# Lattices and perfect form theory

Mathieu Dutour Sikirić

Institute Rudjer Bošković, Croatia and
Universität Rostock

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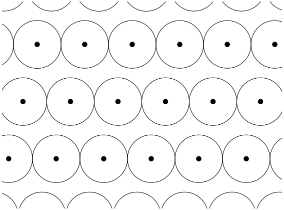
# I. Lattices and Gram matrices

# Lattice packings

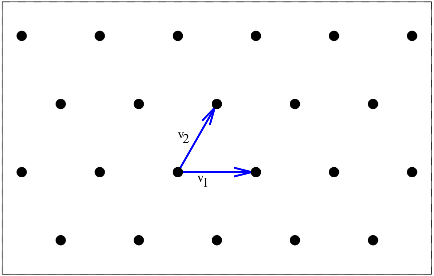
- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ with $(v1, \ldots, v_n)$ independent.
- A packing is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius $r$ and center $x_i$ such that their interiors are disjoint.



- If $L$ is a lattice, the lattice packing is the packing defined by taking the maximal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a packing.
- The maximum $\alpha$ is called $\lambda(L)$ and the determinant of $(v_1, \ldots, v_n)$ is det $L$.

# Gram matrix and lattices

- Denote by $S^n$ the vector space of real symmetric $n \times n$ matrices, $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices and $S^n_{\geq 0}$ the convex cone of real symmetric positive semidefinite $n \times n$ matrices.
- Take a basis $(v_1, \dots, v_n)$ of a lattice $L$ and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$




$$G_{\mathbf{v}} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

# Isometric lattices

- Take a basis $(v_1, \dots, v_n)$ of a lattice $L$ with $v_i = (v_{i,1}, \dots, v_{i,n}) \in \mathbb{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \dots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \dots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T\ V$.
  The matrix $G_{\mathbf{v}}$ is defined by $\frac{n(n+1)}{2}$ variables as opposed to $n^2$ for the basis $V$.
- If $M \in S^n_{>0}$, then there exists $V$ such that $M = V^T\ V$ (Gram Schmidt orthonormalization)
- If $M = V_1^T\ V_1 = V_2^T\ V_2$, then $V_1 = OV_2$ with $O^T\ O = I_n$ (i.e. $O$ corresponds to an isometry of $\mathbb{R}^n$).
- Also if $L$ is a lattice of $\mathbb{R}^n$ with basis $\mathbf{v}$ and $u$ an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

# Arithmetic minimum

- The arithmetic minimum of $A \in S^n_{>0}$ is

$$\min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} x^T A x$$

- The minimal vector set of $A \in S^n_{>0}$ is

$$\mathrm{Min}(A) = \left\{ x \in \mathbb{Z}^n \mid x^T A x = \min(A) \right\}$$

- Both $\min(A)$ and $\mathrm{Min}(A)$ can be computed using some programs (for example SV by Vallentin)
- The matrix $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}.$$

# Re-expression of previous definitions

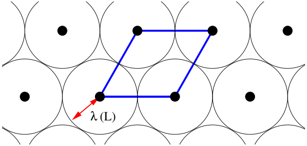
- Take a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. If $x \in L$,

$$x = x_1 v_1 + \cdots + x_n v_n \text{ with } x_i \in \mathbb{Z}$$

we associate to it the column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

- We get $||x||^2 = X^T G_{\mathbf{v}} X$ and

$$\det L = \sqrt{\det G_{\mathbf{v}}} \text{ and } \lambda(L) = \frac{1}{2}\sqrt{\min(G_{\mathbf{v}})}$$




- For $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\det A_{hex} = 3$ and $\min(A_{hex}) = 2$

# Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice $L$ then $V' = VP$ with $P \in \mathrm{GL}_n(\mathbb{Z})$. This implies

$$G_{\mathbf{v}'} = V'^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$A = P^T BP$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken/Souvignier wrote a program ISOM for testing arithmetic equivalence and a program AUTO for computing automorphism group of lattices.
  All such programs take Gram matrices as input.

# Dual lattices

- For a lattice $L$ the dual lattice is

$$L^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L\}$$

- If $L = P\mathbb{Z}^n$ then we can take $L^* = (P^{-1})^T \mathbb{Z}^n$ and we get

$$G(L^*) = (G(L))^{-1}$$

- A lattice $L$ is integral if $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}$.
- This is equivalent to say $L \subset L^*$
- A lattice is self-dual if $L = L^*$.
- A lattice is self-dual if and only if its Gram matrix is integral and of determinant 1.

# Root lattices

- A root lattice is a lattice generated by a root system
- They are integral, $\|x\|^2 \in 2\mathbb{Z}$ and $\mathrm{Min}(L)$ is the root system
- Most classical example is

$$\mathrm{A}_n = \left\{ x \in \mathbb{Z}^{n+1} \text{ s.t. } \sum_{i=1}^{n+1} x_i = 0 \right\}$$

  Possible basis: $v_i = e_{i+1} - e_i$ for $1 \le i \le n$
- They have a strict ADE classification:

| Name | Min | $\lvert Min \rvert$ | det | $\lvert$ Aut $\rvert$ |
|---|---|---|---|---|
| $\mathrm{A}_n$ | $e_i - e_j$ | $2n(n+1)$ | $n+1$ | $2(n+1)!$ |
| $\mathrm{D}_n$ | $\pm e_i \pm e_j$ | $4n(n-1)$ | 4 | $2^n n!$ |
| $\mathrm{E}_6$ | complex | 72 | 3 | 103680 |
| $\mathrm{E}_7$ | complex | 126 | 2 | 2903040 |
| $\mathrm{E}_8$ | complex | 240 | 1 | 696729600 |

# Self-dual even lattice

- A lattice is even if for all $x \in L$, $\langle x, x \rangle \in 2\mathbb{Z}$.
- The Theta function of a self-dual even lattice of dimension $n$ is

$$\Theta(L, q) = \sum_{x \in L} q^{\langle x, x \rangle}$$

  and it is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $n/2$.
- This implies that they exist only for dimension $n$ divisible by 8.

| Dimension | lattices |
|---:|:---:|
| 8 | $\mathrm{E}_8$ |
| 16 | $\mathrm{E}_8 \oplus \mathrm{E}_8$ and $\mathrm{D}_{16}^{+}$ |
| 24 | Leech lattice and 23 Niemeier lattices |
| 32 | at least 40 million lattices |

- The key to above enumeration and estimates are the Siegel Mass formula and Kneser's algorithm
  - M. Kneser, *Quadratische Formen*, Springer Verlag.

# The Leech lattice

- Every non-zero vector $v$ has $\|v\|^2 \geq 4$ and $\det Leech = 1$.
- It is the best lattice packing in dimension 24. Density is

$$\frac{\pi^{12}}{12!} \simeq 0.001930...$$

- There are 196280 shortest vectors (maximal number in dimension 24)
- The covering radius is $\sqrt{2}$ and covering density is

$$\frac{\pi^{12}}{12!}\left(\sqrt{2}\right)^{24} \simeq 7.903536...$$

  It is conjectured to give the best covering in dimension 24.
- Its automorphism group quotiented by $\pm Id_{24}$ is the sporadic simple group $Co_0$ and it contains many sporadic simple groups as subgroups.
- It is also related to some Lorentzian lattices.

# II. Computational techniques

# Polytopes, definition

- A polytope $P \subset \mathbb{R}^n$ is defined alternatively as:
  - The convex hull of a finite number of points $v^1, \dots, v^m$:

$$P = \{v \in \mathbb{R}^n \mid v = \sum_i \lambda_i v^i \text{ with } \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}$$

  - The following set of solutions:

$$P = \{x \in \mathbb{R}^n \mid f_j(x) \geq b_j \text{ with } f_j \text{ linear}\}$$

    with the condition that $P$ is bounded.
- The cube is defined alternatively as
  - The convex hull of the $2^n$ vertices

$$\{(x_1, \dots, x_n) \text{ with } x_i = \pm 1\}$$

  - The set of points $x \in \mathbb{R}^n$ satisfying to

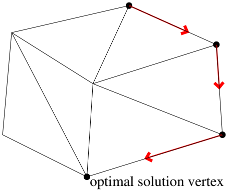
$$x_i \leq 1 \text{ and } x_i \geq -1$$

# Facets and vertices

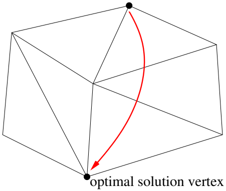
- A vertex of a polytope $P$ is a point $v \in P$, which cannot be expressed as $v = \lambda v^1 + (1-\lambda)v^2$ with $0 < \lambda < 1$ and $v^1 \neq v^2 \in P$.
- A polytope is the convex hull of its vertices and this is the minimal set defining it.
- A facet of a polytope is an inequality $f(x) - b \geq 0$, which cannot be expressed as $f(x) - b = \lambda(f_1(x) - b_1) + (1-\lambda)(f_2(x) - b_2)$ with $f_i(x) - b_i \geq 0$ on $P$.
- A polytope is defined by its facet inequalities. and this is the minimal set of linear inequalities defining it.
- The dual-description problem is the problem of passing from one description to another.
- There are several programs CDD, LRS for computing dual-description computations.
- In case of large problems, we can use the symmetries for faster computation.

# Linear programs

- A linear program is the problem of maximizing a linear function $f(x)$ over a set $\mathcal{P}$ defined by linear inequalities.
$$\mathcal{P} = \{x \in \mathbb{R}^d \quad \text{such that} \quad f_i(x) \geq b_i\}$$
with $f_i$ linear and $b_i \in \mathbb{R}$.
- The solution of linear programs is attained at vertices of $\mathcal{P}$.
- There are two classes of solution methods:




Simplex method




Interior point method

- Simplex methods use exact arithmetic but have bad theoretical complexity
- Interior point methods have good theoretical complexity but only gives an approximate vertex.

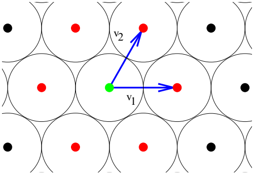
# III. Perfect forms and domains

# Perfect forms

- A form $A$ is extreme if it is a local maximum of the packing density.
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev) if the equation
$$B \in S^n \ \text{ and } \ x^T Bx = \min(A) \text{ for all } x \in \mathrm{Min}(A)$$
implies $B = A$.
- Theorem: (Korkine & Zolotarev) If a form is extreme then it is perfect.
- Up to a scalar multiple, perfect forms are rational.
- All root lattices are perfect, many other families are known.

# A perfect form

- $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ corresponds to the lattice:



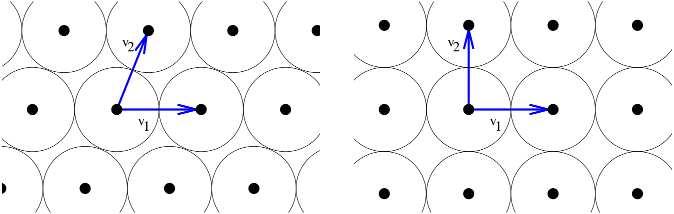

- If $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfies to $x^T Bx = \min(A_{hex})$ for $x \in \mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$, then:

$$a = 2, \quad c = 2 \ \text{ and } \ a - 2b + c = 2$$

which implies $B = A_{hex}$. $A_{hex}$ is perfect.

# A non-perfect form

- $A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\mathrm{Min}(A_{sqr}) = \{\pm(0,1), \pm(1,0)\}$.
- See below lattices $L_B$, $L_{sqr}$ associated to matrices $B, A_{sqr} \in S^2_{>0}$ with $\mathrm{Min}(B) = \mathrm{Min}(A_{sqr})$:



# Perfect domains and arithmetic closure

- If $v \in \mathbb{Z}^n$ then the corresponding rank 1 form is $p(v) = vv^T$.
- If $A$ is a perfect form, its perfect domain is

$$\mathrm{Dom}(A) = \sum_{v \in \mathrm{Min}(A)} \mathbb{R}_+ p(v)$$

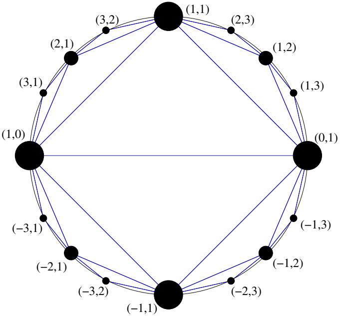
- If $A$ has $m$ shortest vectors then $\mathrm{Dom}(A)$ has $\frac{m}{2}$ extreme rays.
- So actually, the perfect domains realize a tessellation not of $S^n_{>0}$, nor $S^n_{\geq 0}$ but of the rational closure $S^n_{rat,\geq 0}$.
- The rational closure $S^n_{rat,\geq 0}$ has a number of descriptions:
  - $S^n_{rat,\geq 0} = \sum_{v \in \mathbb{Z}^n} \mathbb{R}_+ p(v)$
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if $Ker\ A$ is defined by rational equations.
- So, actually, the stabilizers of some faces of the polyhedral complex are infinite.

# Finiteness

- Theorem:(Voronoi) Up to arithmetic equivalence there is only finitely many perfect forms.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:

$$Q \mapsto P^t Q P$$

and we have $\mathrm{Min}(P^t Q P) = P^{-1}\,\mathrm{Min}(Q)$
- $\mathrm{Dom}(P^T Q P) = c(P)^T\,\mathrm{Dom}(Q) c(P)$ with $c(P) = \left(P^{-1}\right)^T$
- For $n = 2$, we get the classical picture:



# Known results on lattice packing density maximization

| dim. | Nr. of perfect forms | Best lattice packing |
|---|---|---|
| 2 | 1 (Lagrange) | $A_2$ |
| 3 | 1 (Gauss) | $A_3$ |
| 4 | 2 (Korkine & Zolotarev) | $D_4$ |
| 5 | 3 (Korkine & Zolotarev) | $D_5$ |
| 6 | 7 (Barnes) | $E_6$ (Blichfeldt & Watson) |
| 7 | 33 (Jaquet) | $E_7$ (Blichfeldt & Watson) |
| 8 | 10916 (DSV) | $E_8$ (Blichfeldt & Watson) |
| 9 | $\geq$500000 | $\Lambda_9$? |
| 24 | ? | Leech (Cohn & Kumar) |

- The enumeration of perfect forms is done with the Voronoi algorithm.
- The number of orbits of faces of the perfect domain tessellation is much higher but finite (Known for $n \leq 7$)
- Blichfeldt used Korkine-Zolotarev reduction theory.
- Cohn & Kumar used Fourier analysis and Linear programming.

# Some algorithms

- Pb 1: Suppose we have a configuration of vector $\mathcal{V}$. Does there exist a matrix $A \in S^n_{>0}$ such that $\mathrm{Min}(A) = \mathcal{V}$?
- Consider the linear program

$$\begin{array}{rl} \text{minimize} & \lambda \\ \text{with} & \lambda = A[v] \text{ for } v \in \mathcal{V} \\ & A[v] \geq 1 \text{ for } v \in \mathbb{Z}^n - \{0\} - \mathcal{V} \end{array}$$

  The value $\lambda_{opt}$ determines the answer.
- In practice one replaces $\mathbb{Z}^n$ by a finite set and iteratively increases it until a conclusion is reached.
- Pb 2: How given a matrix $A \in S^n_{>0}$ find $B$ perfect with $A \in \mathrm{Dom}(B)$?
- The method is to start from a perfect matrix $B$ and test if $A$ belongs to $\mathrm{Dom}(B)$. If not there exist a facet $F$ of $\mathrm{Dom}(B)$ such that $A$ is on the other side (found by LP).
  We flip over it. Eventually, one finds the right perfect form.

# IV. Ryshkov polyhedron and the Voronoi algorithm

# The Ryshkov polyhedron

- The Ryshkov polyhedron $R_n$ is defined as

$$R_n = \left\{ A \in S^n \text{ s.t. } x^T A x \geq 1 \text{ for all } x \in \mathbb{Z}^n - \{0\} \right\}$$

- The cone is invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$.
- The cone is locally polyhedral, i.e. for a given $A \in R_n$

$$\left\{ x \in \mathbb{Z}^n \text{ s.t. } x^T A x = 1 \right\}$$

  is finite
- Vertices of $R_n$ correspond to perfect forms.
- For a form $A \in R_n$ we define the local cone

$$Loc(A) = \left\{ Q \in S^n \text{ s.t. } x^T Q x \geq 0 \text{ if } x^T A x = 1 \right\}$$

# The Voronoi algorithm

- Find a perfect form (say $\mathsf{A}_n$), insert it to the list $\mathcal{L}$ as undone.
- Iterate
  - For every undone perfect form $A$ in $\mathcal{L}$, compute the local cone $Loc(A)$ and then its extreme rays.
  - For every extreme ray $r$ of $Loc(A)$ realize the flipping, i.e. compute the adjacent perfect form $A' = A + \alpha r$.
  - If $A'$ is not equivalent to a form in $\mathcal{L}$, then we insert it into $\mathcal{L}$ as undone.
- Finish when all perfect forms have been treated.

The sub-algorithms are:

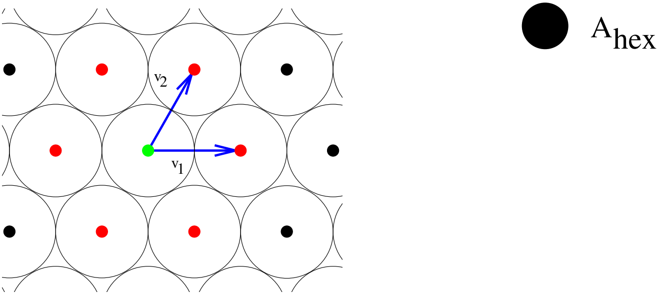
- Find the extreme rays of the local cone $Loc(A)$ (use CDD or LRS or any other program)
- For any extreme ray $r$ of $Loc(A)$ find the adjacent perfect form $A'$ in the Ryshkov polyhedron $R_n$
- Test equivalence of perfect forms using ISOM

# Flipping on an edge I

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$$

with

$$A_{hex} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

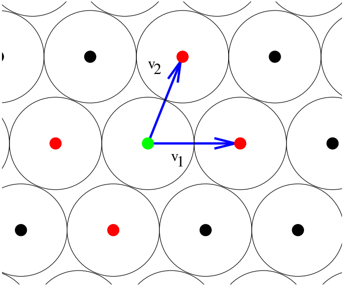


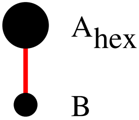
# Flipping on an edge II

$$\mathrm{Min}(B) = \{\pm(1,0), \pm(0,1)\}$$

with

$$B = \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix} = A_{hex} + D/4$$



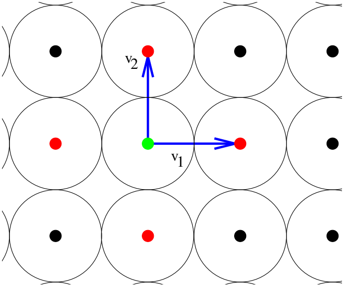



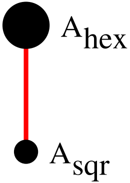
# Flipping on an edge III

$$\mathrm{Min}(A_{sqr}) = \{\pm(1,0), \pm(0,1)\}$$

with

$$A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_{hex} + D/2$$



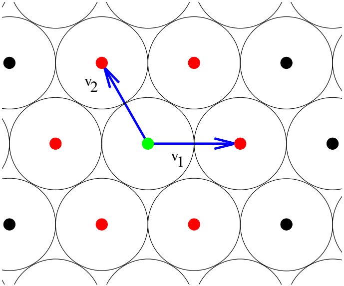



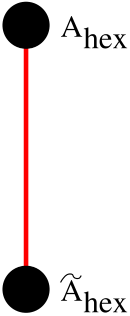
# Flipping on an edge IV

$$\mathrm{Min}(\tilde{A}_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$$

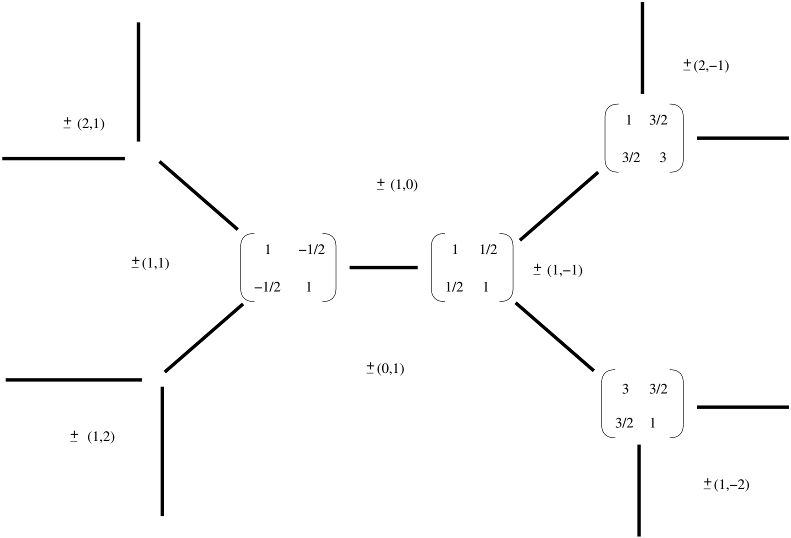
with

$$\tilde{A}_{hex} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = A_{hex} + D$$



# The Ryshkov polyhedron $R_2$



# Well rounded forms and retract

- A form $Q$ is said to be well rounded if it admits vectors $v_1$, $\dots$, $v_n$ such that
  - $(v_1, \dots, v_n)$ form a $\mathbb{R}$-basis of $\mathbb{R}^n$ (not necessarily a $\mathbb{Z}$-basis)
  - $v_1, \dots, v_n$ are shortest vectors of $Q$.
- Well rounded forms correspond to bounded faces of $R_n$.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of $R_n$ onto a polyhedral complex $WR_n$ of dimension $\frac{n(n-1)}{2}$.
- Every face of $WR_n$ has finite stabilizer.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.
- We also cannot reduce ourselves to lattices whose shortest vectors define a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

# Topological applications

- The fact that we have finite stabilizers for all faces means that we can compute rational homology/cohomology of $\mathrm{GL}_n(\mathbb{Z})$ efficiently.
- This has been done for $n \leq 7$
  - P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math 245 (2013) 587–624.
- As an application, we can compute $K_n(\mathbb{Z})$ for $n \leq 8$.
- By using perfect domains, we can compute the action of Hecke operators on the cohomology.
- This has been done for $n \leq 4$:
  - P.E. Gunnells, *Computing Hecke Eigenvalues Below the Cohomological Dimension*, Experimental Mathematics 9-3 (2000) 351–367.
- The above can, in principle, be extended to the case of $\mathrm{GL}_n(R)$ with $R$ a ring of algebraic integers.

# References


On lattice theory:

- J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups* third edition, Springer–Verlag, 1998.

On perfect forms:

- G. Voronoi, *Nouvelles applications des paramètres continues à la théorie des formes quadratiques 1: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math **133** (1908) 97–178.
- A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.
- J. Martinet, *Perfect lattices in Euclidean spaces*, Springer, 2003.
- S.S. Ryshkov, E.P. Baranovski, *Classical methods in the theory of lattice packings*, Russian Math. Surveys **34** (1979) 1–68, translation of Uspekhi Mat. Nauk **34** (1979) 3–63.

# V. Tessellations

## Linear Reduction theories for $S^n$

Some $\mathrm{GL}_n(\mathbb{Z})$ invariant tessellations of $S^n_{rat,\geq 0}$:

- The perfect form theory (Voronoi I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$)
- The central cone compactification (Igusa & Namikawa) (Known for $n \leq 6$)
- The $L$-type reduction theory (Voronoi II) for Delaunay tessellations (Known for $n \leq 5$)
- The $C$-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (Known for $n \leq 5$)
- The Minkowski reduction theory (Minkowski) it uses the successive minima of a lattice to reduce it (Known for $n \leq 7$) not face-to-face
- Venkov's reduction theory also known as Igusa's fundamental cone (finiteness proved by Crisalli)

# Toroidal compactifications of $\mathcal{A}_g$

- A polyhedral $\mathrm{GL}_n(\mathbb{Z})$-tessellation of $S^n_{rat,\geq 0}$ is admissible if it is a face-to-face tessellation and has finite number of orbits.
- Admissible $\mathrm{GL}_n(\mathbb{Z})$ invariant tessellations of $S^n_{rat,\geq 0}$ give rise to toroidal compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties.
- For the perfect form tessellation $\mathcal{A}_g^{Perf}$ is a canonical model in the sense of the minimal model program if $g \geq 12$:
  - N. Shepherd-Barron, *Perfect forms and the moduli space of abelian varieties*, Invent. Math. 163-1 (2006) 25–45
- For Voronoi II tessellation $\mathcal{A}_g^{Vor}$ has its boundary corresponding to semi-abelic varieties:
  - V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. 155-3 (2002) 611–708
- Properties of the compactification being $\mathbb{Q}$-Gorenstein, having canonical singularities, terminal singularities can be read off from properties of the tessellation.

# Geometry of tessellation and compactifications

- Thm: (Namikawa) For a given admissible tessellation $\mathcal{F}$ the corresponding tessellation is smooth if and only if
  - All cones are simplicial
  - For all cones, the set of generators of extreme rays can be extended to a basis of $\mathrm{Sym}^2(\mathbb{Z})$.
- For $\mathcal{A}_g^{Perf}$ we prove
  - Every cone of dimension at most 9 in the perfect cone decomposition is basic. In particular the stack $\mathcal{A}_g^{Perf}$ is smooth for $g \leq 3$ and the codimension of both the singular and the non-simplicial substack of $\mathcal{A}_g^{Perf}$ is 10 if $g \geq 4$.
  - Every cone of dimension 10 is simplicial with the only exception the cone of the root lattice $\mathsf{D}_4$.
- For $\mathcal{A}_g^{Vor}$ we prove
  - For $g \leq 4$ every cone in the second Voronoi compactification is basic.
  - For $g \geq 5$ there are non-simplicial cones in dimension 3, in particular $\mathcal{A}_g^{Vor}$ is singular in dimension 3.

# Self-dual cones

- For an open cone $C$ in $\mathbb{R}^n$ the dual cone is

$$C^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y \rangle > 0 \text{ for } y \in C\}$$

- Such cones are classified by Euclidean Jordan algebras and the classification gives:
  - $S^n$: The cone of positive definite real quadratic forms
  - $H^n$: The cone of positive definite Hermitian quadratic forms
  - $Q^n$: The cone of positive definite quaternionic quadratic forms
  - The cone of $3 \times 3$ positive definite octonion matrices.
  - The hyperbolic cone $H_n$

$$H_n = \left\{(x_1, \ldots, x_n) \text{ s.t. } x_1 > 0 \text{ and } x_1^2 - x_2^2 - \cdots - x_n^2 > 0\right\}$$

- References

  - A. Ash, D. Mumford, M. Rapoport, Y. Tai *Smooth compactifications of locally symmetric varieties*, Cambridge University Press
  - M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivtätsbereichan I/II*, Math. Annalen 141, 384–432, 144, 175–182

# $T$-space theory

- A $T$-space $\mathcal{F}$ is a vector space in $S^n$ with $\mathcal{F}_{>0} = \mathcal{F} \cap S^n_{>0}$ being non-empty.
- All above reduction theories apply to that case.
- But some dead ends exist to the polyhedral tessellations.
- Relevant group is $\mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \text{ s.t. } g\mathcal{F}g^T = \mathcal{F}\}$.
- For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ of space

$$\mathcal{F}(G) = \left\{A \in S^n \text{ s.t. } gAg^T = A \text{ for } g \in G\right\}$$

  we have $\mathrm{Aut}(\mathcal{F}(G)) = Norm(G, \mathrm{GL}_n(\mathbb{Z}))$ (Zassenhaus) and a finite number of $\mathcal{F}$-perfect forms.
- There exist some $T$-spaces having a rational basis and an infinity of perfect forms.
- Another finiteness case is for spaces obtained from $\mathrm{GL}_n(R)$ with $R$ number ring.

# Non-polyhedral reduction theories

- Some works with non-polyhedral, but still manifold domains:
  - R. MacPherson and M. McConnel, *Explicit reduction theory for Siegel modular threefolds*, Invent. Math. **111** (1993) 575–625.
  - D. Yasaki, *An explicit spine for the Picard modular group over the Gaussian integers*, Journal of Number Theory, **128** (2008) 207–234.
- Other works in complex hyperbolic space using Poincaré polyhedron theorem:
  - M. Deraux, *Deforming the $\mathbb{R}$-fuchsian* $(4,4,4)$*-lattice group into a lattice*.
  - E. Falbel and P.-V. Koseleff, *Flexibility of ideal triangle groups in complex hyperbolic geometry*, Topology **39** (2000) 1209–1223.
- Other works for non-manifold setting would be:
  - T. Brady, *The integral cohomology of $Out_+(F_3)$*, Journal of Pure and Applied Algebra **87** (1993) 123–167.
  - K.N. Moss, *Cohomology of* $\mathrm{SL}(n, \mathbb{Z}[1/p])$, Duke Mathematical Journa **47-4** (1980) 803–818.

# VI. Central cone compactification

# Central cone compactification

- We consider the space of integral valued quadratic forms:

$$I_n = \{A \in S^n \text{ s.t. } A[x] \in \mathbb{Z} \text{ for all } x \in \mathbb{Z}^n\}$$

  All the forms in $I_n$ have integral coefficients on the diagonal and half integral outside of it.
- The centrally perfect forms are the elements of $I_n$ that are vertices of conv $I_n$.
- For $A \in I_n$ we have $A[x] \geq 1$. So, $I_n \subset R_n$
- Any root lattice is a vertex both of $R_n$ and conv $I_n$.
- The centrally perfect forms are known for $n \leq 6$:

| dim. | Centrally perfect forms |
|---|---|
| 2 | $A_2$ (Igusa) |
| 3 | $A_3$ (Igusa) |
| 4 | $A_4$, $D_4$ (Igusa) |
| 5 | $A_5$, $D_5$ (Namikawa) |
| 6 | $A_6$, $D_6$, $E_6$ (Dutour Sikirić) |

- By taking the dual we get tessellations of $S^n_{rat,\geq 0}$.

# Enumeration of centrally perfect forms

- Suppose that we have a conjecturally correct list of centrally perfect forms $A_1, \dots, A_m$. Suppose further that for each form $A_i$ we have a conjectural list of neighbors $N(A_i)$.
- We form the cone

$$C(A_i) = \{X - A_i \text{ for } X \in N(A_i)\}$$

  and we compute the orbits of facets of $C(A_i)$.
- For each orbit of facet of representative $f$ we form the corresponding linear form $f$ and solve the Integer Linear Problem

$$f_{opt} = \min_{X \in I_n} f(X)$$

  We have to use GLPK program for that. It is done iteratively since $I_n$ is defined by an infinity of inequalities.
- If $f_{opt} = f(A_i)$ always then the list is correct. If not then the $X$ realizing $f(X) < f(A_i)$ need to be added to the full list.

# VII. Voronoi II theory

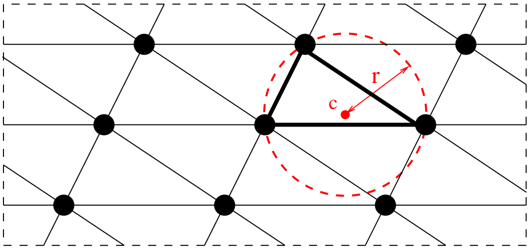
# Empty sphere and Delaunay polytopes

A sphere $S(c, r)$ of radius $r$ and center $c$ in an $n$-dimensional lattice $L$ is said to be an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope $P$ in a lattice $L$ is a polytope, whose vertex-set is $L \cap S(c, r)$.



# Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice $L$.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$||w_i - c|| = r \ \text{ i.e. } \ {w_i}^T M w_i - 2{w_i}^T Mc + c^T Mc = r^2$$

- Subtracting one obtains

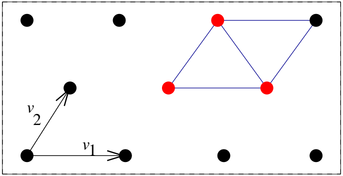
$$\{{w_i}^T M w_i - {w_j}^T M w_j\} - 2\{{w_i}^T - {w_j}^T\} Mc = 0$$

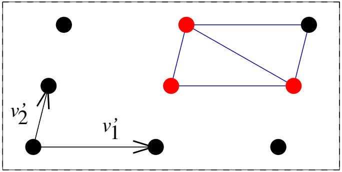
- Inverting matrices, one obtains $Mc = \psi(M)$ with $\psi$ linear and so one gets linear equalities on $M$.
- Similarly $||w - c|| \geq r$ translates into linear inequalities on $M$: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$, then one has

$$||w - c|| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i {v_i}^T M v_i \geq 0$$

# Iso-Delaunay domains

- Take a lattice $L$ and select a basis $v_1, \dots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that






are part of the same iso-Delaunay domain.

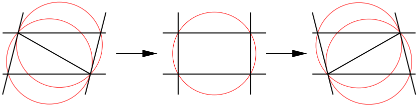
- An iso-Delaunay domain is the assignment of Delaunay polytopes. It is a polyhedral domain of $S^n_{rat,\geq 0}$.

Primitive iso-Delaunay

- If one takes a generic matrix $M$ in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on $M$.
- Hence the corresponding iso-Delaunay domain is of dimension $\frac{n(n+1)}{2}$, they are called primitive

# Equivalence and enumeration

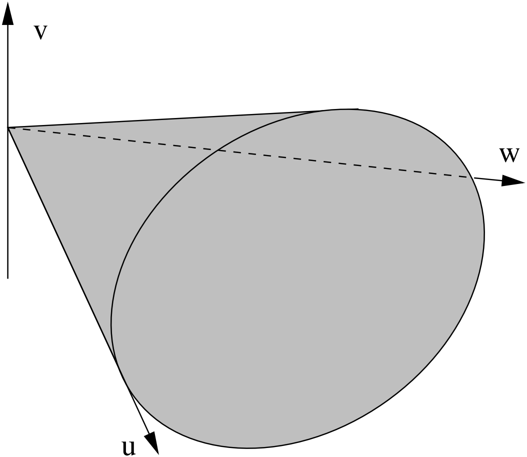
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive iso-Delaunay domains.
- Voronoi proved that after this action, there is a finite number of primitive iso-Delaunay domains.
- Bistellar flipping creates one iso-Delaunay from a given iso-Delaunay domain and a facet of the domain. In dim. 2:



- Enumerating primitive iso-Delaunay domains is done classically:
  - Find one primitive iso-Delaunay domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.
- This is very similar to the Voronoi algorithm for perfect forms.

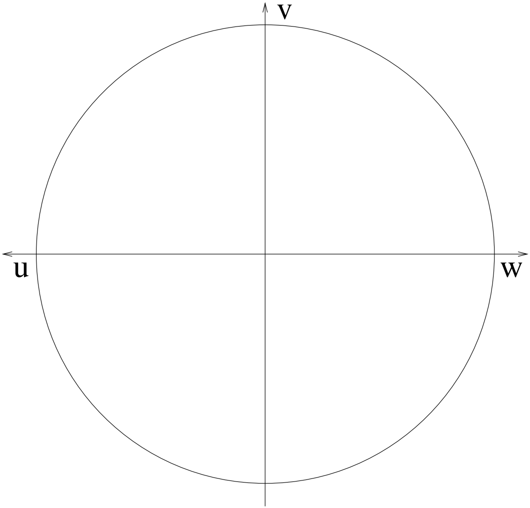
# The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ I

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



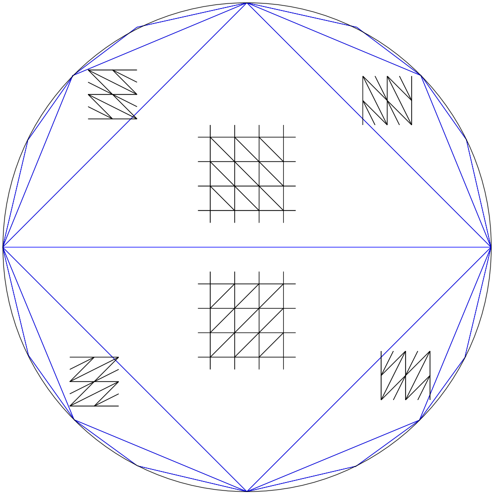
# The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ II

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



# The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ III

Primitive iso-Delaunay domains in $S^2_{rat,\geq 0}$:



# Enumeration of iso-Delaunay domains

- The covering density is equal to the maximum of the circumradius of the Delaunay polytopes.
- In principle if one knows all primitive iso-Delaunay then one can find the best covering lattice.
- A lattice is rigid (Grishukhin & Baranovski) if it is determined by its Delaunay polytopes (iso-Delaunay domain of dimension 1).

| dim. | Best covering | Nr. of primitive iso-Delaunay | Nr. of rigid lattices |
|---|---|---|---|
| 2 | $\mathsf{A}_2$ (Kershner) | 1 (Voronoi) | 0 |
| 3 | $\mathsf{A}_3^*$ (Bambah) | 1 (Voronoi) | 0 |
| 4 | $\mathsf{A}_4^*$ (Delone & Ryshkov) | 3 (Voronoi) | 1 |
| 5 | $\mathsf{A}_5^*$ (Ryshkov & Baranovski) | 222 (Engel) | 7 |
| 6 | $L_6$ (conj. Vallentin)? | $\geq 2.10^8$ (Engel) | $\geq 20000$ |

- See for more details
  - A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.

THANK

YOU