Delaunay and Voronoi partitions of lattices

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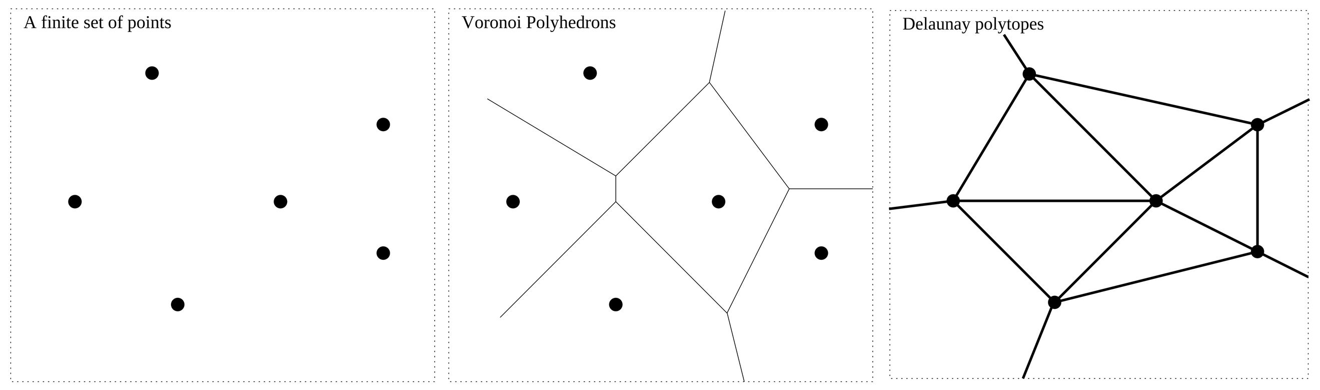
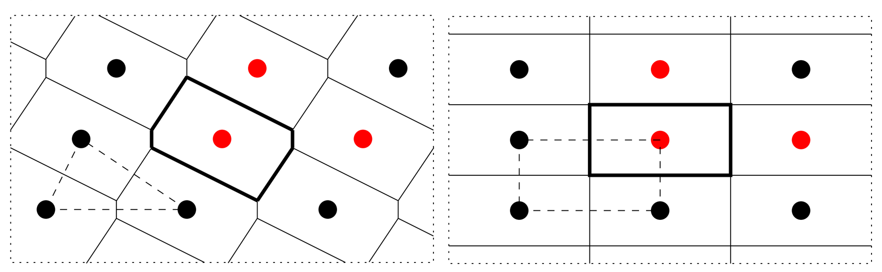
Introduction



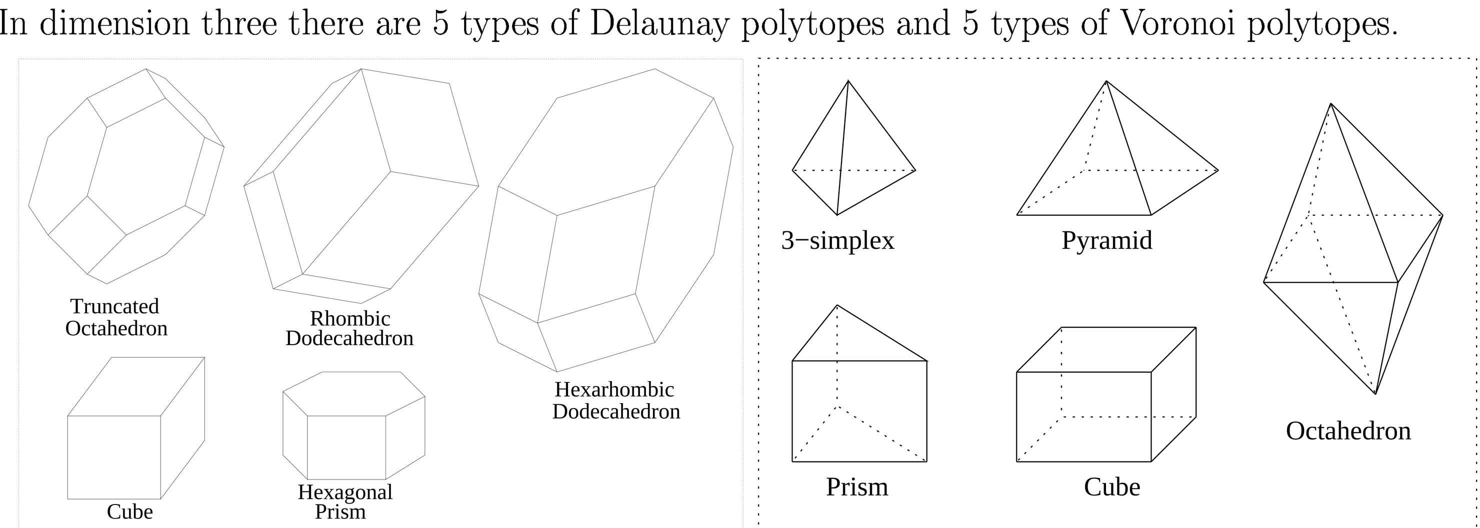
Name	Relevant field of knowledge
Dirichlet domains	lattice theory, 2-dimensional case
Voronoi polytope	n-dimensional lattice, computational geometry
Thiessen polygons	geography
Wigner-Seitz cell	solid state physic, crystallography
First Brillouin zone	solid state physic, momentum space
Domain of influence	politics

Low dimension result

A lattice of R^n is a discrete subgroup of R^n. Lattice appear in Lie theory as root lattice, Coding Theory, Crystallography, Number Theory.
There are two kinds of Voronoi polytope (Hexagon and rectangle) and two kinds of Delaunay polytopes (triangle and rectangle) in dimension two:



In dimension three there are 5 types of Delaunay polytopes and 5 types of Voronoi polytopes.



Voronoi Polytopes   Delaunay polytopes

dimension	1	2	3	4	5	6	7
Nr. Voronoi polytopes	1	2	5 Fedorov	52 DeSh	179377 Engel	?	?
Nr. Delaunay polytopes	1	2	5 Fedorov	19 Erdahl Ryshkov	138 Kononenko	6241 Dutour	?
Nr. extreme Delaunay	1	0	0	0	0	1 DeDu	≥ 1

The hypermetric cone

An abstract metric on $n+1$ point $\{0, 1, \dots, n\}$ is an application

$$\begin{array}{l} d: R^{n(n+1)/2} \to R \\ \quad (i,j) \mapsto d(i,j) \end{array} \text{, which satisfy to} \quad \begin{array}{c} d(i,i) = 0 \quad \text{and} \\ d(i,j) \leq d(i,k) + d(k,j) \end{array}$$

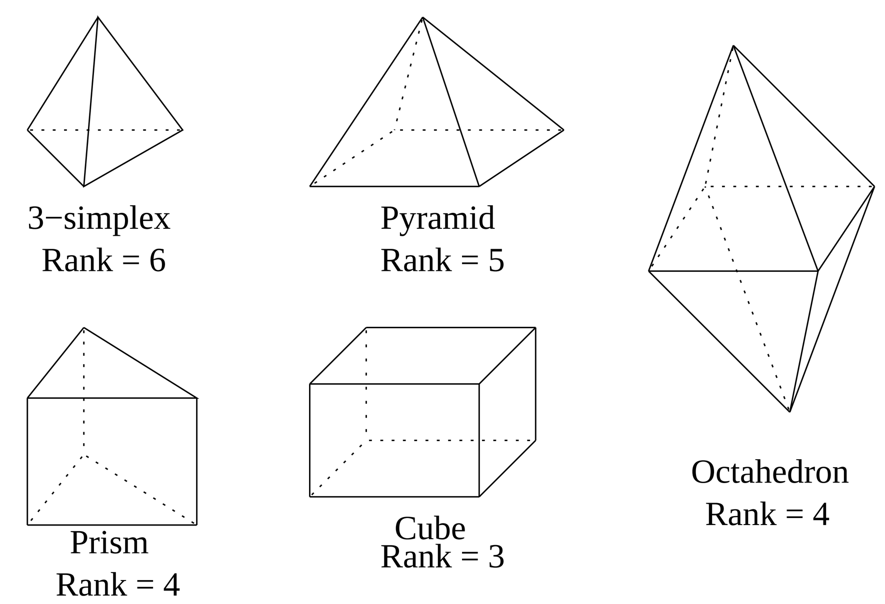
Given a vector $b = (b_0, b_1, \dots, b_n)$ with $b_i \in Z$ and $\sum_{i=0}^n b_i = 1$, one defines the hypermetric inequality for a metric d:

$$H(b)d = \sum_{0 \leq i < j \leq n} b_i b_j d(i,j) \leq 0$$

The hypermetric cone HYP_{n+1} is defined as the set of metrics on $\{0, \dots, n\}$ that satisfy to all hypermetric inequalities.
HYP_{n+1} is a polyhedral cone, which means we can use the notion of dimension. The dimension of faces of HYP_{n+1} varies from 1 to $\frac{n(n+1)}{2}$.

Theorem 1 *There is a correspondence between Delaunay polytopes and hypermetrics: given a Delaunay polytope P, we can select an affine basis $\{v_0, \dots, v_n\}$. The distance $d(i,j) = \|v_i - v_j\|^2$ is an hypermetric.*

This allows to define the rank of a Delaunay polytope as the dimension of the face in which it is contained or in other words as its number of degrees of freedom.



Extreme Delaunay in high dimension

A Delaunay polytope whose rank is equal to 1 is called *extreme*. A Delaunay polytope P is extreme if and only if the only transformations $f: R^n \to R^n$ such that $f(P)$ is a Delaunay polytope are isometries and homotheties.
The first non-trivial extreme Delaunay polytope is the Schläfli polytope, which is the unique Delaunay polytope of the root lattice E_6.

$$E_6 = \{x \in E_8 : x_1 + x_2 = x_3 + \dots + x_8 = 0\}$$
$$E_8 = \{x \in Z^8 \cup (\tfrac{1}{2} + Z)^8 \text{ and } \textstyle\sum_i x_i \in 2Z\}$$

Schläfli polytope has a symmetry group of size 51840.
Root lattice E_6 is obtained by a section of root lattice E_8 (which has an even bigger symmetry group).
So, an idea would be to consider other highly symmetric lattices, do section of them and then, possibly, obtain extreme Delaunay polytopes:

Name	dimension	Nr. vertices	section of
Schläfli	6	27	E_8
Gosset	7	56	E_8
	16	512	Barnes Wall
B_{15}	15	135	Barnes Wall
	22	275	Leech
	23	552	Leech

This idea works!

The polyhedral method

The condition of being extreme correspond to being an extreme ray (think about one vertex in the cube) in the hypermetric cone HYP_{n+1}.
Using standard polyhedral technique it is possible, given an extreme ray e of a cone to find the adjacent extreme rays of this cone (think about the 3 adjacent vertex of a given vertex v of the cube).

The only drawback is that we need to select the right inequality $H(b)d$ of the hypermetric cone HYP_{n+1}. This is done by using the solution of the Closest Vector Problem:

Problem 1 *Given a lattice L in R^n and a vector v, find one vector $l \in L$ (or all vectors $l \in L$), which has $\|v - l\|$ minimal.*

This is a classic problem of geometry; it is proven to be Non-Polynomial, i.e. one cannot avoid doing an exhaustive enumeration of all possibilities and the number of vectors to consider grow exponentially with n.
This problem is key to all polyhedral computation with Voronoi and Delaunay, it is also used in Cryptography and in Coding Theory.
So, we take the Delaunay polytope B_{15} find the corresponding extreme ray and find the adjacent extreme rays. It happens that 77 of those extreme rays correspond to an extreme Delaunay polytope ED_8 of dimension 8.
The f-vector of this polytope is

$$\begin{array}{lll} f_0 = & 79 & \text{number of vertices of } ED_8 \\ f_1 = & 1268 & \text{number of edges of } ED_8 \\ f_2 = & 7896 & \text{number of 2-faces of } ED_8 \\ f_3 = & 23520 & \vdots \\ f_4 = & 36456 & \vdots \\ f_5 = & 29876 & \vdots \\ f_6 = & 11364 & \vdots \\ f_7 = & 1131 & \text{number of facets of } ED_8 \\ f_8 = & 1 & ED_8 \text{ itself} \end{array}$$

Infinite construction

We analyse mathematically ED_8 and manage to build an infinite sequence of extreme Delaunay polytopes.
This construction uses root lattice D_n:

$$D_n = \{x \in Z^n \text{ such that } \sum_{i=1}^n x_i \text{ is even}\}$$

Proposition 1 *If n even, $n \geq 6$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice*

- *a vertex*
- *the $n-1$ half-cube*
- *the $n-1$ cross-polytope*

Case $n = 6$ corresponds to Schläfli polytope.

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