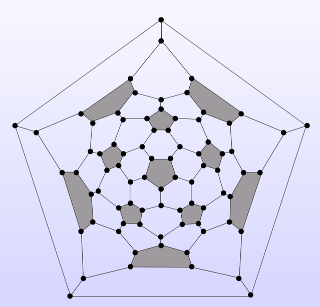
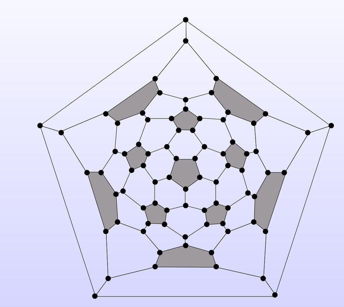


Goldberg-Coxeter construction for 3- or 4-valent plane graphs



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History

1. **Mathematics**: construction of planar graphs

   M. Goldberg, *A class of multisymmetric polyhedra*, Tohoku Math. Journal, **43** (1937) 104–108.

   Objective was to maximize the interior volume of the polytope, i.e. to find 3-dimensional analogs of regular polygons.

   ➠ search of equidistributed systems of points on the sphere for application to Numerical Analysis

2. **Biology**: explanation of structure of icosahedral viruses

   D.Caspar and A.Klug, *Physical Principles in the Construction of Regular Viruses*, Cold Spring Harbor Symp. Quant. Biol., **27** (1962) 1-24.

(k,l)	symmetry	capsid of virion
$(1,0)$	I_h	*gemini virus*
$(2,0)$	I_h	*hepathite B*
$(2,1)$	I, laevo	*HK97, rabbit papilloma virus*
$(3,1)$	I, laevo	*rotavirus*
$(4,0)$	I_h	*herpes virus, varicella*
$(5,0)$	I_h	*adenovirus*
$(6,3)$?	I, laevo	*HIV-1*

3. **Architecture**: construction of geodesic domes, Patent by Buckminster Fuller

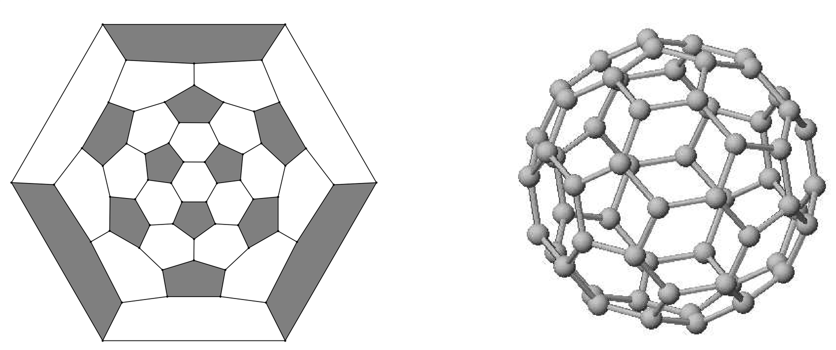
EPCOT in Disneyland.

4. **Mathematics**:

   H.S.M. Coxeter, *Virus macromolecules and geodesic domes*, in *A spectrum of mathematics*; ed. by J.C.Butcher, Oxford University Press/Auckland University Press: Oxford, U.K./Auckland New-Zealand, (1971) 98–107.

5. **Chemistry**: Buckminsterfullerene C_{60} (football, Truncated Icosahedron)

   Kroto, Kurl, Smalley (Nobel prize 1996) synthetized in 1985 a new molecule, whose graph is $GC_{1,1}(Dodecahedron)$. Osawa constructed theoretically C_{60} in 1984.



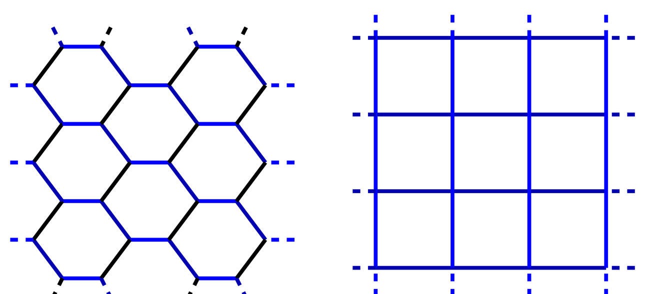
Euler formula

If G is a plane graph, then one has

$$V - E + F = 2$$

with V=Nr. vertices, E=Nr. edges and F=Nr. faces.

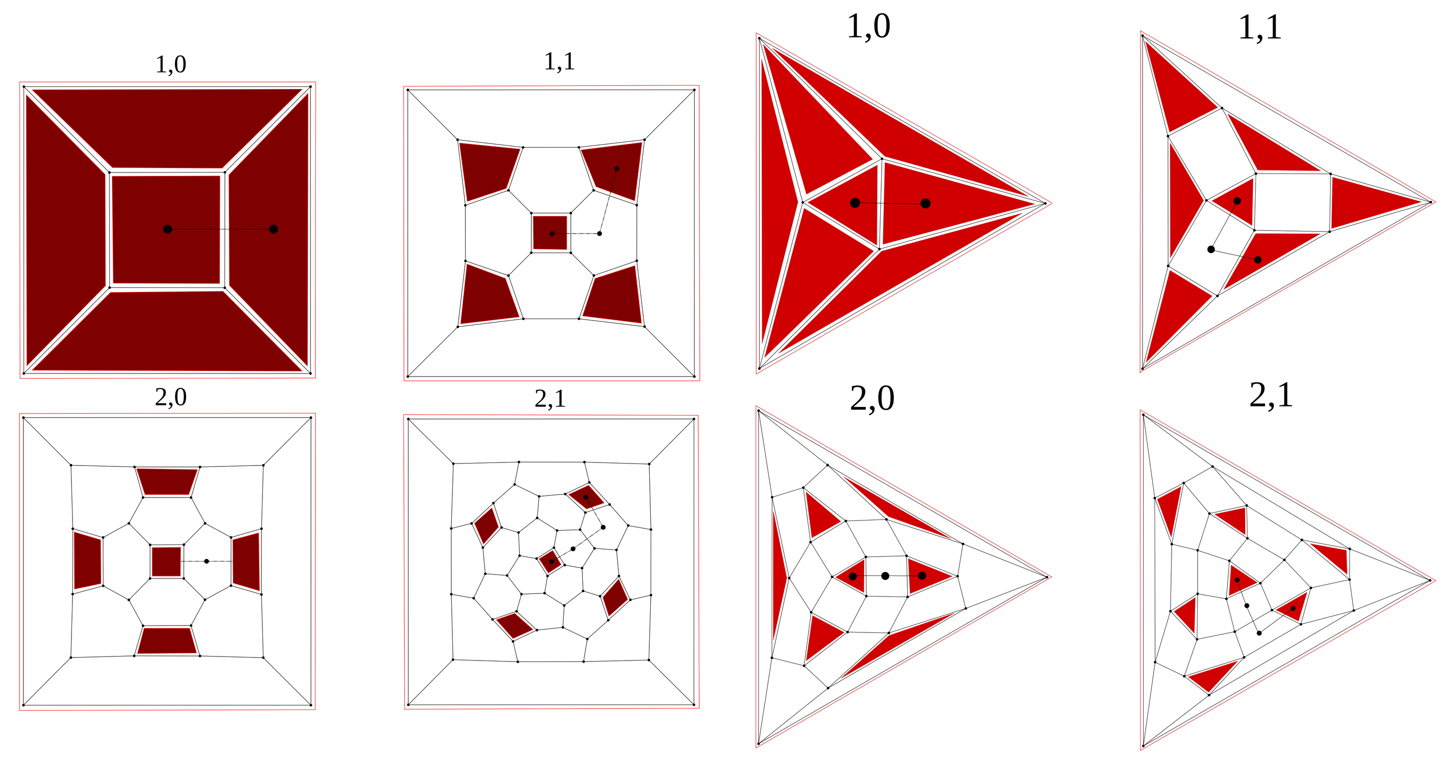
3-valent case $12 = \sum_i (6-i)p_i$
4-valent case $8 = \sum_i (4-i)p_i$



hexagon and squares are said to be of zero-curvature

The Goldberg-Coxeter construction

The Goldberg-Coxeter contruction take a 3- or 4-valent plane graph G_0, two integers k, l and build another 3- or 4-valent plane graph $GC_{k,l}(G_0)$.



If G_0 has n vertices, then $GC_{k,l}(G_0)$ has

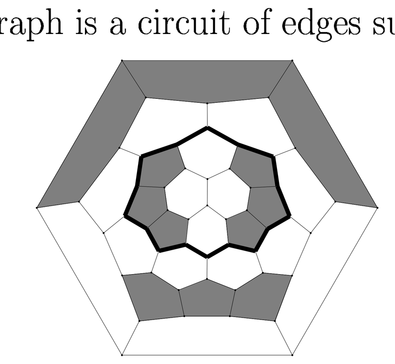
$$n(k^2+kl+l^2) \text{ vertices if } G_0 \text{ is 3-valent} \quad n(k^2+l^2) \text{ vertices if } G_0 \text{ is 4-valent}$$

- q_n is the class of 3-valent plane graphs having only q- and 6-gonal faces.
- The class of 4-valent plane graphs having only 3- and 4-gonal faces is called Octahedrites.

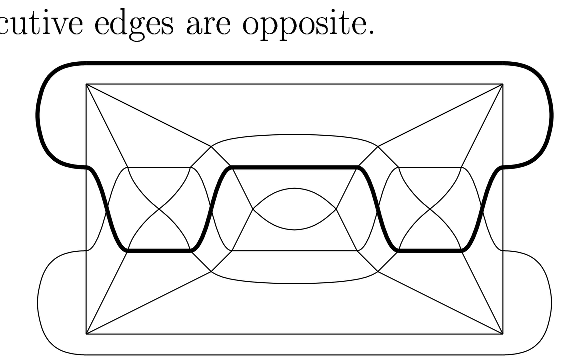
Class		Groups	Construction
3_n	$p_3 = 4$	T, T_d	$GC_{k,l}$(Tetrahedron)
4_n	$p_4 = 6$	O, O_h	$GC_{k,l}$(Cube)
4_n	$p_4 = 6$	D_6, D_{6h}	$GC_{k,l}$(Prism$_6$)
5_n	$p_5 = 12$	I, I_h	$GC_{k,l}$(Dodecahedron)
Octahedrites	$p_3 = 8$	O, O_h	$GC_{k,l}$(Octahedron)

The zigzags and central circuits

1. (i) a zigzag (also called petrie polygon, left-right path, geodesic) in a 3-valent plane graph is a circuit of edges such that any two, but no three, consecutive edges belong to the same face.
   (ii) a central circuit (also called traverse, straight-ahead, straight Eulerian, cut-though) in a 4-valent plane graph is a circuit of edges such that any two consecutive edges are opposite.



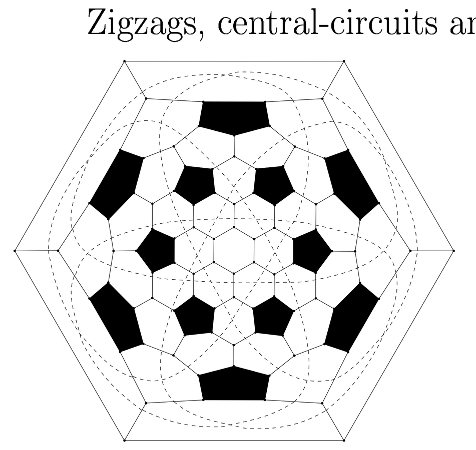
Example of a zigzag in a 3-valent plane graph.



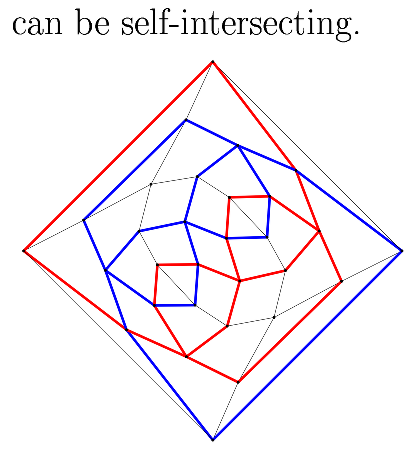
Example of a central-circuit in a 4-valent plane graph.

2. A railroad in a 3- or 4-valent map is a circuit of hexagons or squares, adjacent on two of its neighbors by opposite edges.

   Zigzags, central-circuits and railroad can be self-intersecting.



A self-intersecting railroad in 3-valent graph



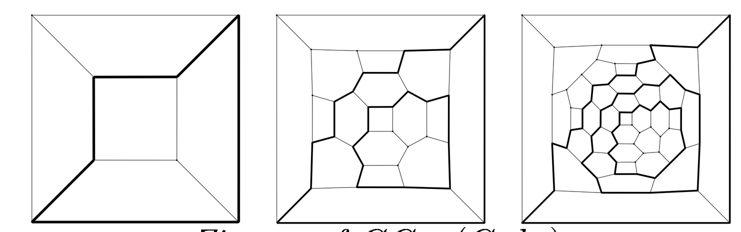
A self-intersecting railroad in 4-valent graph.

3. A graph is tight if it has no railroad.

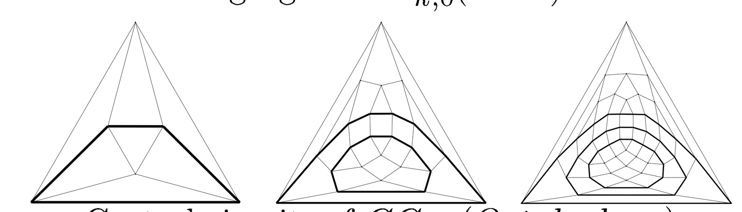
Class	maximal number of zigzags/central circuits
octahedrite	6
3_n	3
4_n	at most 9 and conjecturally 8
5_n	at most 30 and conjecturally 15

Computing ZC-circuits of $GC_{k,l}(G_0)$

1. **If** $gcd(k,l) = 1$, then $GC_{k,l}(G_0)$ is tight.
2. **The special case** $GC_{k,0}$: Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.



Zigzags of $GC_{k,0}(Cube)$



Central circuits of $GC_{k,0}(Octahedron)$

3. $gcd(k,l) = 1$**:** We associate to G_0 two elements L and R, which are permutation of the directed edges of G_0. The length of ZC-circuits of $GC_{k,l}(G_0)$ is obtained from the cycle structure of $L \odot_{k,l} R$. The product $L \odot_{k,l} R$

$$(k,l) = (5,2), \text{ Sequence: } 1,3,5,7,2,4,6,1 \text{ product: } L \odot_{5,2} R = RLLRLLL$$

$$\begin{cases} L \odot_{k,l} R = L \odot_{k-ql,l} RL^q & \text{if } k - ql \geq 0 \\ L \odot_{k,l} R = R^q L \odot_{k,l-qk} R & \text{if } l - qk \geq 0 \end{cases}$$

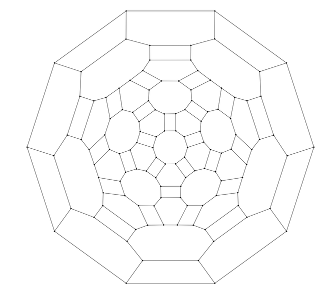
4. **Cube Case**:
   - L and R do not commute ➠ $L \odot_{k,l} R \neq Id$.
   - $Mov(Cube) = \langle L, R \rangle = Alt(4)$
   - $K = \langle (1,2)(3,4), (1,3)(2,4) \rangle$ normal subgroup of index 3 of $Alt(4)$. $\overline{L}$ is of order 3.

$$\begin{cases} \overline{L \odot_{k,l} R} = \overline{L}^k \overline{R}^l = \overline{L}^{k-l} \\ L \odot_{k,l} R \in K \Leftrightarrow k - l \text{ divisible by } 3 \end{cases}$$

   - Elements of $Alt(4) - K$ have order 3. Elements of $K - \{Id\}$ have order 2.

   ➠ $GC_{k,l}$(Cube) has 6 zigzags if $k \equiv l \pmod 3$ and 4 zigzags, otherwise

5. **For a given graph** G_0, one can compute the list of possible lengths of zigzags in finite time.



$2^{30}, 3^{40}$	$2^{30}, 5^{24}$	$3^{20}, 5^{24}$
$2^{60}, 3^{20}$	$2^{60}, 5^{12}$	$3^{40}, 5^{12}$
2^{90}	3^{60}	5^{36}
9^{20}	6^{30}	15^{12}

References

[DeDu02] M. Deza and M. Dutour, *Zigzag structure of Simple Two-faced Polyhedra*, http://www.arxiv.org/abs/math.GT/0212352, to appear in Combinatorics, Probability & Computing, Special Issue in memory of W. Deuber (2004).

[DDF] M.Deza, M. Dutour and P. Fowler, *Zigzags, Railroads and Knots in Fullerenes*, Journal of Chemical Information and Computer Sciences, in press, http://pubs.acs.org/cgi-bin/asap.cgi/jcisd8/asap/abs/ci049955h.html

[DDS] M. Deza, M.Dutour and M. Shtogrin, *4-valent plane graphs with 2-, 3- and 4-gonal faces*, "Advances in Algebra and Related Topics" (in memory of B.H.Neumann; Proceedings of ICM Satellite Conference on Algebra and Combinatorics, Hong Kong 2002), World Scientific Publ. Co. (2003) 73–97.

[DeSt03] M. Deza and M. Shtogrin, *Octahedrites*, Polyhedra. Symmetry: Culture and Science. The Quarterly of the International Society for the Interdisciplinary Study of Symmetry, **11/1-4** (2000) 27–64.

[DD04] M. Dutour and M. Deza, *Goldberg-Coxeter construction for 3- or 4-valent plane graphs*, Electronic Journal of Combinatorics, **11-1** (2004) R20.