Simple random sequential packing of cubes

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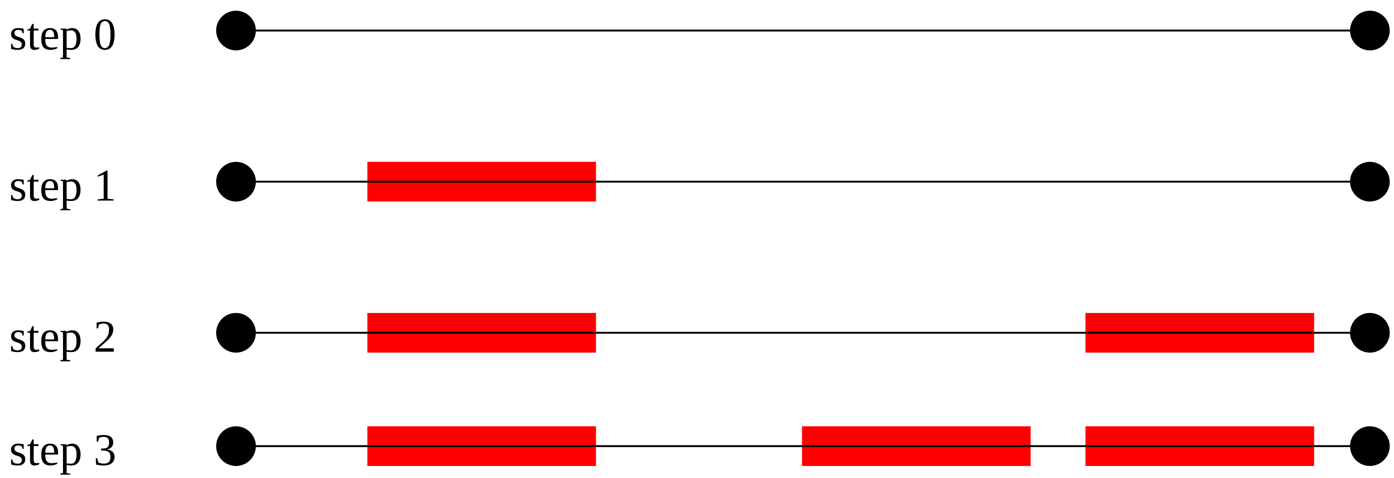
I. Packing with rigid boundaries

1-dim. random packing

- Put sequentially at random intervals $[0,1]$ into $[0,x]$ until one cannot do it any more.



- Denote by $M(x)$ the number of intervals put in $[0,x]$.
- Renyi (1958) proved that

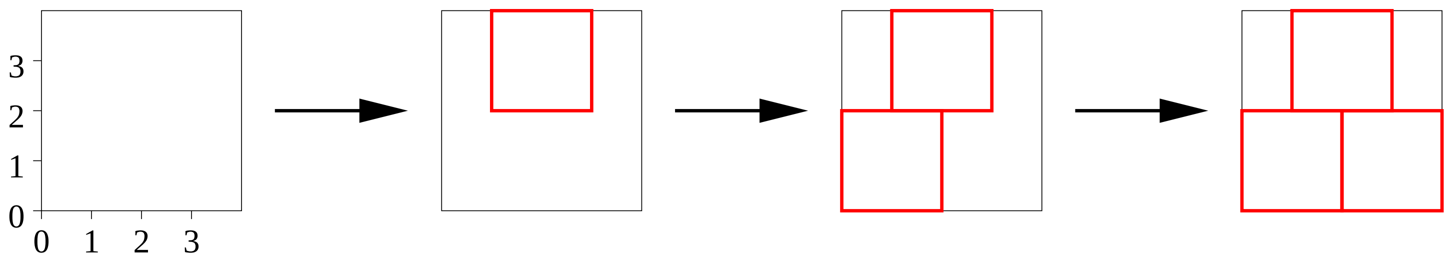
$$\lim_{x\to\infty}\frac{E(M(x))}{x}=\beta_1 \quad = \quad \int_0^\infty \exp\{-2\int_0^t \frac{1-e^{-u}}{u}du\}dt$$
$$= \quad 0.748...$$

d-dimensional random packing

- Put sequentially at random cubes $[0,1]^d$ into $[0,x]^d$ until one cannot do it any more.
- Denote by $M_d(x)$ the number of cubes put in $[0,x]^d$.
- Palasti conjectured that $\lim_{x\to\infty} \frac{E(M_d(x))}{x^d}$ exists and is equal to β_1^d.
- Existence of the limit was proved by Penrose (2001) but the second conjecture is probably false.

A simplified model

- We consider the cube $[0, 4]^d$.
- We put sequentially at random cubes $z + [0, 2]^d$ with $z \in \mathbf{Z}^d$ in it



until one cannot insert cubes any more.

- Denote by M_d the number of cubes in the obtained non-extendible packing.
- We want to estimate the packing density

$$\gamma_d = \frac{1}{2^d} E(M_d)$$

Value of γ_d

- Computer simulations (Itoh and Ueda, (1983)) suggest that γ_d is asymptotically $d^{-\alpha}$ with $\alpha = 0.44\ldots$

dim.	γ_d	$d^{-\alpha}$	γ_1^d
1	0.8348	1	0.8348
2	0.7112	0.736113	0.696891
3	0.6157	0.615336	0.581765
4	0.5481	0.541863	0.485657
5	0.4927	0.49097	0.405427
6	0.4508	0.452957	0.33845
7	0.4212	0.423123	0.282538
8	0.3958	0.398873	0.235863
9	0.3762	0.378639	0.196898
10	0.3631	0.36141	0.164371
11	0.3516	0.346501	0.137217

Extended model

- Consider the cube $[0, 2N]^d$.
- We put sequentially at random cubes $z + [0, N]^d$ with $z \in \mathbf{Z}^d$ in it until one cannot insert cubes any more.
- Denote by $M_d(N)$ the number of cubes in the obtained non-extendible packing.
- The main problem is to estimate the packing density:

$$\gamma_d(N) = \frac{1}{2^d} E(M_d(N))$$

- Poyarkov, (2004) proved that $\gamma_d \geq (1 + \frac{1}{N})^d$

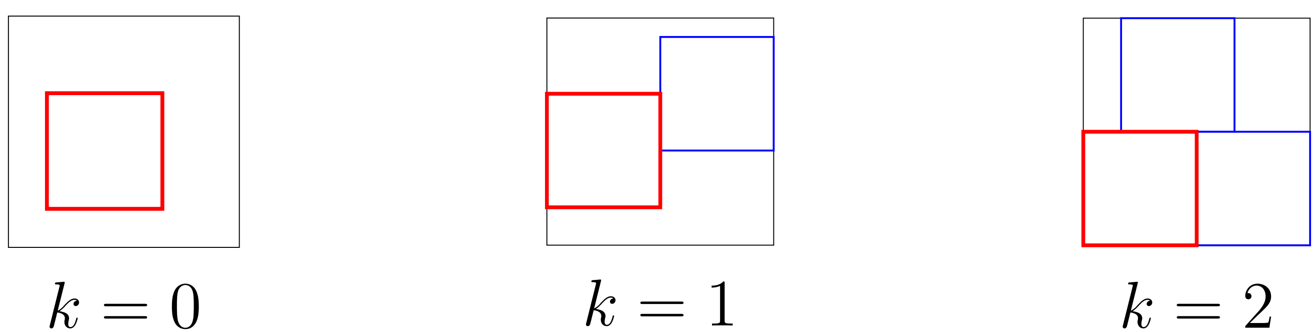
Lemma

- Lemma First put the cube $z+[0,N]^d$ in $[0,2N]^d$ and write

$$k = \#\{i \mid z_i = 0 \text{ or } N\}$$

Second put cube at random until one cannot do it any more. Then:

  - The minimal number of cubes is $k+1$.
  - The expectation of the number of cubes put is $k+1+O(\frac{1}{N+1})$



Expansion of $E(M_d(N))$

- Denote by $E(M_d(N)|k)$ the expected number of cubes in a non-extendible random packing, while imposing that the first cube $z+[0,N]^d$ has z with k coordinates equal to 0 or N.
- One has the expression

$$E(M_d(N)) = \sum_{k=0}^{d} (\frac{2}{N+1})^k (\frac{N-1}{N+1})^{d-k} \binom{k}{d} E(M_d(N)|k)$$

- So, one gets

$$\begin{aligned} E(M_d(N)) &= (\tfrac{N-1}{N+1})^d E(M_d(N)|0) + d\tfrac{2}{N+1}(\tfrac{N-1}{N+1})^{d-1} E(M_d(N)|1) \\ &+ d(d-1)\tfrac{2}{(N+1)^2}(\tfrac{N-1}{N+1})^{d-2} E(M_d(N)|2) + O(\tfrac{1}{N+1})^3 \end{aligned}$$

Expansion of $E(M_d(N))$

- Clearly $E(M_d(N)|0) = 1$ and $E(M_d(N)|1) = 1 + E(M_{d-1}(N))$.
- By above lemma, $E(M_d(N)|2) = 3 + O(\frac{1}{N+1})$
- First, one has $E(M_d(N)) = 1 + O(\frac{1}{N+1})$
- Second, one gets

$$\begin{aligned} E(M_d(N)) &= (1 - \tfrac{2}{N+1})^d + \tfrac{2d}{N+1}(1 - \tfrac{2}{N+1})^{d-1}(2 + O(\tfrac{1}{N+1})) \\ &+ O(\tfrac{1}{(N+1)^2}) \\ &= \{1 - \tfrac{2d}{N+1} + O(\tfrac{1}{(N+1)^2})\} + \tfrac{4d}{N+1} + O(\tfrac{1}{(N+1)^2}) \\ &= 1 + \tfrac{2d}{N+1} + O(\tfrac{1}{(N+1)^2}) \end{aligned}$$

Expansion of $E(M_d(N))$

- Inserting this expression one gets

$$E(M_d(N)) = 1 + \frac{2d}{N+1} + \frac{4d(d-1)}{(N+1)^2} + O(\frac{1}{N+1})^3$$

- One proves that for fixed d, there exists an asymptotic expansion

$$E(M_d(N)) = \sum_{k=0}^{\infty} c_{k,d} \frac{1}{(N+1)^k}$$

- But finding the coefficients $c_{k,d}$ for $k \geq 3$ is less easy.

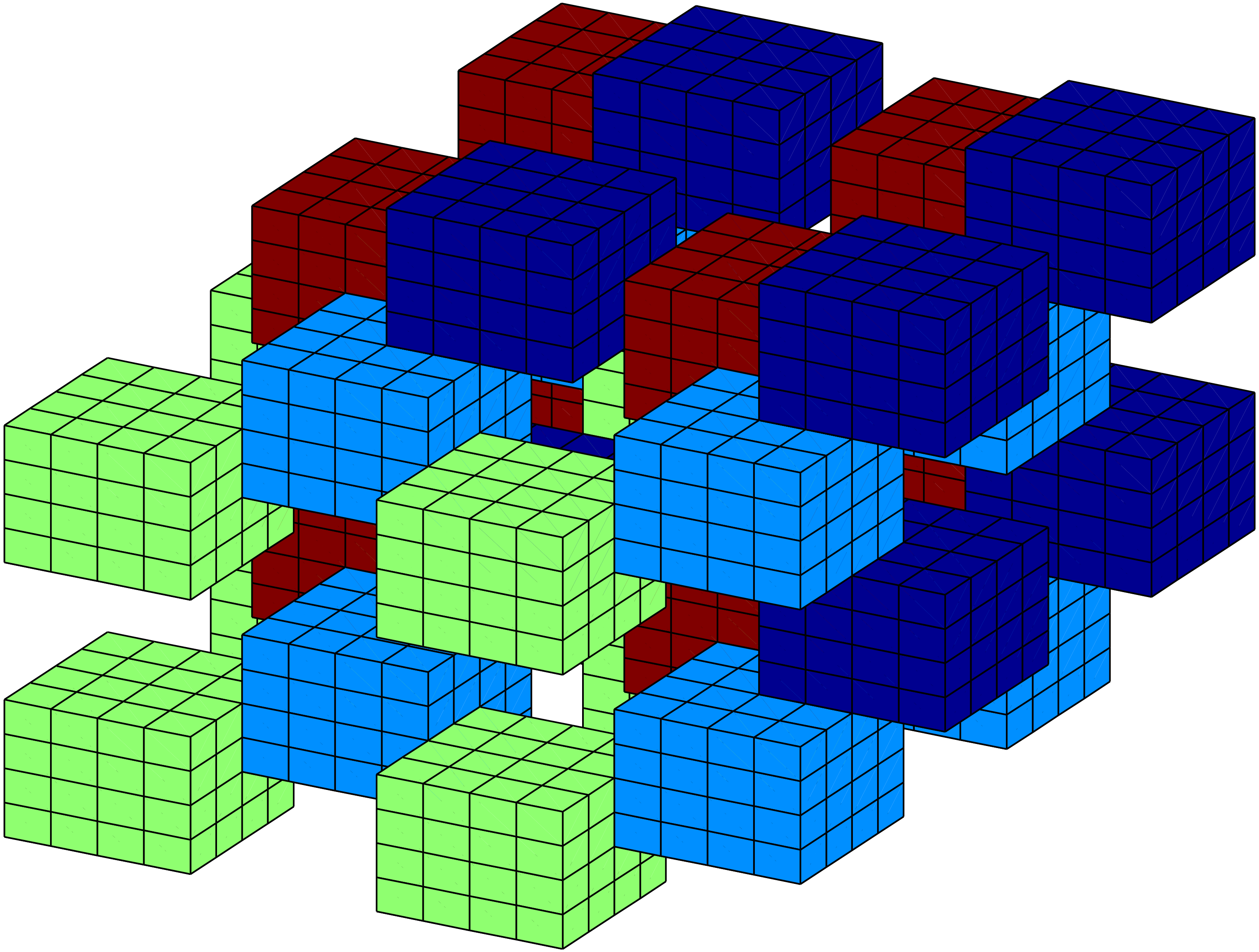
II. Torus cube tilings and packings

Torus cube tilings and packings

- A cube tiling is a $4\mathbf{Z}^d$-periodic tiling of $\mathbf{R}^d$ by integral translates of the cube $[0,2]^d$.
- There is only one cube tiling in dimension 1.
- There are two cube tilings in dimension 2:

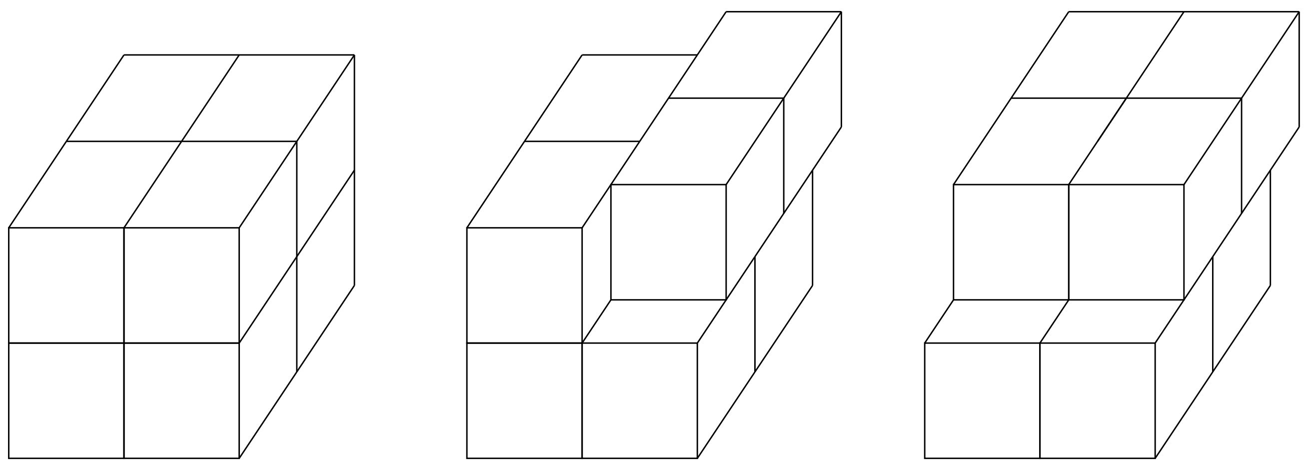
- A cube packing is a packing of $\mathbf{R}^d$ by integral translates of cubes $[0,2]^d$, which is $4\mathbf{Z}^d$-periodic.
- If we cannot extend a cube packing by adding another cube, then it is called non-extendible.
- No non-extendible cube packings in dimension 1 and 2.

3-dim. non-extendible cube packing



Results for $d \leq 4$

- In dimension 3, there is a unique non-extendible cube packing and there are 9 types of cube tilings.



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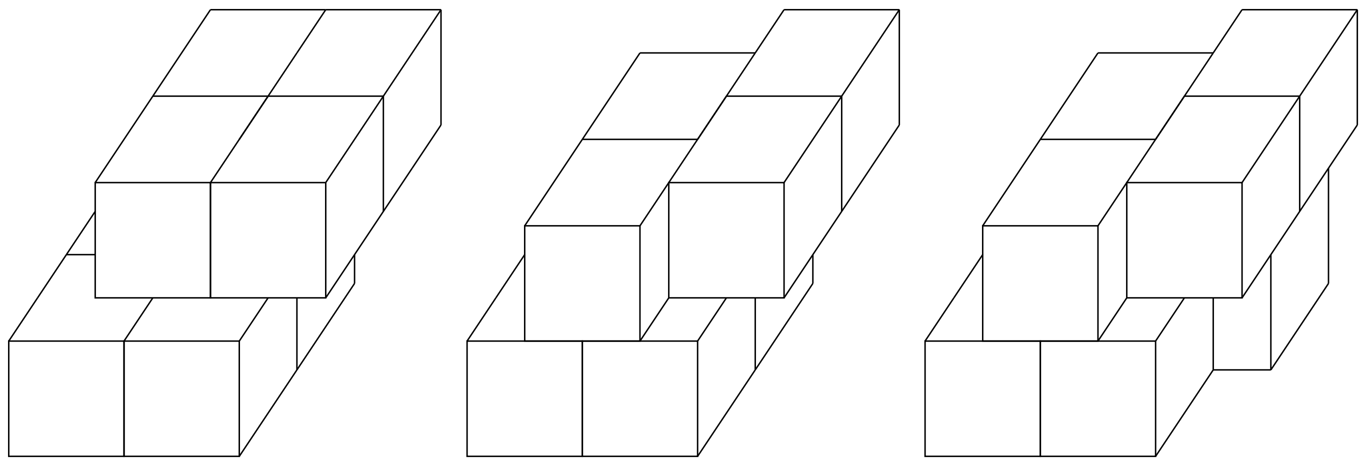

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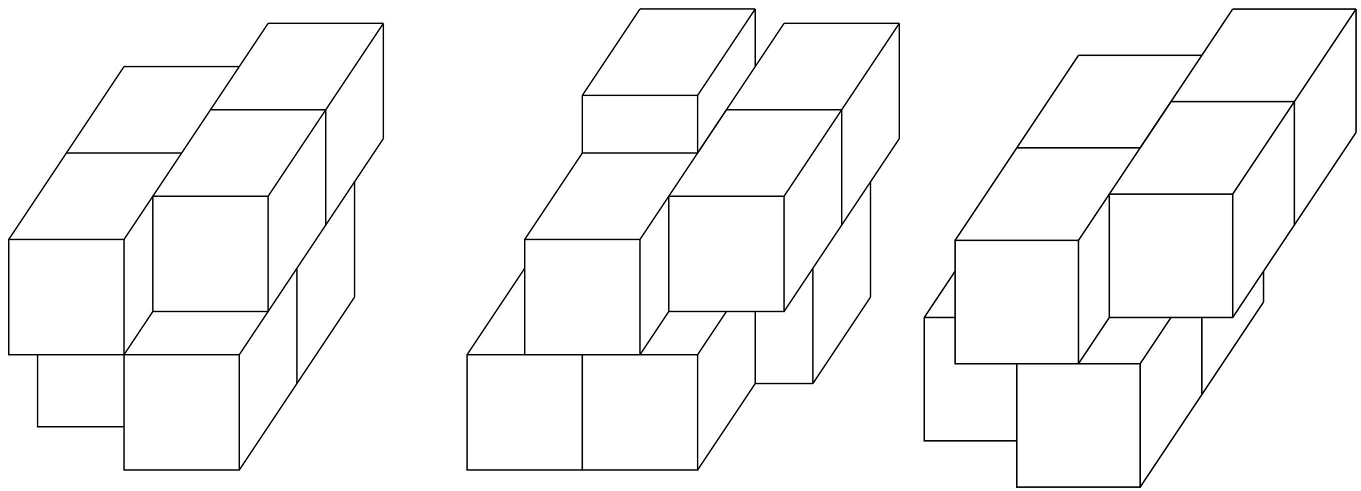


- In dimension 4, the repartition is as follows:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
nb	0	0	0	0	0	0	0	38	6	24	0	71	0	0	0	744

Low density cube packings

- Denote by $f(d)$ the smallest number of cubes of non-extendible cube packing.
- $f(3) = 4$ and $f(4) = 8$.
- For any $n, m \in \mathbf{N}$, the following inequality holds:

$$f(n+m) \le f(n)f(m) \ .$$

  The cube packing realizing this is constructed by "product" of two cube packings of $\mathbf{R}^n$ and $\mathbf{R}^m$
- Conjecture: $f(5) = 12$ and $f(6) = 16$.

III. Continuous torus cube packings

Torus cube packings

- We consider the torus $\mathbf{Z}^d/2N\mathbf{Z}^d$ and do sequential random packing by cubes $z + [0, N]^d$ with $z \in \mathbf{Z}^d$.
- We denote $M_d(N)$ the number of cubes in the obtained torus cube packing and

$$\alpha_d(N) = E(M_d(N))$$

- As in the case of rigid boundaries, we are interested in the limit $N \to \infty$.
- In the rigid boundary case, in the limit $N \to \infty$, one has a single cube in the middle of another cube and no possibility of adding any other cube.
- But for torus case, the limit $N \to \infty$ is more interesting.

Continuous cube packings

- We consider the torus $\mathbf{R}^d/2\mathbf{Z}^d$ and do sequential random packing by cubes $z+[0,1]^d$ with $z \in \mathbf{R}^d$.
- Two cubes $z+[0,1]^d$ and $z'+[0,1]^d$ are non-overlapping if and only if there is $1 \leq i \leq d$ with $z'_i \equiv z_i + 1 \pmod 2$
- Fix a cube $C = z+[0,1]^d$.
  - We want to insert a cube $z'+[0,1]^d$, which do not overlap with C.
  - The condition $z'_i = z_i + 1$ defines an hyperplane in the torus $\mathbf{R}^d/2\mathbf{Z}^d$.
  - Those d hyperplanes have the same $(d-1)$-dimensional volume.
  - In doing the sequential random packing, every one of the d hyperplanes is chosen with equal probability.

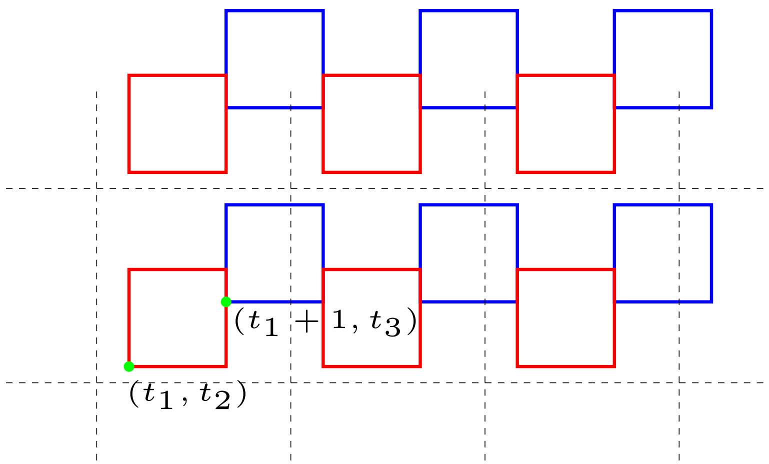
Several cubes

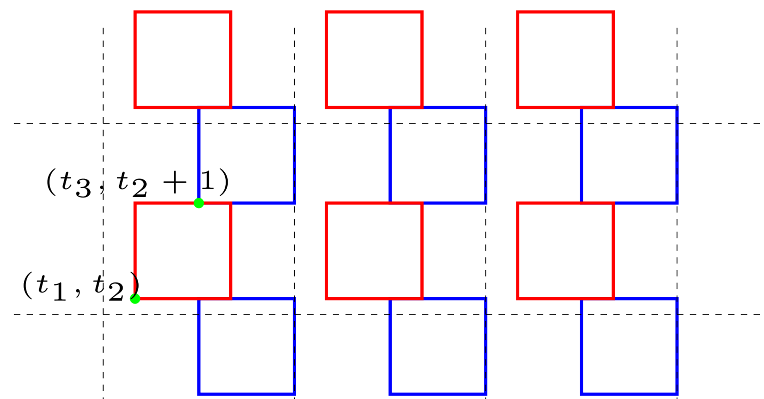
One has several non-overlapping cubes $z^1 + [0,1]^d$, ..., $z^r + [0,1]^d$. We want to add one more cube $z + [0,1]^d$.

- For every cube $z^j + [0,1]^d$, there should exist some $1 \leq i \leq d$ such that $z_i^j \equiv z_i + 1 \pmod 2$
- After enumerating all possible choices, one gets different planes.
- Their dimension might differ.
- Only the one with maximal dimension have strictly positive probability of being attained.
- All planes of the highest dimension have the same volume in the torus $\mathbf{R}^d/2\mathbf{Z}^d$ and so, the same probability of being attained.

The two dimensional case

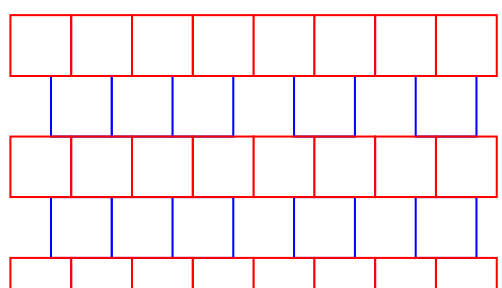
- Put a cube $z + [0,1]^2$ in $\mathbf{R}^2/2\mathbf{Z}^2$. $z = (t_1, t_2)$
- In putting the next cube, two possibilities: $(t_1 + 1, t_3)$ or $(t_3, t_2 + 1)$. They correspond geometrically to:





and they are equivalent.

- Continuing the process, up to equivalence, one obtains:



The 3-dimensional case

- At first step, one puts the vector $z^1 = (t_1, t_2, t_3)$
- At second step, up to equivalence, $z^2 = (t_1 + 1, t_4, t_5)$
- At third step, one generates six possibilities, all with equal probabilities:

$$\begin{matrix} (t_1+1, t_4+1, t_6) & (t_1, t_2+1, t_6) & (t_1, t_6, t_3+1) \\ (t_1+1, t_6, t_5+1) & (t_6, t_2+1, t_5+1) & (t_6, t_4+1, t_3+1) \end{matrix}$$

- Up to equivalence, those possibilities split into 2 cases:
  - $\{(t_1, t_2, t_3), (t_1+1, t_4, t_5), (t_1, t_6, t_3+1)\}$ with probability $\frac{2}{3}$
  - $\{(t_1, t_2, t_3), (t_1+1, t_4, t_5), (t_6, t_2+1, t_5+1)\}$ with probability $\frac{1}{3}$

The 3-dimensional case

- Possible extensions of $\{(t_1, t_2, t_3), (t_1+1, t_4, t_5), (t_1, t_6, t_3+1)\}$ with probability $\frac{2}{3}$ are:
  - (t_1+1, t_7, t_5+1) with 1 parameter
  - (t_1+1, t_4+1, t_7) with 1 parameter
  - (t_1, t_2+1, t_3) with 0 parameter
  - (t_1, t_6+1, t_3+1) with 0 parameter
- Cases with 0 parameters have probability 0, so can be neglected.
- So, up to equivalence, one obtains
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2+1, t_5+1), (t_1, t_6, t_3+1), (t_1+1, t_7, t_5+1)$ with probability $\frac{1}{3}$
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2+1, t_5+1), (t_1, t_6, t_3+1), (t_1+1, t_4+1, t_7)$ with probability $\frac{1}{3}$

The 3-dimensional case

- Possible extensions of $\{(t_1, t_2, t_3), (t_1 + 1, t_4, t_5), (t_6, t_2 + 1, t_5 + 1)\}$ with probability $\frac{1}{3}$ are:

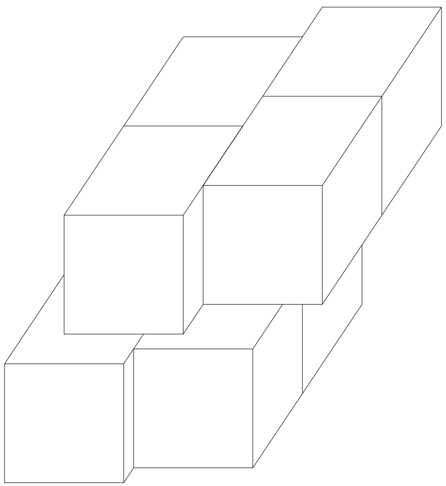
$$(t_6 + 1, t_4 + 1, t_3 + 1) \quad (t_1 + 1, t_2, t_5 + 1) \quad (t_1, t_2 + 1, t_5)$$
$$(t_6 + 1, t_2 + 1, t_5 + 1) \quad (t_1 + 1, t_4 + 1, t_5) \quad (t_1, t_2, t_3 + 1)$$

  All those choices have 0 parameter.

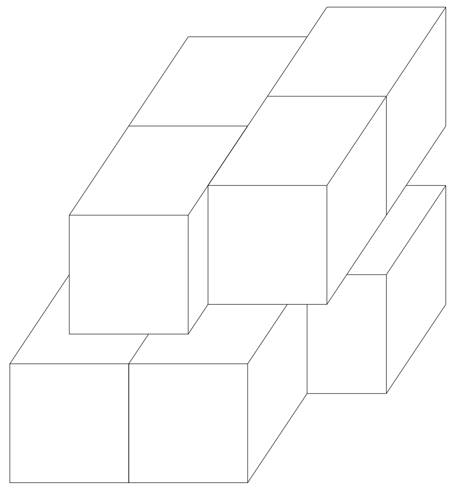
- Those possibilities are in two groups:
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2 + 1, t_5 + 1), (t_6 + 1, t_4 + 1, t_3 + 1)\}$ with probability $\frac{1}{18}$
  - 5 other cases with probability $\frac{5}{18}$.

The 3-dimensional case

- At the end of the process, one obtains



7 parameters, probability $\frac{1}{3}$



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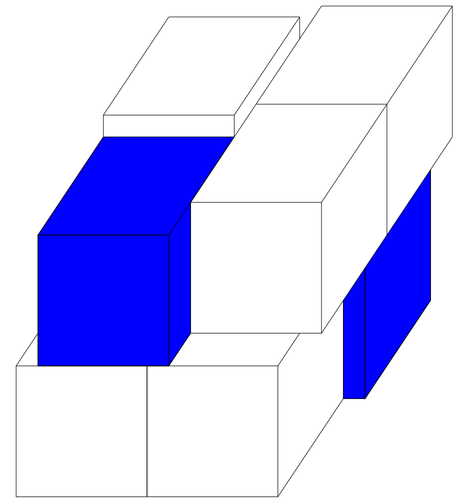

6 parameters, probability $\frac{5}{18}$

- Also, with probability $\frac{1}{18}$, one obtains the non-extendible cube packing with 4 cubes and 6 parameters.

The 4-dimensional case

- Doing the enumeration by computer, one obtains
  - 31 non-extendible continuous cube packings
  - 32 continuous cube tilings.
- The number are lower than in the case $N = 2$, since we do the enumeration only of the one with strictly positive probability.
- One of 31 non-extendible continuous cube-packing has 6 cubes:

z^1	z^2	z^3	z^4	z^5	z^6
t_1	$t_1 + 1$	t_8	t_{10}	$t_8 + 1$	$t_{10} + 1$
t_2	t_5	$t_2 + 1$	$t_5 + 1$	$t_2 + 1$	t_2
t_3	t_6	$t_6 + 1$	$t_3 + 1$	t_3	$t_6 + 1$
t_4	t_7	t_9	$t_9 + 1$	$t_7 + 1$	$t_4 + 1$

Packing density

- Denote by $\alpha_d(\infty)$ the packing density for continuous cube packing and $\alpha_d(N)$ the packing density in $\mathbf{Z}^d/2N\mathbf{Z}^d$.
- One has

$$\alpha_1(\infty) = \alpha_2(\infty) = 1, \quad \alpha_3(\infty) = \frac{35}{36} = 0.972.$$

$$\text{and} \quad \alpha_4(\infty) = \frac{15258791833}{16102195200} = 0.947\ldots$$

- Thm. For any $d \geq 3$, one has $\alpha_d(\infty) < 1$.
- Thm. One has the limit

$$\lim_{N\to\infty} \alpha_d(N) = \alpha_d(\infty)$$

Non-extendible cube packings

- The number of cubes in non-extendible cube packings is at least $n+1$.
- Problem find better lower bound on size of non-extendible cube packing.
- In dimension 5, we found a continuous non-extendible cube packing with 8 cubes. But is it with strictly positive probability?
- Problem Does there exist continuous non-extendible cube packing of lower number of cubes than the one of strictly positive probability?
- Problem Prove that there is no non-extendible cube packing with $2^n - \delta$ cubes with $\delta \leq 3$.