# Torus Cube packings

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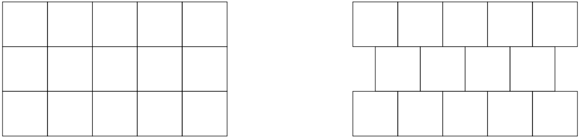
# I. Torus cube tilings and packings

# Cube packings

- A *N-cube packing* is a $2N\mathbb{Z}^n$-periodic packing of $\mathbb{R}^n$ by integral translates of the cube $[0, N]^n$.
- A *N-cube tiling* is a $N$-cube packing with $2^n$ translation classes of cubes.
- There are two types of 2-cube tilings in dimension 2:



- Any $N$-cube packing with $m$ translation classes corresponds in the torus $(\mathbb{Z}/2N\mathbb{Z})^n$ to a packing with $m$ cubes.

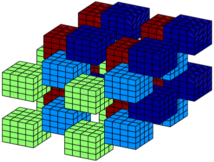
# Keller conjecture

- Conjecture: for any cube tiling of $\mathbb{R}^n$, there exist at least one face-to-face adjacency.
- This conjecture was proved by Perron (1940) for dimension $n \leq 6$.
- Szabo (1986): if there is a counter-example to the conjecture, then there is a counter-example, which is 2-cube tiling.
  - Lagarias & Shor (1992) have constructed counter-example to the Keller conjecture in dimension $n \geq 10$
  - Mackey (2002) has constructed a counter-example in dimension $n \geq 8$.
  - Dimension $n = 7$ remains open.

# Extensibility

- If we cannot extend a $N$-cube packing by adding another cube, then it is called non-extensible.

No non-extensible $N$-cube packings in dimensions 1 and 2.



- Denote by $f_N(n)$ the smallest number of cubes of non-extensible cube packing.
- $f_N(3) = 4$ and $6 \leq f_N(4) \leq 8$.
- For any $n, m \in \mathbb{N}$, the following inequality holds:

$$f_N(n+m) \leq f_N(n) f_N(m) .$$

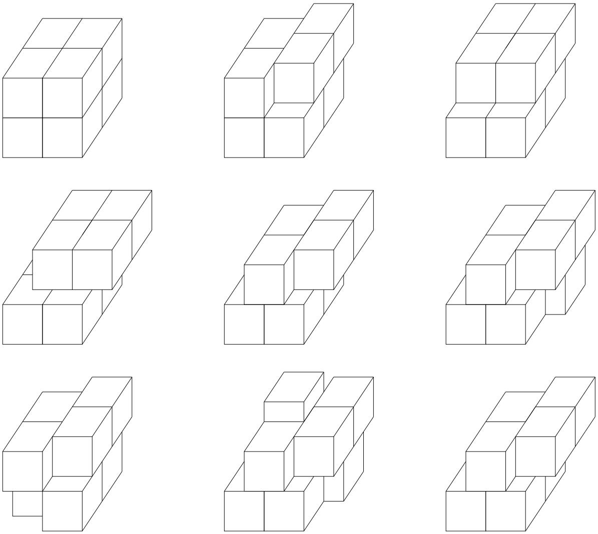
- Conjecture: $f_2(5) = 12$ and $f_2(6) = 16$.

# Clique formalism (case $N = 2$)

- Associate to every cube $C$ its center $c \in \{0, 1, 2, 3\}^n$
- Two cubes with centers $c$ and $c'$ are non-overlapping if and only if there exist a coordinate $i$, such that $|c_i - c'_i| = 2$.
- The graph $G_n$ is the graph with vertex set $\{0, 1, 2, 3\}^n$ and two vertices being adjacent if and only if the corresponding cubes do not overlap. $|Aut(G_n)| = n!8^n$.
- A clique $S$ in a graph is a set of vertices such that any two vertices in $S$ are adjacent.
- Cube tilings correspond to cliques of size $2^n$ in the graph $G_n$.
- We set $L_1 = \{\{v\}\}$ and iterate $i$ from 2 to $2^n$:
  - For every subset in $L_{i-1}$, consider all vertices, which are adjacent to all element in $L_{i-1}$.
  - Test if they are isomorphic to existing elements in $L_i$ and if not, insert them into $L_i$.
- Isomorphism tests are done using the action OnSets of `GAP`, which uses backtrack and is very efficient.

# Results in dimension 3 (case $N = 2$)

In dimension 3, there is a unique non-extensible cube packing and there are 9 types of cube tilings.

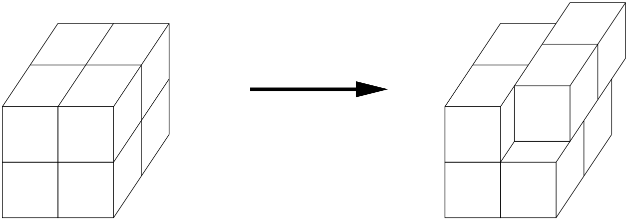


# Results in dimension 4 (case $N = 2$)

- In dimension 4, the number of combinatorial types of cube packings with $N$ cubes is as follows:

| $N$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|
| $nb$ | 38 | 6 | 24 | 0 | 71 | 0 | 0 | 0 | 744 |

- Furthermore all cube packings in dimension 4 can be obtained from the regular cube tiling by following operations:



# Complement of cube packings (case $N = 2$)

- The complement of an non-extensible cube packing $\mathcal{CP}$ is the set $\mathbb{R}^n - \mathcal{CP}$.
- Theorem: There is no complement of size smaller than 4.
- Conjecture: If $\mathcal{CP}$ is a non-extensible cube packing with $2^n - 4$ tiles, then its complement has "same shape", as the one in dimension 3.
- Conjecture: If $\mathcal{CP}$ is a cube packing with $2^n - 5$ cubes, then it is extensible by at least one cube.
- Conjecture: If $\mathcal{CP}$ is a non-extensible cube packing with $2^n - 6$ or $2^n - 7$ cubes, then its complement has "same shape", as the ones in dimension 4.

# II. Continuous torus cube packings

# Torus cube packings

- We consider the torus $\mathbb{Z}^n/2N\mathbb{Z}^n$ and do sequential random packing by cubes $z + [0, N]^n$ with $z \in \mathbb{Z}^n$.
- We denote $M_N^T(n)$ the number of cubes in the obtained torus cube packing and the average density of cube packing:

$$\frac{1}{2^n} E(M_N^T(n))$$

- We are interested in the limit $N \to \infty$.
- The cube packings obtained in the limit will be called continuous cube packings and we will develop a combinatorial formalism for dealing with them.

# Continuous cube packings

- We consider the torus $\mathbb{R}^n/2\mathbb{Z}^n$ and do sequential random packing by cubes $z + [0,1]^n$ with $z \in \mathbb{R}^n$.
- Two cubes $z + [0,1]^n$ and $z' + [0,1]^n$ are non-overlapping if and only if there is $1 \leq i \leq n$ with $z'_i \equiv z_i + 1 \pmod 2$
- Fix a cube $C = z^1 + [0,1]^n$.
  - We want to insert a cube $z + [0,1]^n$, which do not overlap with $C$.
  - The condition $z_i = z^1_i + 1$ defines an hyperplane in the torus $\mathbb{R}^n/2\mathbb{Z}^n$.
  - Those $n$ hyperplanes have the same $(n-1)$-dimensional volume.
  - In doing the sequential random packing, every one of the $n$ hyperplanes is chosen with equal probability.

# Several cubes

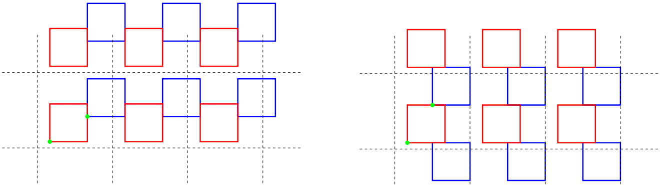
One has several non-overlapping cubes $z^1 + [0,1]^n$, ..., $z^r + [0,1]^n$. We want to add one more cube $z + [0,1]^n$.

- For every cube $z^j + [0,1]^n$, there should exist some $1 \leq i \leq n$ such that $z_i \equiv z_i^j + 1 \pmod 2$
- After enumerating all possible choices, one gets different planes.
- Their dimension might differ.
- Only the one with maximal dimension have strictly positive probability of being attained.
- All planes of the highest dimension have the same volume in the torus $\mathbb{R}^n / 2\mathbb{Z}^n$ and so, the same probability of being attained.

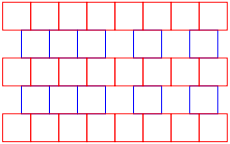
Definition: The number of cubes of a continuous cube packing is $N(\mathcal{CP})$ and its number of parameters is $m(\mathcal{CP})$.

# The two dimensional case

- Put a cube $z + [0,1]^2$ in $\mathbb{R}^2/2\mathbb{Z}^2$. $z = (t_1, t_2)$
- In putting the next cube, two possibilities: $(t_1 + 1, t_3)$ or $(t_3, t_2 + 1)$. They correspond geometrically to:



  and they are equivalent.
- Continuing the process, up to equivalence, one obtains:



# The 3-dimensional case

- At first step, one puts the vector $c^1 = (t_1, t_2, t_3)$
- At second step, up to equivalence, $c^2 = (t_1 + 1, t_4, t_5)$
- At third step, one generates six possibilities, all with equal probabilities:

$$\begin{array}{ccc} (t_1+1, t_4+1, t_6) & (t_1, t_2+1, t_6) & (t_1, t_6, t_3+1) \\ (t_1+1, t_6, t_5+1) & (t_6, t_2+1, t_5+1) & (t_6, t_4+1, t_3+1) \end{array}$$

- Up to equivalence, those possibilities split into 2 cases:
  - $\{(t_1, t_2, t_3), (t_1 + 1, t_4, t_5), (t_1, t_6, t_3 + 1)\}$ with probability $\frac{2}{3}$
  - $\{(t_1, t_2, t_3), (t_1 + 1, t_4, t_5), (t_6, t_2 + 1, t_5 + 1)\}$ with probability $\frac{1}{3}$

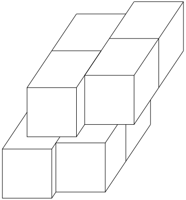
- Possible extensions of $\{(t_1, t_2, t_3), (t_1+1, t_4, t_5), (t_1, t_6, t_3+1)\}$ with probability $\frac{2}{3}$ are:
  - $(t_1+1, t_7, t_5+1)$ with 1 parameter
  - $(t_1+1, t_4+1, t_7)$ with 1 parameter
  - $(t_1, t_2+1, t_3)$ with 0 parameter
  - $(t_1, t_6+1, t_3+1)$ with 0 parameter
- Cases with 0 parameters have probability 0, so can be neglected.
- So, up to equivalence, one obtains
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2+1, t_5+1), (t_1, t_6, t_3+1), (t_1+1, t_7, t_5+1)$ with probability $\frac{1}{3}$
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2+1, t_5+1), (t_1, t_6, t_3+1), (t_1+1, t_4+1, t_7)$ with probability $\frac{1}{3}$

- Possible extensions of $\{(t_1, t_2, t_3), (t_1+1, t_4, t_5), (t_6, t_2+1, t_5+1)\}$ with probability $\frac{1}{3}$ are:

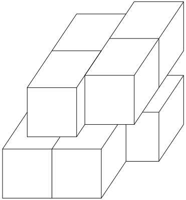
$$\begin{array}{ccc} (t_6+1, t_4+1, t_3+1) & (t_1+1, t_2, t_5+1) & (t_1, t_2+1, t_5) \\ (t_6+1, t_2+1, t_5+1) & (t_1+1, t_4+1, t_5) & (t_1, t_2, t_3+1) \end{array}$$

  All those choices have 0 parameter.
- Those possibilities are in two groups:
  - $\{(t_1, t_2, t_3), \ldots, (t_6, t_2+1, t_5+1), (t_6+1, t_4+1, t_3+1)\}$ with probability $\frac{1}{18}$
  - 5 other cases with probability $\frac{5}{18}$.

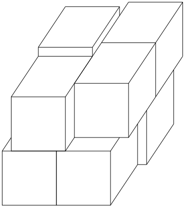
- At the end of the process, one obtains



7 parameters, probability $\frac{1}{3}$

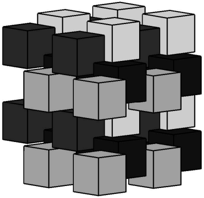


7 parameters, probability $\frac{1}{3}$



6 parameters, probability $\frac{5}{18}$

- Also, with probability $\frac{1}{18}$, one obtains the non-extensible cube packing with 4 cubes and 6 parameters.



# IV. Computer methods

# Automorphy/isomorphy questions

- The program `nauty` of Brendan McKay allows to find the automorphism group of a finite graph $G$ and to test if two graphs are isomorphic.
- Those two problems are not expected to be solvable in polynomial time, but `nauty` is extremely efficient in doing those computations.
- If one has a finite combinatorial object (edge colored graphs, set-system, etc.), we associate to it a graph, which encodes all its properties.
- We then use `nauty` to test if the combinatorial objects are isomorphic, to compute their automorphism groups, etc.
- `nauty` can deal with directed graph but this is not recommended, it can also deal with vertex colors.
- Another feature is to be able to get a canonical representative of a graph, which is helpful in enumeration purposes.

# The combinatorial object used here

- We want to define a characteristic graph $G(\mathcal{CP})$ for any continuous cube packing $\mathcal{CP}$ such that:
  - if $\mathcal{CP}_1$ and $\mathcal{CP}_2$ are two cube packings, then $\mathcal{CP}_1$ and $\mathcal{CP}_2$ are isomorphic if and only if $G(\mathcal{CP}_1)$ and $G(\mathcal{CP}_2)$ are isomorphic.
  - If $\mathcal{CP}$ is a cube packing, then the group $Aut(\mathcal{CP})$ is isomorphic to the group $Aut(G(\mathcal{CP}))$.
- If $\mathcal{CP}$ is a $n$-dimensional cube packing then we define a graph with $n \times N(\mathcal{CP}) + 2 \times m(\mathcal{CP})$ vertices:
  - Every cube of center $v^i = (v^i_1, \dots, v^i_n)$ correspond to $n$ vertices $v^i_j$.
  - Every parameter $t_i$ correspond to two vertices $t_i$, $t_i + 1$.

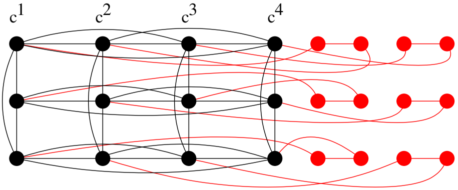
  and the following edges:
  - Every $v^i_j$ is adjacent to all $v^{i'}_j$ and to all $v^i_{j'}$
  - The vertices $t_i$ and $t_i + 1$ are adjacent.
  - If $v^i_j$ is $t_i$, then we make it adjacent to the vertex $t_i$.

# Example of the non-extensible cube packing in dimension 3

- The non-extensible cube packing of dimension 3 with 4 cubes:

$$\begin{array}{lcllll}
c^1 & = & ( & t_1, & t_2, & t_3 & ) \\
c^2 & = & ( & t_1+1, & t_4, & t_5 & ) \\
c^3 & = & ( & t_6, & t_2+1, & t_5+1 & ) \\
c^4 & = & ( & t_6+1, & t_4+1, & t_3+1 & )
\end{array}$$

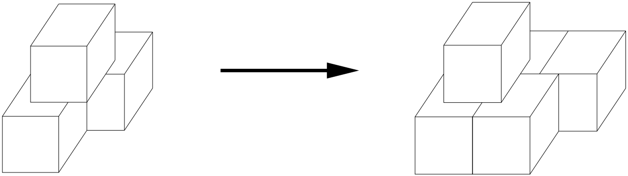
- The corresponding graph is




- The symmetric group of the structure is Sym(4), the group on the cubes is Sym(4), the group on the coordinates is Sym(3) and the group on the parameter is Sym(4) acting on 6 points.

# How to enumerate continuous cube packings

- The basic technique is to enumerate all continuous cube packings with $n$ cubes and then to add in all possible ways another cube.
- We use nauty to resolve isomorphy questions between the generated cube packings.
- In dimension 4 this technique works in 4 days.
- Suppose that we have a hole in a cube packing and that there is only one way to put a cube and that this choice will not overlap with other choices:



Enumeration time reduces to 5 minutes.

# Enumeration results

| $N$ | $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\infty$ | Nr cube tilings | 1 | 1 | 3 | 32 | ? |
| | Nr non-extensible cube packings | 0 | 0 | 1 | 31 | ? |
| | $f_{>0,\infty}(n)$ | 2 | 4 | 4 | 6 | 6 |
| | $f_{\infty}(n)$ | 2 | 4 | 4 | 6 | 6 |
| | $E(M_{\infty}^{T}(n))$ | 1 | 1 | $\frac{35}{36}$ | $\frac{15258791833}{16102195200}$ | ? |
| 2 | Nr cube tilings | 1 | 2 | 9 | 744 | ? |
| | Nr cube packings | 0 | 0 | 1 | 139 | ? |
| | $f_2(n)$ | 2 | 4 | 4 | 8 | $10 \leq f_2(5) \leq 12$ |

- $f_{\infty}(n)$ is the minimum number of cubes of non-extensible $n$-dimensional continuous cube packings.
- $f_{>0,\infty}(n)$ is the minimum number of cubes of non-extensible $n$-dimensional continuous cube packings, obtained with strictly positive probability.

# Test positive probability

- Suppose that one has a continuous cube packing $\mathcal{CP}$ and we want to check if it can be obtained with strictly positive probability.
- The cubes are of the form $z^1 + [0,1]^n, \ldots, z^M + [0,1]^n$.
- To be obtainable with strictly positive probability, means there exist a permutation $\sigma \in \mathrm{Sym}(M)$ such that we can obtain

$$z^{\sigma(1)} + [0,1]^n,\ z^{\sigma(2)} + [0,1]^n,\ \ldots,\ z^{\sigma(M)} + [0,1]^n$$

  in this order.
- The method of obtention is to consider all possibilities sequentially and backtrack when

$$z^{\sigma(1)} + [0,1]^n,\ \ldots,\ z^{\sigma(M')} + [0,1]^n \text{ with } M' \leq M$$

  is obtained with zero probability.

# III. Lamination and packing density

# Product construction

If $\mathcal{CP}$, $\mathcal{CP}'$ are $n$-, $n'$-dimensional continuous cube packings, we want to construct a product $\mathcal{CP} \times \mathcal{CP}'$.

- If the cubes of $\mathcal{CP}$, $\mathcal{CP}'$ are $(z^i + [0,1]^n)_{1 \leq i \leq N}$, $(z'^j + [0,1]^{n'})_{1 \leq j \leq N'}$ then:
  - We form $N$ independent copies of $\mathcal{CP}'$: $({z'}^{i,j} + [0,1]^{n'})_{1 \leq j \leq N'}$
  - We form the cube packing $\mathcal{CP} \ltimes \mathcal{CP}'$ with cubes $(z^i, {z'}^{i,j}) + [0,1]^{n+n'}$ for $1 \leq i \leq N$ and $1 \leq j \leq N'$.
- This product $\mathcal{CP} \ltimes \mathcal{CP}'$ has the following properties:
  - $m(\mathcal{CP} \ltimes \mathcal{CP}') = m(\mathcal{CP}) + N(\mathcal{CP}) m(\mathcal{CP}')$
  - If $\mathcal{CP}$ and $\mathcal{CP}'$ are non-extensible then $\mathcal{CP} \ltimes \mathcal{CP}'$ is non-extensible.
  - If $\mathcal{CP}$ and $\mathcal{CP}'$ are obtained with strictly positive probability and $\mathcal{CP}$ is non-extensible then $\mathcal{CP} \ltimes \mathcal{CP}'$ is obtained with strictly positive probability.

# Packing density

- Denote by $\alpha_n(\infty)$ the packing density for continuous cube packing and $\alpha_n(N)$ the packing density in $\mathbb{Z}^n/2N\mathbb{Z}^n$.
- One has

$$\alpha_1(\infty) = \alpha_2(\infty) = 1, \quad \alpha_3(\infty) = \frac{35}{36} = 0.972.$$

$$\text{and} \quad \alpha_4(\infty) = \frac{15258791833}{16102195200} = 0.947...$$

- Theorem: For any $n \geq 3$, one has $\alpha_\infty(n) < 1$.
  Proof: Take the 3-dimensional continuous non-extensible cube packing $\mathcal{CP}_1$ with 4 cubes (with $> 0$). Take a continuous non-extensible cube packing $\mathcal{CP}_2$ of dimension $n-3$ (with $> 0$). Then the product $\mathcal{CP}_1 \ltimes \mathcal{CP}_2$ is non-extensible and obtained with strictly positive probability, which proves $\alpha_n(\infty) < 1$.
- Theorem: One has the limit

$$\lim_{N\to\infty} \alpha_N(n) = \alpha_\infty(n)$$

# Non-extensible cube packings

- Theorem: If a $n$-dimensional continuous cube packing has $2^n - \delta$ cubes with $\delta \leq 3$ then it is extensible.
- Take $\mathcal{CP}$ such a continuous cube packing and assign a value $\alpha_i$ to the parameters $t_i$ such that if $i \neq j$ then $\alpha_i \neq \alpha_j, \alpha_j + 2$ (mod 2).
- Lemma: Given
  - a cube packing with $2^n - \delta$ cubes of coordinates $x^i$, $1 \leq i \leq 2^n - \delta$,
  - a coordinate $k$ and a value $\alpha \in \mathbb{R}$

  The induced cube packing is the cube packing of $\mathbb{R}^{n-1}$ obtained by taking all vectors $x^i$ with $x^i_k \in [\alpha, \alpha + 1[$ and removing the $k$-th coordinate.
  Such cube packings have at least $2^{n-1} - \delta$ tiles.
- The proof is then by induction.

# II. Number of parameters

# The numbers $N_k(\mathcal{CP})$

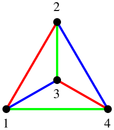
- Let $\mathcal{CP}$ be a non-extensible cube packing obtained with strictly positive probability.
  - We denote by $N_k(\mathcal{CP})$ the number of cubes which occurs with $k$ new parameters.
  - We have $N_n(\mathcal{CP}) = 1$ and $N_{n-1}(\mathcal{CP}) = 1$.
  - $N_k(\mathcal{CP}) \geq 1$
- The total number of cubes is $N(\mathcal{CP}) = \sum_{k=0}^{n} N_k(\mathcal{CP})$; $N(\mathcal{CP}) \geq n + 1$.
- The total number of parameters is $m(\mathcal{CP}) = \sum_{k=1}^{n} kN_k(\mathcal{CP})$; $m(\mathcal{CP}) \geq \frac{n(n+1)}{2}$.
- Conjecture: If $\mathcal{CP}$ is a non-extensible continuous cube packing obtained with strictly positive probability then:
  - For all $k \geq 1$ we have $\sum_{l=0}^{k} N_{n-l} \leq 2^k$
  - We have $m(\mathcal{CP}) \leq 2^n - 1$

# Minimal number of cubes

- Theorem: If $\mathcal{CP}$ is a non-extensible cube packing with $n+1$ cubes then:
  - Its number of parameter is $\frac{n(n+1)}{2}$
  - In every coordinate a parameter appear exactly one time as $t$ and exactly one time as $t+1$
- Consequences:
  - If $n$ is even there is no such cube packing
  - If $n$ is odd such cube packings correspond to 1-factorization of the graph $\mathrm{K}_{n+1}$, i.e. a set of $n$ perfect matching in $\mathrm{K}_{n+1}$, which partitions the edge set.

$$
\begin{array}{llll}
c^1=( & t_1, & t_2, & t_3 \quad ) \\
c^2=( & t_1+1, & t_4, & t_5 \quad ) \\
c^3=( & t_6, & t_2+1, & t_5+1 \quad ) \\
c^4=( & t_6+1, & t_4+1, & t_3+1 \quad )
\end{array}
$$

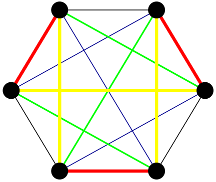
The 3-dim. non-extensible cube packing



The 1-factorization of $\mathrm{K}_4$

# One-factorizations of $K_{n+1}$

- The graph $K_6$ has exactly one 1-factorization with symmetry group Sym(5), i.e. the group Sym(5) acts on 6 elements.



| graph | Nr | authors |
|---|---|---|
| $K_6$ | 1 | |
| $K_8$ | 6 | 1906, Dickson, Safford |
| $K_{10}$ | 396 | 1973, Gelling |
| $K_{12}$ | 526915620 | 1993, Dinitz, Garnick, McKay |

- Every graph $K_{2p}$ has at least one 1-factorization.
- So, for $n$ odd, there is a non-extensible cube packing with $n+1$ cubes.

# Minimal number of cubes in even dimension

- If $n$ is even, then $f_\infty(n) \geq n+1$.
- For $n=4$, this minimum is attained by the following structure:

$$H = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_4+1 \\ t_1+1 & t_8 & t_7+1 & t_9 \\ t_5+1 & t_8+1 & t_3+1 & t_{10} \\ t_1+1 & t_6+1 & t_7 & t_{10}+1 \\ t_5 & t_2+1 & t_7+1 & t_9+1 \end{pmatrix}$$

with probability $\frac{1}{480}$. $|Aut(H)| = 4$ and $m(H) = \frac{4(4+1)}{2} = 10$

- Conjecture: If $n$ is even then $f_\infty(n) = n+2$ and one of the structures realizing it has $\frac{n(n+1)}{2}$ parameters.

# Minimal 6-dimensional non-extensible cube packings

- Instead of adding rows, we add columns. We first determine columns types and then add columns in all possible ways and reduce by isomorphism.
- We find 9 non-extensible continuous cube packings with at least $\frac{6(6+1)}{2} = 21$ parameters, all with zero probability. So, $8 = f_\infty(6) < f_{>0,\infty}(6)$.
- One of them has 21 parameters and $|Aut| = 4$:

$$\begin{pmatrix}
t_1 & t_5 & t_9 & t_{14}+1 & t_{17}+1 & t_{19} \\
t_1+1 & t_6 & t_{10} & t_{13}+1 & t_{16}+1 & t_{19} \\
t_2 & t_5+1 & t_{11} & t_{13} & t_{18} & t_{20} \\
t_2+1 & t_7 & t_9+1 & t_{15} & t_{16} & t_{21} \\
t_3 & t_6+1 & t_{12} & t_{14} & t_{18}+1 & t_{21}+1 \\
t_3+1 & t_8 & t_{10}+1 & t_{15}+1 & t_{17} & t_{20}+1 \\
t_4 & t_7+1 & t_{12}+1 & t_{13}+1 & t_{17}+1 & t_{19}+1 \\
t_4+1 & t_8+1 & t_{11}+1 & t_{14}+1 & t_{16}+1 & t_{19}+1
\end{pmatrix}$$

Column types:

$(1,1)^4$ (3 times), $(1,1),(2,1)^2$ (2 times), $(1,1)^2,(2,2)$ (1 time).

# Full cube tilings with minimal number of parameters

- Question For which $n$, there is a non-extensible cube tiling with $\frac{n(n+1)}{2}$ parameters?
- There is existence and unicity for $n \leq 4$.
- We concentrate on the existence question.
- For $n = 5$, we obtain by random computer search one such structure.
- The first 5 cubes are organized in the following way.

$$H_5 = \begin{pmatrix} t'_1 & t_3+1 & t_6+1 & t_8 & t_9 \\ t_1 & t'_2 & t_5+1 & t_8+1 & t_{10} \\ t_2 & t_3 & t'_3 & t_7+1 & t_{10}+1 \\ t_2+1 & t_4 & t_5 & t'_4 & t_9+1 \\ t_1+1 & t_4+1 & t_6 & t_7 & t'_5 \end{pmatrix}$$

This block structure can be generalized immediately for $n$ odd. Its symmetry group is the dihedral group $D_{2n}$.

# Search of structures

- The next 5 cubes have a specific form:

$$H_5 + I_5 = \begin{pmatrix} t_1' + 1 & t_3 + 1 & t_6 + 1 & t_8 & t_9 \\ t_1 & t_2' + 1 & t_5 + 1 & t_8 + 1 & t_{10} \\ t_2 & t_3 & t_3' + 1 & t_7 + 1 & t_{10} + 1 \\ t_2 + 1 & t_4 & t_5 & t_4' + 1 & t_9 + 1 \\ t_1 + 1 & t_4 + 1 & t_6 & t_7 & t_5' + 1 \end{pmatrix}$$

- Then we have 2 cubes of coordinates

$$\begin{pmatrix} t_1 & t_3 & t_5 & t_7 & t_9 \\ t_1 + 1 & t_3 + 1 & t_5 + 1 & t_7 + 1 & t_9 + 1 \end{pmatrix}$$

- Then we have 2 orbits of 10 cubes with a more complicate structure.

# Permutation formalism

- Consider the block structure

$$\begin{pmatrix} t'_1 & t_3+1 & t_6+1 & t_8 & t_9 \\ t_1 & t'_2 & t_5+1 & t_8+1 & t_{10} \\ t_2 & t_3 & t'_3 & t_7+1 & t_{10}+1 \\ t_2+1 & t_4 & t_5 & t'_4 & t_9+1 \\ t_1+1 & t_4+1 & t_6 & t_7 & t'_5 \end{pmatrix}$$

  of 5 cubes $C_i$, $1 \leq i \leq 5$.
- If $C$ is non-overlapping cube, then for every $i$ it should have a coordinate $\sigma(i)$, different from 1 with the cube $C_i$.
- A coordinate can differ from 1 with only one cube. This means that no new parameter can show up and that new cubes are encoded by a permutation $\sigma$ of Sym(5).

# Equivariant computer search

- We consider the $n + n + 2$ cubes obtained in case $n = 5$. We impose the symmetry $D_{2n}$ and search for all possibilities of extension.
- For $n = 7$ and $n = 9$ we found exactly one such continuous cube tiling.
- For $n = 11$, the number of possibilities is much larger. We needed to reprogram in C++ and doing a computer search we found no such cube tiling.

# II. Further research

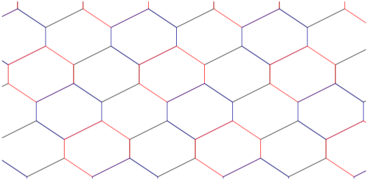
# Parallelotope extension

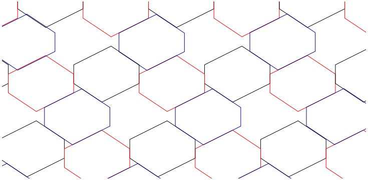
- A parallelotope is a polytope $P$, which tiles the space by translation.

| Dimension | Nr. types | Authors |
|---|---|---|
| 2 | 2 (hexagon, parallelogram) | Dirichlet (1860) |
| 3 | 5 | Fedorov (1885) |
| 4 | 52 | Delaunay, Shtogrin (1973) |
| 5 | 179377 | Engel (2000) |

The set of translation vector form a lattice

- If $P$ is a parallelotope in $\mathbb{R}^n$ of lattice $L$, then we consider random packing of $P + 2L$ in $\mathbb{R}^n$:





# THANK

# YOU


- M. Dutour Sikirić, Y. Itoh and A. Poyarkov, *Cube packing, second moment and holes*, European Journal of Combinatorics **28-3** (2007) 715–725.
- M. Dutour Sikirić and Y. Itoh, *Continuous random cube packings in cube and torus*, in preparation.


- Programs at `http://www.liga.ens.fr/~dutour/Documents/PackingProbability.tar.gz`