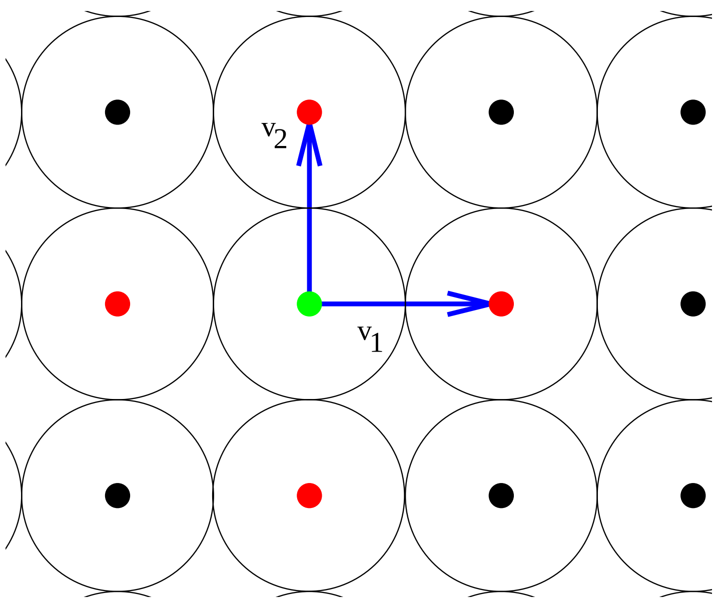


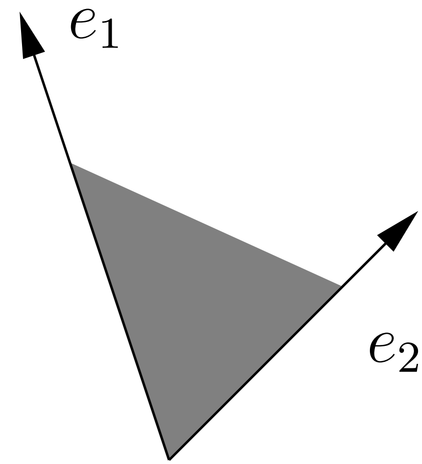


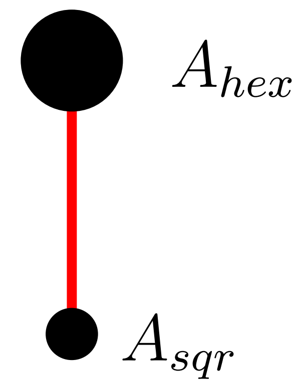
$$e_1 = (1,0)(1,0)^T$$
$$e_2 = (0,1)(0,1)^T$$

Flipping on a facet

$$Min(A_{sqr}) = \{\pm(1,0), \pm(0,1)\}$$



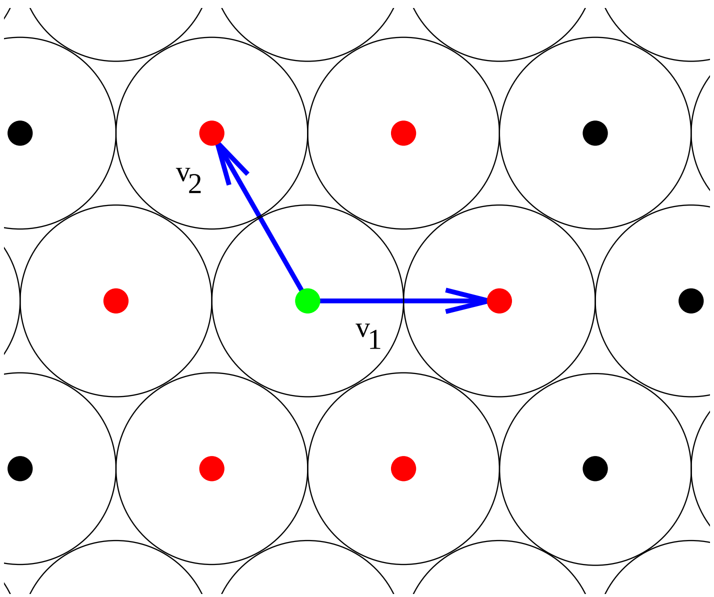


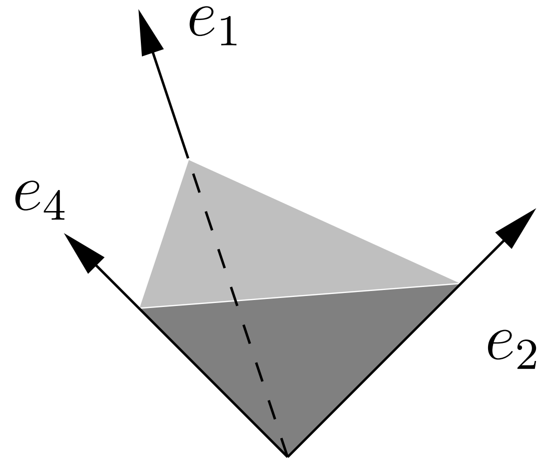


$$e_1 = (1,0)(1,0)^T$$
$$e_2 = (0,1)(0,1)^T$$

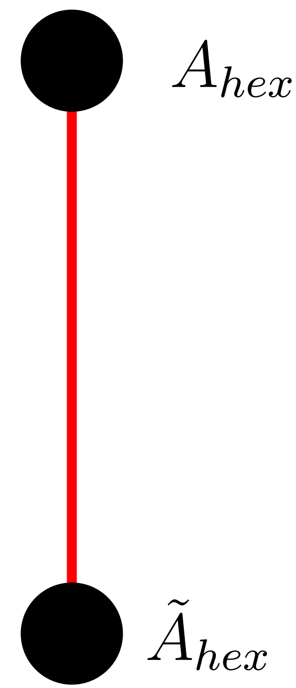
Flipping on a facet

$$Min(\tilde{A}_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$$

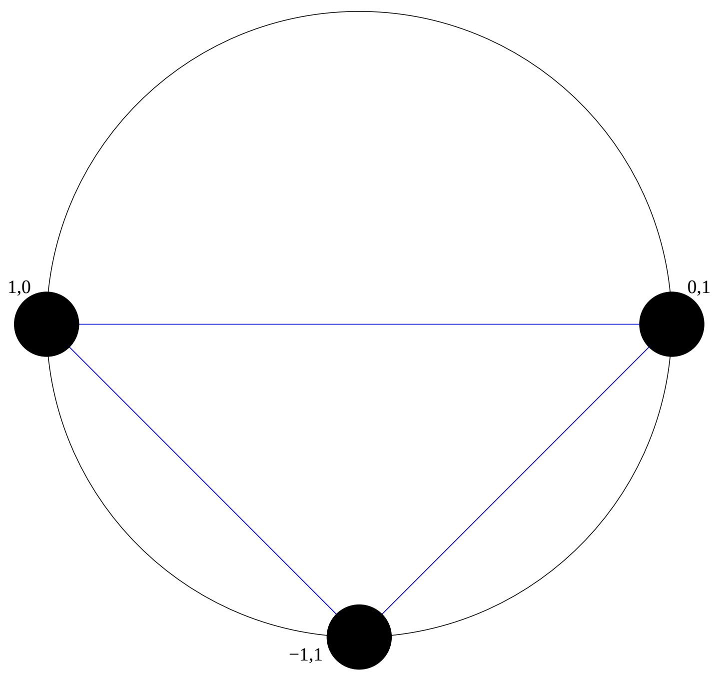




$$e_1 = (1,0)(1,0)^T$$
$$e_2 = (0,1)(0,1)^T$$
$$e_4 = (1,1)(1,1)^T$$

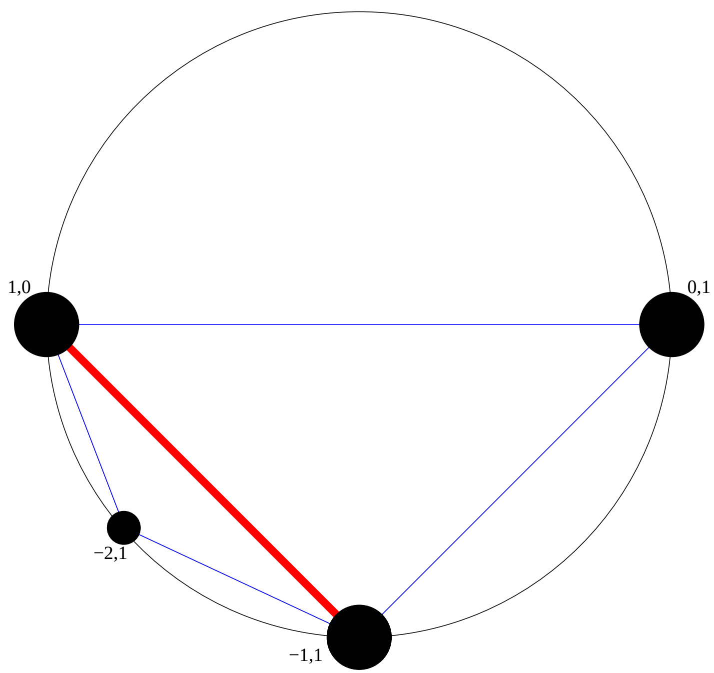


Adjacency of perfect domains



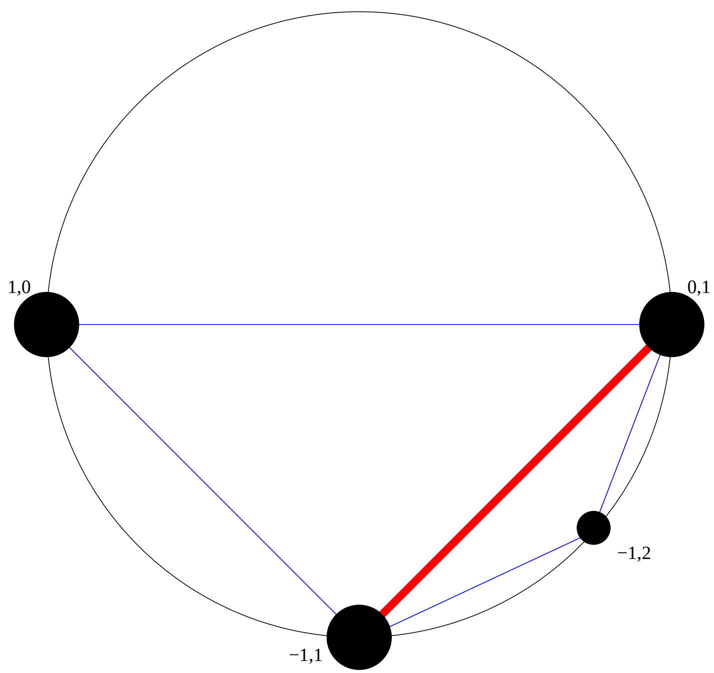
- $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- $Min(A_{hex}) = \{\pm(0,1), \pm(1,0), \pm(-1,1)\}$.
- $Dom(A_{hex})$ has three facets.

Adjacency of perfect domains



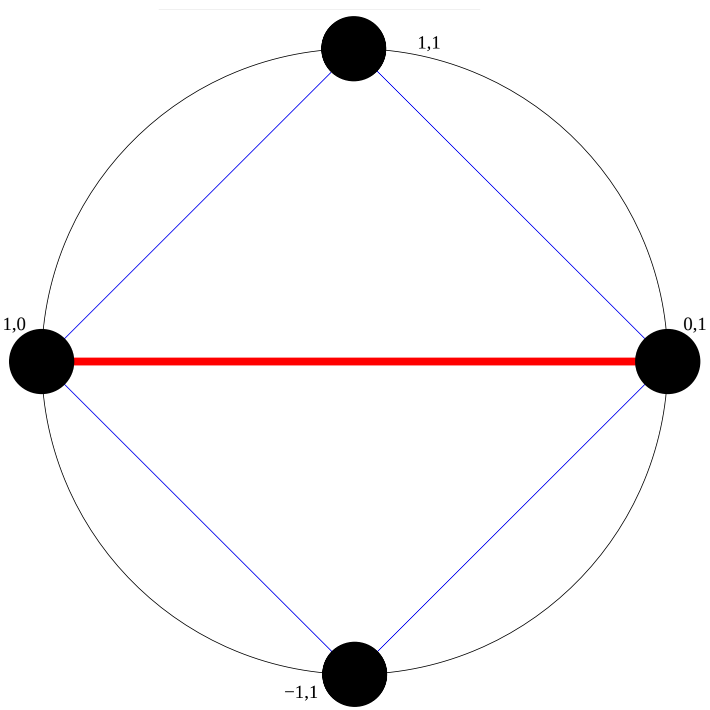
- $B_1 = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$
- $Min(B_1) = \{\pm(1,0), \pm(-2,1), \pm(-1,1)\}$.
- $Dom(A_{hex})$ and $Dom(B_1)$ share a facet.
- $B_1 = P^T A_{hex} P$ with $P = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$

Adjacency of perfect domains



- $B_2 = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$
- $Min(B_2) = \{\pm(0,1), \pm(-1,2), \pm(-1,1)\}$.
- $Dom(A_{hex})$ and $Dom(B_2)$ share a facet.
- $B_2 = P^T A_{hex} P$ with $P = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

Adjacency of perfect domains



- $B_3 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
- $Min(B_3) = \{\pm(0,1), \pm(1,0), \pm(1,1)\}$.
- $Dom(A_{hex})$ and $Dom(B_3)$ share a facet.
- $B_3 = P^T A_{hex} P$ with $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

This completes the enumeration of perfect form in dimension 2: matrices arithmetically equivalent to A_{hex}.

V. Korkine Zolotarev method

Korkine Zolotarev reduction

A matrix $A \in S^n_{>0}$ is called KZ-reduced if $f(x) = x^T A x$ can be expressed as

$$\begin{aligned} f(x) &= A_1(x_1 - \alpha_{12}x_2 \pm \cdots \pm \alpha_{1n}x_n)^2 \\ &+ A_2(x_2 - \alpha_{23}x_3 \pm \cdots \pm \alpha_{2n}x_n)^2 \\ &+ \cdots + A_{n-1}(x_{n-1} - \alpha_{n-1n}x_n)^2 + A_n x_n^2 \end{aligned}$$

with the following properties:

- $0 \leq \alpha_{ij} \leq \frac{1}{2}$
- A_1 is the minimum of the matrix A.
- If we put $x_n = 0$, then the above is KZ-reduced.

Properties

- Every matrix $A \in S^n_{>0}$ is arithmetically equivalent to a KZ-reduced one.
- Due to the invariance of γ by arithmetic equivalence, it suffices to solve the packing problem for KZ-reduced matrices.
- A_1 is the minimum of A.
- $A_1 \dots A_n$ is the determinant of A.
- The Hermite constant is expressed as

$$\gamma_n^n = \max_{A \ KZ\text{-reduced}} \frac{A_1^n}{A_1 \dots A_n}$$

In dimension 2

- We have $f(x_1, x_2) = A_1(x_1 - \alpha_{12}x_2)^2 + A_2x_2^2$
- $f(0,1) \geq A_1$ implies:

$$\left\{ \begin{array}{rcl} A_1(0 - \alpha_{12} \times 1)^2 + A_2 1^2 & \geq & A_1 \\ A_1\alpha_{12}^2 + A_2 & \geq & A_1 \\ A_2 & \geq & A_1(1 - \alpha_{12}^2) \end{array} \right.$$

- $0 \leq \alpha_{12} \leq \frac{1}{2}$ implies $1 - \alpha_{12}^2 \geq \frac{3}{4}$ and $A_2 \geq A_1 \frac{3}{4}$
- This implies

$$\gamma(A)^2 = \frac{A_1^2}{A_1 A_2} = \frac{A_1}{A_2} \leq \frac{4}{3} = \gamma(A_{hex})^2$$

- So, $\gamma_2 = \frac{2}{\sqrt{3}}$.

Best lattice packing

- Conjecture best lattice packing

dim.	Symbol	Name
9	Λ_9	Laminated lattice of dim. 9
10	K'_{10}	Coxeter Todd lattice of dim. 10
11	Λ_{11}	Laminated lattice of dim. 11
12	K_{12}	Coxeter Todd lattice of dim. 12
16	BW_{16}	Barnes-Wall lattice

- It is mysterious that there are so many beautiful lattices that are most likely of highest density but that we have no proof of their optimality.
- Perhaps global optimization using Korkine-Zolotarev method can help?

VI. Elkies-Cohn method

Fourier transform

- If f is an integrable function over $\mathbf{R}^n$, then

$$\hat{f}(t) = \int_{\mathbf{R}^n} f(x) e^{2\pi i \langle x,t \rangle} dx$$

- If L is a lattice, then the dual lattice is

$$L^* = \{ y \mid \langle x, y \rangle \in \mathbf{Z} \text{ for all } x \in L \}$$

- Poisson summation formula

$$\sum_{x \in L} f(x+v) = \frac{1}{det\, L} \sum_{t \in L^*} e^{-2\pi i \langle v,t \rangle} \hat{f}(t)$$

Fundamental theorem

- A function is admissible if $f(x)$ and $\hat{f}(x)$ are $O(|x|+1)^{-n-\delta}$ for some $\delta > 0$.
- Suppose $0 \neq f : \mathbf{R}^n \to \mathbf{R}$ is admissible and
  - $f(x) \leq 0$ for all x with $||x|| \geq 1$.
  - $\hat{f}(t) \geq 0$ for all t

  then the density of sphere packings (not only lattice sphere packings) is bounded from above by

$$\mathbf{upp}_n(f) = \frac{f(0)\kappa_n}{2^n \hat{f}(0)}$$

Partial proof of theorem

- Take a lattice L and suppose that $\min_{x\in L-\{0\}} ||x|| = 1$.
- Poisson formula and hypothesis implies:

$$\begin{array}{rcll} f(0) & \geq & \sum_{x\in L} f(x) & \text{by } f(x)\leq 0 \text{ if } ||x||\geq 1 \\ & \geq & \frac{1}{det\, L}\sum_{x\in L^*} \hat{f}(x) & \text{by Poisson formula} \\ & \geq & \frac{\hat{f}(0)}{det\, L} & \text{by } \hat{f}\geq 0 \end{array}$$

- As a consequence

$$\delta(L) = \frac{\lambda(L)^n \kappa_n}{det\, L} = \frac{\kappa_n}{2^n det\, L} \leq \frac{\kappa_n f(0)}{2^n \hat{f}(0)} = \mathbf{upp}_n(f)$$

- The above proof extends to general packings.

Finding good functions

- Cohn, Elkies & Kumar found functions f_2, f_8, f_{24} using polynomials of degree 800 such that

$$\begin{aligned} \mathbf{upp}_2(f_2) &\leq \delta(A_{hex})(1+10^{-10}) \\ \mathbf{upp}_8(f_8) &\leq \delta(E_8)(1+10^{-10}) \\ \mathbf{upp}_{24}(f_{24}) &\leq \delta(Leech)(1+10^{-30}) \end{aligned}$$

- Conjecture There exist some functions g_2, g_8, g_{24} that realize

$$\mathbf{upp}_2(g_2) = \delta(A_{hex}), \quad \mathbf{upp}_8(g_8) = \delta(E_8)$$
$$\text{and } \mathbf{upp}_{24}(g_{24}) = \delta(Leech)$$

- This would imply that A_{hex}, E_8 and $Leech$ lattice are the unique best packings in dimension 2, 8 and 24.

Application to lattice

- The lattices A_{hex}, E_8 and $Leech$ are extreme lattices (local maximum of packing density)
- Cohn & Kumar were able to prove that if a lattice has a higher density than A_{hex}, E_8 and $Leech$, then it is “not too far” from those lattices.
- Using the fact that an extreme matrix is a local maximum of the packing density and careful analysis, they were able to prove that

E_8 is best lattice packing in dimension 8

Leech is best lattice packing in dimension 24

VII. Symmetry method

Symmetry of lattices

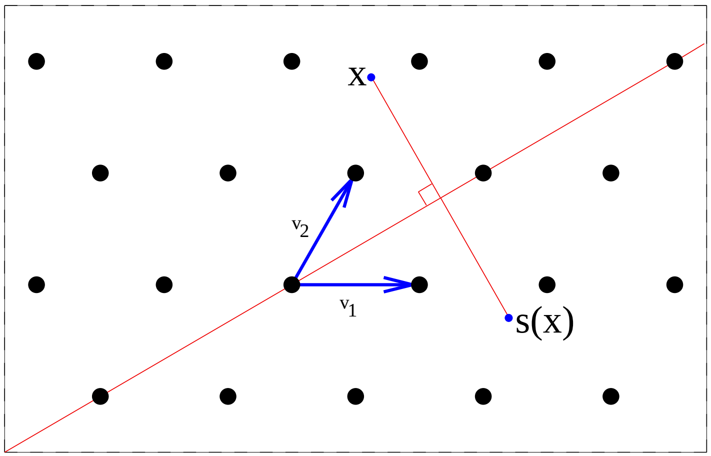
- A symmetry of a lattice L is an isometry u of $\mathbf{R}^n$ preserving 0 such that $L = u(L)$.
- If one selects a basis $\mathbf{v}$ of L and consider the Gram matrix $G_{\mathbf{v}}$, then a u corresponds to a matrix $P \in GL_n(\mathbf{Z})$ such that $G_{\mathbf{v}} = P^T G_{\mathbf{v}} P$.
- If $A \in S^n_{>0}$, then the symmetry group

$$Aut(A) = \{P \in GL_n(\mathbf{Z}) \mid A = P^T A P\}$$

is finite.

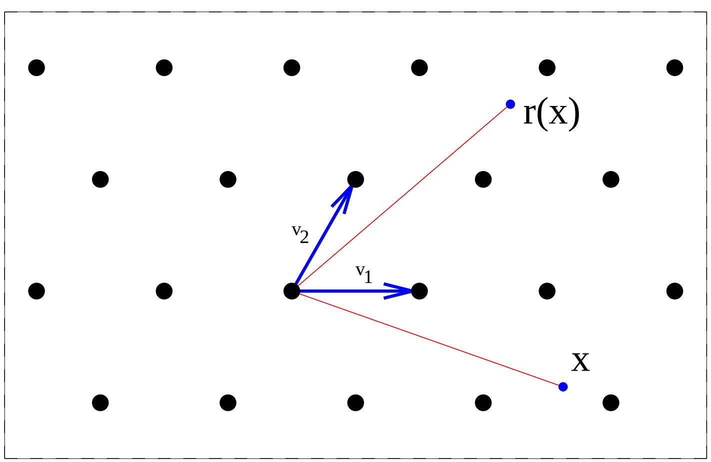
Hexagonal symmetries

- An orthogonal symmetry s:



$$Mat_{\mathbf{v}} s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- A rotation r of angle $\frac{\pi}{3}$:



$$Mat_{\mathbf{v}} r = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Matrix integral groups

- We want to consider lattices having a fixed symmetry group.
- We use the Gram matrix formalism.
- If G is a finite subgroup of $GL_n(\mathbf{Z})$, denote by

$$\mathcal{SP}(G) = \{A \in S^n \mid A = P^T A P \text{ for all } P \in G\}$$

- Two subgroups G_1, G_2 of $GL_n(\mathbf{Z})$ are conjugate if there exist $P \in GL_n(\mathbf{Z})$ such that $G_1 = P G_2 P^{-1}$
- Thm.(Zassenhaus): In dimension n, there is a finite number of subgroups of $GL_n(\mathbf{Z})$ (up to conjugacy).
- They are classified up to dimension 6.

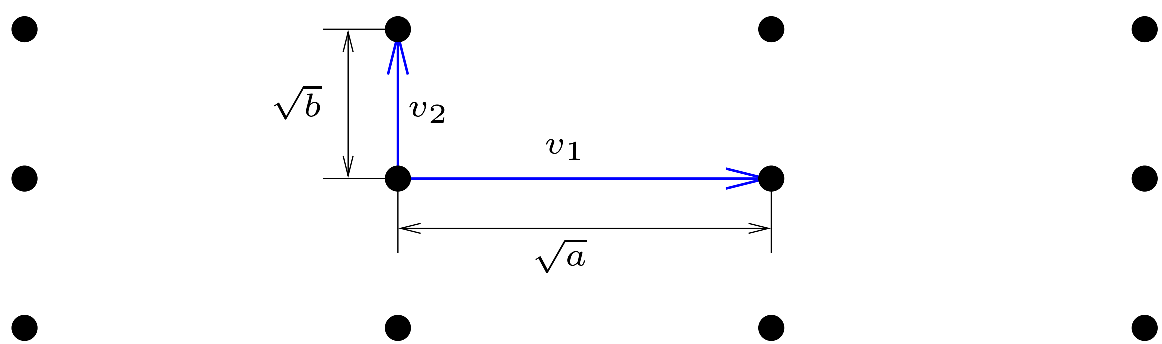
Two dimensional example

- Take the group G_4 formed by the 4 integral matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

- One has $\mathcal{SP}(G_4) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbf{R}\}$

- $\mathcal{SP}(G_4) \cap S^2_{>0} = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a > 0, b > 0\}$



G-perfect matrices

- If G is a finite subgroup of $GL_n(\mathbf{Z})$, we want to describe the packing density of Gram matrices corresponding to elements in $\mathcal{SP}(G)$.
- A matrix $A \in \mathcal{SP}(G)$ is G-perfect if:

$$B \in \mathcal{SP}(G) \text{ and } x^T Bx = min(A) \text{ for all } x \in Min(A)$$

  implies $B = A$.
- A matrix $A \in \mathcal{SP}(G)$ is G-extreme if it is a local maximum in $\mathcal{SP}(G)$ of γ.
- G-extreme $\Rightarrow$ G-perfect.

G-perfect domains

- If A is G-perfect then:
  - Partition $Min(A)$ into $Min(A) = O_1 \cup O_2 \cup \cdots \cup O_r$,
  - with $O_i = \cup_{g \in G} gx$ for some $x \in Min(A)$ (O_i is an orbit).
  - Define $p_i = \sum_{x \in O_i} xx^T$
  - Define the G-perfect domain by

$$Dom_G(A) = \{\sum_{i=1}^{r} \lambda_i p_i \text{ with } \lambda_i \geq 0\}$$

- Thm. (Bergé, Martinet & Sigrist): G-perfect domains realize a polyhedral subdivision of $\mathcal{SP}(G) \cap S^n_{>0}$.
- We can enumerate all G-perfect matrices with analogs of Voronoi algorithm.

VIII. Periodic structures

Motivation

- A packing $\mathcal{P}$ of $\mathbf{R}^n$ is called periodic if there is a lattice L such that the set of centers of $\mathcal{P}$ is of the form

$$\cup_{i=1}^{m} x^i + L \text{ with } x^i \in \mathbf{R}^n$$

  Lattices correspond to the case $m = 1$.

- If $\mathcal{P}$ is a packing of density δ, then for any $\epsilon > 0$, one can find a periodic packings of density $\geq \delta - \epsilon$. Hence, periodic packings approximate packings.

- We hope that by cleverly choosing $\{x^1, \dots, x^m\}$, we would be able to find packings of higher density than any lattice packings, i.e. that $\delta_n > \delta_n^*$. Possible dimensions are 4, 5, 6 and 7.

Matrix setting

- We want to vary the lattice, while keeping the same structure of the periodic structure.
- In algebraic terms we select some vector $x^i \in [0,1[^n$ and consider the set

$$X = \cup_{i=1}^m x^i + \mathbf{Z}^n$$

  with the norm $||x-y||_A = \sqrt{(x-y)^T A(x-y)}$ with $A \in S^n_{>0}$.
- If one authorizes some variation in x^i, then the setting becomes non-linear and almost impossible to compute.

Perfection

- Denote by $min(x^i, A)$, the shortest norm of the set $\{x^i\}_{i=1}^m$ is smallest norm $||\ .\ ||_A$ between any two elements of X.
- To $i \leq j$, one associates the set $X_{i,j}$ of vectors $v \in \mathbf{Z}^n$ such that $||v + x^i - x^j||_A = min(x^i, A)$
- A matrix $A \in S^n_{>0}$ is called (x^i)-perfect if the equation

$$B \in S^n \text{ and } ||v + x^i - x^j||_B = min(x^i, A) \text{ for all } v \in X_{i,j}$$

  implies $B = A$.
- One has analog of perfect domain, flipping and so on in that context.

THANK

YOU