Classifying spaces from polyhedral tesselations: the perfect form method

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I. Problem setting

Group Homology

- Take G a group, suppose that:
  - X is a contractible space.
  - G act fixed point free on X.

  Then we define the group homologies of G to be $H_p(G) = H_p(X/G)$.
- The space X is then a classifying space.
- Examples:
  - The bar construction gives a classifying space which can be used to compute with general groups.
  - If G is a Bieberbach group (acts fixed point free on $\mathbb{R}^n$) then $\mathbb{R}^n$ is the classifying space and the homology is the one of a flat manifold.
- Getting workable classifying space for a group is not easy:
  - If G is finite then $H_i(G) \neq 0$ for an infinity of i and thus X is infinite dimensional.
  - Thus one hopes to work out some "approximate classifying space" and obtain the homology by perturbation arguments.

The case of $\mathrm{GL}_n(\mathbb{Z})$

- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{R}^n$.
- So a priori, it would seem that the approximate classifying space would be $\mathbb{R}^n$. But the stabilizer of a point $x \in \mathbb{R}^n$ can be infinite or $\mathrm{GL}_n(\mathbb{Z})$ itself.
- So, we would like another space X on which $\mathrm{GL}_n(\mathbb{Z})$ could act. Our wishes are for:
  - X to be contractible.
  - X to admit a cell decomposition (polyhedral tesselation) invariant under $\mathrm{GL}_n(\mathbb{Z})$.
  - That every face F of the tesselation has finite stabilizer under $\mathrm{GL}_n(\mathbb{Z})$.

Positive definite quadratic forms

- A matrix Q is called positive definite, respectively positive semidefinite, if for every $x \in \mathbb{R}^n - \{0\}$ we have

$$x^t Qx > 0, \text{ respectively } x^t Qx \geq 0.$$

- Denote by $S^n_{>0}$, respectively $S^n_{\geq 0}$ the cones of positive definite, respectively positive semidefinite $n \times n$-matrices.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by the relation

$$(P, Q) \mapsto P^t QP$$

- For any $Q \in S^n_{>0}$ the automorphism group

$$\mathrm{Aut}(Q) = \{P \in \mathrm{GL}_n(\mathbb{Z}) \text{ such that } P^t QP = Q\}$$

is finite.

Why use perfect forms?

- They satisfy the necessary condition of being a polyhedral tesselation with finite stabilizers (More to that later).
- They are computationally expensive, i.e. only up to dimension 8.
- But other decomposition are worse:
    - The L-type domain tesselation is not effective beyond dimension 5.
    - The Minkovski domain method gives only one domain and with trivial stabilizer but it has a lot of facets and extreme rays.
- We do not explain the geometric aspect of perfect forms.

References

- G. Voronoi, *Nouvelles applications des paramètres continues à la théorie des formes quadratiques 1: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math **133** (1908) 97–178.
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- A. Schuermann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.
- J. Martinet, *Perfect lattices in Euclidean spaces*, Springer, 2003.
- S.S. Ryshkov, E.P. Baranovski, *Classical methods in the theory of lattice packings*, Russian Math. Surveys **34** (1979) 1–68, translation of Uspekhi Mat. Nauk **34** (1979) 3–63.

II. Perfect forms

Perfect form

- If $A \in S^n_{>0}$ then define $\min(A) = \min_{v \in \mathbb{Z}^n \neq 0} A[v]$ and

$$\mathrm{Min}(A) = \{x \in \mathbb{Z}^n \text{ such that } A[x] = \min(A)\}$$

- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:

$$Q \mapsto P^t QP$$

  and we have $\mathrm{Min}(P^t QP) = P^{-1}\,\mathrm{Min}(Q)$.
- A form is called perfect if the equation in B

$$B[v] = \min(A) \text{ for all } v \in \mathrm{Min}(A)$$

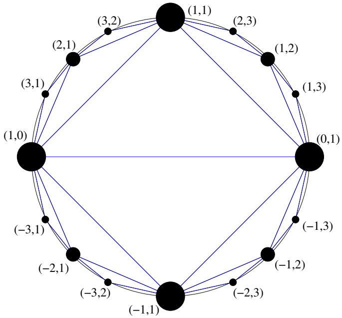
  implies $B = A$.
- A perfect form is necessarily rational and thus up to a multiple integral.
- There is a finite number of perfect forms up to $GL_n(\mathbb{Z})$ equivalence.

Perfect domains

- If $v \in \mathbb{Z}^n$ then the corresponding rank 1 form is $p(v) = {}^t vv$.
- If A is a perfect form, its perfect domain is

$$\mathrm{Dom}(A) = \sum_{v \in \mathrm{Min}(A)} \mathbb{R}_+ p(v)$$

- If A has m shortest vectors then $\mathrm{Dom}(A)$ has $\frac{m}{2}$ extreme rays.
- The perfect domains define a polyhedral tesselation of $S^n_{>0}$.
- For $n = 2$, we get the classical picture:



The Voronoi algorithm

The algorithm itself is:

- Find a perfect form, insert it to the list $\mathcal{L}$ as undone.
- Iterate
  - For every undone perfect form Q in $\mathcal{L}$, compute the perfect domain $\mathrm{Dom}(Q)$ and then its facets.
  - For every facet F of $\mathrm{Dom}(Q)$ realize the flipping, i.e. compute the adjacent perfect form Q' such that $\mathrm{Dom}(Q) \cap \mathrm{Dom}(Q') = F$.
  - If Q' is not equivalent to a form in $\mathcal{L}$, then we insert it into $\mathcal{L}$ as undone.
- Finish when all perfect domains have been treated.

The subalgorithms are:

- Find the dual description of the perfect domain $Dom(A)$
- For a facet F of $\mathrm{Dom}(A)$ find the adjacent perfect form A'.
- Test equivalence of perfect forms.

Enumeration of Perfect forms

dim	Nr of forms	forms	Authors
1	1	A_1	
2	1	A_2	Lagrange
3	1	A_3	Gauss
4	2	D_4, A_4	Korkine & Zolotareff
5	3	D_5, A_5, ...	Korkine & Zolotareff
6	7	E_6, E_6^*, ...	Barnes
7	33	E_7, ...	Jaquet
8	10916	E_8, ...	Dutour, Schürmann & Vallentin

Remarks

- This gives the number of perfect domains.
- The number of orbits of faces of the perfect domain tesselation is much higher but finite. It has been enumerated up to dimension 7.

III. Well rounded retract

Arithmetic closure

- For A a perfect quadratic form, the perfect domain $\mathrm{Dom}(A)$ contains some rank 1 forms, for example $p(v)$.
- So actually, the perfect domains realize a tiling not of $S^n_{>0}$, nor $S^n_{\geq 0}$ but of the rational closure $S^n_{rat,\geq 0}$.
- The rational closure $S^n_{rat,\geq 0}$ has a number of descriptions:
  - $S^n_{rat,\geq 0} = \sum_{v\in\mathbb{Z}^n} \mathbb{R}_+ p(v)$
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if $Ker\ A$ is defined by rational equations.
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if it defines a tesselation of $\mathbb{Z}^n$ by Delaunay polyhedra.
- So, actually, the stabilizers of some faces of the polyhedral complex are infinite.

Well rounded forms

- A form Q is said to be well rounded if it admits vectors v_1, ..., v_n such that
  - $(v_1, \dots, v_n)$ form a basis of $\mathbb{R}^n$
  - $v_1, \dots, v_n$ are shortest vectors.
  - $Q[v_1] = \cdots = Q[v_n]$.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of $S^n_{>0}$ onto a polyhedral complex of dimension $\frac{n(n-1)}{2} + 1$.
- So, by killing the faces of the perfect form tesselation that contain some degenerate form we keep only the one that have finite stabilizers and we get the decomposition that we want.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.

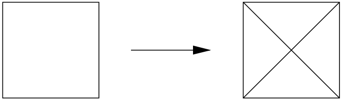
IV. Hacking tesselations

Special tesselations

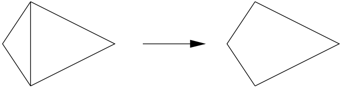
- A polyhedral decomposition is called special if for all faces F of the tesselation and every $g \in \mathrm{Stab}(F)$ the element g stabilizes F pointwise.
- In particular, top dimensional faces have trivial stabilizers and codimension 1 faces have stabilizer of order 1 or 2.
- This property is not achieved by the perfect form tesselation.
- So, we have to modify the tesselation in order to achieve this.
- A weaker property that we may wish is that the top-dimensional faces have small stabilizers.
- We cannot get rid of stabilizers, but we have some degree of freedom for the face that they stabilize.

Some operations

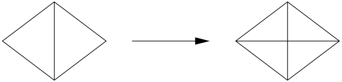
- We can add a ray in the middle of the perfect domain. The operation is as follows:



- We may merge back some faces. As follows:



- We can also add a ray on a face:



Perfect forms in dimension 4

- Initially there are 2 orbits of perfect forms so full dimensional cells are:
  - O_1: full dimensional cell with 64 facets and stabilizer of size 1152 (perfect domain of D_4).
  - O_2: full dimensional cell with 10 facets and stabilizer of size 240 (perfect domain of A_4).
- Now split both O_1 and O_2 by adding a central ray. We then get as orbits of full dimensional cells:
  - $O_{1,1}$: full dimensional cell with 10 facets and stabilizer of size 24.
  - $O_{1,2}$: full dimensional cell with 10 facets and stabilizer of size 8.
  - $O_{2,1}$: full dimensional cell with 10 facets and stabilizer of size 24.
- Every cell $O_{1,1}$ is adjacent to a unique cell $O_{2,1}$. Join them:
  - O'_1: full dimensional cell with 18 facets and stabilizer of size 24.
  - O'_2: full dimensional cell with 10 facets and stabilizer of size 8.

Perfect forms in dimension 4

- Now we put a central ray in O'_1 and get the following decomposition:
  - $O'_{1,1}$: full dimensional cell with 10 facets and stabilizer of size 2.
  - $O'_{1,2}$: full dimensional cell with 10 facets and stabilizer of size 4.
  - O'_2: full dimensional cell with 10 facets and stabilizer of size 8.

  This decomposition is much more manageable.

V. Other tesselations

The case of $\mathrm{GL}_n(\mathbb{Z}[i])$

- We can make $\mathrm{GL}_n(\mathbb{Z}[i])$ act on $\mathbb{R}^{2n}$ and more precisely on the quadratic forms corresponding to hermitian forms.
- All the theory follow as before, but the dimension is n^2.
- See for more details.
  - A. Schürmann, *Enumerating perfect forms*, Contemporary Mathematics
- The method applies to $\mathrm{GL}_n(\mathbb{Z}[\omega])$ with $\mathbb{Z}[\omega]$ the Eisenstein integers.

Other techniques I

- Some methods based on the Poincare polyhedron theorem have been devised. Example of application:
  - R. Riley, *Applications of a computer implementation of Poincare theorem on fundamental polyhedra*, Mathematics of Computation **40** (1983) 607–632.
  - A. Rahm and M. Fuchs, *The integral homnology of PSL_2 of imaginary quadratic integers with non-trivial class group*.
- More sophisticated applications of Poincare polyhedron theorem to complex hyperbolic spaces are:
  - M. Deraux, *Deforming the $\mathbb{R}$-fuchsian $(4, 4, 4)$-lattice group into a lattice*.
  - E. Falbel and P.-V. Koseleff, *Flexibility of ideal triangle groups in complex hyperbolic geometry*, Topology **39** (2000) 1209–1223.

Other techniques II

- As far as we know there as only two work for non-polyhedral, but still manifold, domains.
  - R. MacPherson and M. McConnel, *Explicit reduction theory for Siegel modular threefolds*, Invent. Math. **111** (1993) 575–625.
  - D. Yasaki, *An explicit spine for the Picard modular group over the Gaussian integers*, Journal of Number Theory, **128** (2008) 207–234.
- Other works for non-manifold setting would be:
  - T. Brady, *The integral cohomology of* $Out_+(F_3)$, Journal of Pure and Applied Algebra **87** (1993) 123–167.
  - H.-W. Henn, *The cohomology of* $\mathrm{SL}_3(\mathbb{Z}[1/2])$, K-theory **16** (1999) 299–359.