# The homology of $\mathrm{PSL}_4(\mathbb{Z})$


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# I. $G$-module and group resolutions

# $G$-modules

- We use the GAP notation for group action, on the right.
- A $G$-module $M$ is a $\mathbb{Z}$-module with an action

$$\begin{array}{rcl} M \times G & \rightarrow & M \\ (m, g) & \mapsto & m.g \end{array}$$

- The group ring $\mathbb{Z}G$ formed by all finite sums

$$\sum_{g \in G} \alpha_g g \text{ with } \alpha_g \in \mathbb{Z}$$

  is a $G$-module.
- If the orbit of a point $v$ under a group $G$ is $\{v_1, \dots, v_m\}$, then the set of sums

$$\sum_{i=1}^{m} \alpha_i v_i \text{ with } \alpha_i \in \mathbb{Z}$$

  is a $G$-module.
- We can define the notion of generating set, free set, basis of a $G$-module. But not every finitely generated $G$-module admits a basis.

# Resolutions

Take $G$ a group.

- A resolution of a group $G$ is a sequence of $G$-modules $(M_i)_{i\geq 0}$:
$$\mathbb{Z} \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$
together with a collection of $G$-linear operators $d_i : M_i \to M_{i-1}$ such that Ker $d_i = \mathrm{Im}\, d_{i-1}$
- What is useful to homology computations are free resolutions with all $M_i$ being free $G$-modules.
- The homology is then obtained by killing off the $G$-action of a free resolution, i.e replacing the $G$-modules $(\mathbb{Z}G)^k$ by $\mathbb{Z}^k$, replacing accordingly the $d_i$ by $\tilde{d}_i$ and getting
$$H_i(G) = \text{Ker } \tilde{d}_i / \mathrm{Im}\, \tilde{d}_{i-1}$$

# Examples of resolutions

- If $C_m =< x >$ is a cyclic group then a resolution $(R_n)_{n\geq 0}$ with $R_n = \mathbb{Z}C_m$ is obtained by taking the differential $d_{2n+1} = 1 - x$ and $d_{2n} = 1 + x + \dots + x^{m-1}$
- For the dihedral groups $D_{2m}$ of order $2m$ with $m$ odd there exist periodic resolutions.
- If a group $G$ admits an action on a $m$-dimensional polytope $P$ which is fixed point free then it admits a periodic free resolution of length $m$.
  The technique is to glue together the cell complexes, which are free:

$$\mathbb{Z} \leftarrow C_0(P) \leftarrow C_1(P) \leftarrow \dots \leftarrow C_{m-1}(P) \leftarrow C_0(P) \leftarrow \dots$$

# Products

- If $G$ and $G'$ are two groups of resolution $R$ and $R'$ then the tensor product $S = R \otimes R'$ is a resolution for $G \times G'$. That is

$$S_n = \oplus_{i+j=n} R_i \otimes R'_j$$

with the differential at $R_i \otimes R'_j$

$$D_{ij} = d_i + (-1)^i d'_j$$

- If $N$ is a normal subgroup of $G$ and we have a free resolution $R$ for $N$ and a free resolution $R'$ for the quotient $G/N$ then it is possible to put together the twisted tensor product $S = R \tilde{\otimes} R'$ and get a resolution for $G$. That is

$$S_n = \oplus_{i+j=n} R_i \otimes R'_j$$

with the differential at $R_i \otimes R'_j$

$$D_{ij} = d_0 + d_1 + \cdots + d_i$$

with $d_k : R_i \otimes R'_j \to R_{i-k} \otimes R'_{j+k-1}$

# The $p$-rank

- A $p$-group $G$ is called elementary Abelian if it is isomorphic to $(\mathbb{Z}_p)^n$ with $n$ being called the rank of $G$.
- If $G$ is a finite group then the $p$-rank of $G$ is the largest rank of its elementary Abelian $p$-subgroups.
- For a finite group $G$ of maximum $p$-rank $m$ the dimension of its free resolutions grows at least like $n^{m-1}$.
- For products $C_{m_1} \times \dots C_{m_p}$ of cyclic groups the dimensions of resolutions is at least $\binom{n+p}{p}$.
- One example:

```
gap> GRP:=AlternatingGroup(4);;
gap> R:=ResolutionFiniteGroup(GRP, 20);;
gap> List([1..20], R!.dimension);
[ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 11, 11, 12, 12, 11, 11,
12, 12, 12, 13 ]
```

# II. The CTC Wall lemma

# Using non-free resolutions

- Suppose we have a $G$-resolution,

$$\mathbb{Z} \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

  which is not free
- Then we find free resolutions for every module $M_i$ and sum it to get a resolution of the form

$$\begin{array}{ccccc} R_{0,2} & \leftarrow & R_{1,2} & \leftarrow & R_{2,2} \\ \downarrow & & \downarrow & & \downarrow \\ R_{0,1} & \leftarrow & R_{1,1} & \leftarrow & R_{2,1} \\ \downarrow & & \downarrow & & \downarrow \\ R_{0,0} & \leftarrow & R_{1,0} & \leftarrow & R_{2,0} \end{array}$$

- The differentials involved are $d_k : R_{i,j} \to R_{i-k,j+k-1}$ and the summands are

$$S_n = \oplus_{i+j=n} R_{i,j}$$

# CTC Wall lemma

- We denote $d_1$ the operator of the $R_{i,0} \to R_{i-1,0}$.
- We can find free resolutions of the $R_{i,0}$ $G$-modules by $G$-modules

$$R_{i,0} \leftarrow R_{i,1} \leftarrow R_{i,2} \leftarrow \dots$$

  with the boundary operators being named $d_0$.
- Then we search for operators $d_k : R_{i,j} \to R_{i-k,j-1+k}$ such that

$$D = \sum_{i=0}^{\infty} d_i$$

  realize a free resolutions of $G$-modules $\sum_{i+j=k} R_{i,j}$.
- It suffices to solve the equations

$$\sum_{h=0}^{k} d_h d_{k-h} = 0$$

# CTC Wall lemma

- One way is to have the expression

$$d_k = -h_0(\sum_{h=1}^{k} d_h d_{k-h})$$

  with $h_0$ a contracting homotopy for the $d_0$ operator, i.e. an operator $h_0 : R_{i,j} \to R_{i,j+1}$ such that $d_0(h_0(x)) = x$ if $x$ belongs to the image of $d_0$.
- This gives a recursive method for computing first $d_1$ from the relations

$$d_1 d_0 + d_0 d_1 = 0$$

- Then $d_2$, $d_3$, ...
- CTC Wall also gives a contracting homotopy for the obtained resolution.

# CTC Wall lemma for polytopes: right cosets

- Take $R_{k,0}$, which is sum of orbits $O_i$ of faces of dimension $k$.
- We compute resolutions $\tilde{R}_{k,i,l}$ of Stab $f_l$ with $f_l$ representative of $O_l$.

$$R_{k,i} = \oplus_{l=1}^{r} \tilde{R}_{k,i,l}$$

- The matrix of the operator $d_0 : R_{k,i} \to R_{k,i-1}$ is then a block matrix of the $\tilde{d}_0 : \tilde{R}_{k,i,l} \to \tilde{R}_{k,i-1,l}$
- For the contracting homotopy of a vector $v \in R_{k,i}$:
  - Decompose $v$ into components $v_l \in \tilde{R}_{k,i,l} \otimes \mathbb{Z}G$
  - Decompose $v_l$ into right cosets

    $$v_l = \sum_s v_{s,l} g_s$$

    with $g_s \in G$ distinct right Stab $f_l$-cosets and $v_{s,l} \in \tilde{R}_{k,i,l}$.
  - Apply the contracting homotopy $h_0$ of the resolution $\tilde{R}_{k,i,l}$ to $v_{s,l}$ and sum

    $$h_0(v_l) = \sum_s h_0(v_{s,l}) g_s$$

# CTC Wall lemma for polytopes: twisting

- If $F$ is a face of a polyhedral tesselation, $H = \mathrm{Stab}(F)$ and $f \in \mathrm{Stab}(F)$ then we need to consider whether or not $f$ inverts the orientation of $F$.
- This define a subgroup $\mathrm{Rot}(F)$ of $\mathrm{Stab}(F)$ formed by all elements stabilizing the face $F$ and its orientation.
- We define a twisting accordingly:

$$\sigma(f) = \begin{cases} 1 & \text{if } f \in \mathrm{Rot}(F) \\ -1 & \text{otherwise} \end{cases}$$

- The twisting $\sigma$ acts on $\mathbb{Z}G$ and it can twist accordingly a given resolution of $\mathrm{Stab}(F)$.
- For the contracting homotopy, the solution to the twisting problem is accordingly:

$$\sigma(d_i)y = x \Leftrightarrow d_i\sigma(y) = \sigma(x)$$

- Twisting does not change the dimension of the resolutions, but it does change their homology.

# III. The $PSL_4(\mathbb{Z})$ case

# Resolutions for $A_5$, $A_4$ and $S_4$

- The symmetry group of the Icosahedron is the group $W(H_3) = \mathbb{Z}_2 \times A_5$. The subgroup $A_5$ acts on it with stabilizers
  - $C_5$ for vertices (1 orbit) nontwisted
  - $C_2$ for edges (1 orbit) twisted
  - $C_3$ for faces (1 orbit) nontwisted

  Thus by applying the CTC Wall lemma we get a resolution $(R_n)_{n \geq 0}$ with dim $R_n = n + 1$. This is about the best possible since the 2-rank is 2.
- Similarly the symmetry group of the Tetrahedron is $S_4$ and $A_4$ acts on the Tetrahedron with stabilizers $C_3$, $C_2$ and $C_3$.
- The symmetry group of the cube is $W(B_3)$ of size 48 and it contains two conjugacy classes of subgroups isomorphic to $S_4$, one comes from the tetrahedron and the other has stabilizers $C_4$, $C_2$ and $C_3$.

# Perfect forms in dimension 4

- Tessel1: Initially there are 2 orbits of perfect forms so full dimensional cells are:
  - $O_1$: full dimensional cell with 64 facets and stabilizer of size 288.
  - $O_2$: full dimensional cell with 10 facets and stabilizer of size 60.
- Tessel2: Now split both $O_1$ by adding a central ray. We then get as orbits of full dimensional cells:
  - $O_{1,1}$: full dimensional cell with 10 facets and stabilizer of size 6.
  - $O_{1,2}$: full dimensional cell with 10 facets and stabilizer of size 2.
  - $O_{2,1}$: full dimensional cell with 10 facets and stabilizer of size 6.
- Tessel3: Every cell $O_{1,1}$ is adjacent to a unique cell $O_{2,1}$. Join them:
  - $O'_1$: full dimensional cell with 18 facets and stabilizer of size 6.
  - $O'_2$: full dimensional cell with 10 facets and stabilizer of size 2.

# The stabilizers in Tessel1

For the first tesselation Tessel1, we have the following orbits of faces and stabilizers:

0 $A_5$, $G_{288} = (A_4 \times A_4) : C_2$.

1 $S_3$, $S_3 \times S_3$.

2 $C_2 \times C_2$, $C_2 \times C_2$.

3 $C_2 \times C_2$, $C_2 \times C_2$, $S_4$, $S_4$.

4 $S_3$, $D_8$, $S_4$, $C_2 \times S_3 \times S_3$.

5 $D_{24}$, $S_4$, $A_5$.

6 $G_{96} = ((C_2 \times C_2 \times C_2 \times C_2) : C_3) : C_2$.

Since we have the quotients

$$G_{288}/A_4 \simeq S_4 \quad \text{and} \quad G_{96}/C_2 \times C_2 \simeq S_4$$

by combining everything we can find resolutions of asymptotically optimal dimensions for all stabilizers.

# For Tessel3

0 $S_3$, $S_3 \times S_3$.

1 $C_2 \times C_2$, $C_2$, $C_2$, $C_2$.

2 $C_2$, $C_2$, $C_2$, $C_2 \times C_2$, $S_3$, $C_3$, $C_2 \times C_2$, $C_2 \times C_2$, $C_2$, $C_2$, $C_2$, $S_3$, $C_4$.

3 $S_3$, $C_2$, $C_2$, 1, $C_2 \times C_2$, 1, $C_2 \times C_2$, $C_2 \times C_2$, $S_4$, $A_4$, $S_4$, $S_4$, 1, $C_2$, 1, $D_{12}$, $S_3$, $D_{12}$, $S_4$.

4 1, $S_4$, $D_{10}$, 1, $C_2$, $S_3$, $C_2$, $D_{10}$, $C_2$, $D_8$, $C_2$, $C_2 \times S_3 \times S_3$, $C_4$, $S_3$, $C_2$, $C_2$, $C_2$, $C_2$, $C_2 \times C_2$.

5 $C_2$, $C_2 \times C_2$, $C_2$, 1, $S_4$, $A_5$, 1, $S_3$, $D_{24}$, $A_4$, $C_2$, $D_8$, $C_3$, $D_8$, $D_8$, $C_2$.

6 $C_2$, $C_2$, $((C_2 \times C_2 \times C_2 \times C_2) : C_3) : C_2$, $C_2$, $C_3$, $S_3$, $C_2$, $S_3$, $S_4$, $(C_3 \times C_3) : C_2$.

7 $C_2$, $C_2 \times C_2$, $S_3$, $C_2 \times D_8$.

8 $S_3$, $S_4$.

9 $A_5$, $(A_4 \times A_4) : C_2$.

# Low dimensional results

- By using the tesselation Tessel3 we can compute the following homology groups:
  - $H_0(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}$
  - $H_1(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = 0$
  - $H_2(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = 3\mathbb{Z}_2$
  - $H_3(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = 2\mathbb{Z}_4 + \mathbb{Z} + 2\mathbb{Z}_3 + \mathbb{Z}_5$
  - $H_4(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = 4\mathbb{Z}_2 + \mathbb{Z}_5$
  - $H_5(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = 13\mathbb{Z}_2$
- Further computations are difficult and constrained by
  - Choosing tesselations influence the occurring stabilizers.
  - Dimensions is one factor, another is the length of the boundaries.
  - Memory is the main problem of such computations.

# Cohomology rings

- If $G$ is a finite group then $H^*(G, \mathbb{Z})$ is a ring and also for any prime $p$ $H^*(G, \mathbb{Z}_p)$.
- This ring can be computed very efficiently by first computing the cohomology of the sylow $p$-subgroup and then by computing the stable classes.
- But this is valid only for nontwisted cohomology. If $u$ and $u'$ are cohomology classes in $H^n(G, \mathbb{Z}_\sigma)$ and $H^{n'}(G, \mathbb{Z}_{\sigma'})$ then their cup products $u \cup u'$ is a cohomology class in $H^{n+n'}(G, \mathbb{Z}_{\sigma\sigma'})$.
- So, one possible strategy if $f \in H^*(G, \mathbb{Z}_\sigma)$ is to use the following identities:

$$f \cup H^*(G, \mathbb{Z}) \subset H^*(G, \mathbb{Z}_\sigma) \text{ and } f \cup H^*(G, \mathbb{Z}_\sigma) \subset H^*(G, \mathbb{Z})$$

# The equivariant spectral sequence

- It is also named the Leray spectral sequence, Normalizer spectral sequence, etc.
- Denote by $Repr_p(X)$ the list of representative of orbits of $p$ dimensional faces of the polyhedral complex $X$.
- In the $E^1$ page of the spectral sequence we have

$$E^1_{pq} = \oplus_{F \in Repr_p(X)} H_q(\mathrm{Stab}(F), \mathbb{Z}_\sigma)$$

- The differential $d_i$ that allows to define the higher spectral sequences are determined by the CTC Wall lemma.

# 5-part of the homology I

- We have

$$H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} = \begin{cases} \mathbb{Z}_5 & \text{if } n = 3 + 4k \\ \mathbb{Z}_5 & \text{if } n = 4 + 4k \\ 0 & \text{otherwise} \end{cases}$$

- First of all $A_5$ is a subgroup of $\mathrm{PSL}_4(\mathbb{Z})$ and it appears in the top dimensional cells.
- Thus we have mapping in homology $H_*(A_5, \mathbb{Z}) \to H_*(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})$ and another mapping in cohomology $H^*(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) \to H^*(A_5, \mathbb{Z})$.
- Since $A_5$ is directly embedded in the resolution, we can readily check that the mapping $H_n(A_5, \mathbb{Z}) \to H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})$ are injective if $n \leq 3$.
- So we get that for all $k$, $H^{4k}(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)}$ contains a non-zero class.

# 5-part of the homology II

- For the tesselation Tessel1 the only stabilizer groups with order divisible by 5 are $A_5$ and they have no twisting.
- Thus we get the following picture for the $E^1$ page:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | $\mathbb{Z}_5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}_5$ | 7 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |
| 0 | $\mathbb{Z}_5$ | 0 | 0 | 0 | 0 | $\mathbb{Z}_5$ | 3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $\mathbb{Z}$ | $\mathbb{Z}^3$ | $\mathbb{Z}^4$ | $\mathbb{Z}^4$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | 0 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | |

- Either $d_5$ is non-zero and it kills two $\mathbb{Z}_5$ or it is zero and they stay.
- But we know that two $H^8(\mathrm{PSL}_4, \mathbb{Z})_{(5)}$ is nontrivial thus $d_5$ has to be zero which proves the result.

# Towards the 3-part

- For the 3-part we have the relations

$$
\begin{array}{ll}
H_1(G_{288},\mathbb{Z})_{(3)} = 0 & H_6(G_{288},\mathbb{Z})_{(3)} = 3\mathbb{Z}_3 \\
H_2(G_{288},\mathbb{Z})_{(3)} = \mathbb{Z}_3 & H_7(G_{288},\mathbb{Z})_{(3)} = 5\mathbb{Z}_3 \\
H_3(G_{288},\mathbb{Z})_{(3)} = 3\mathbb{Z}_3 & H_8(G_{288},\mathbb{Z})_{(3)} = 0 \\
H_4(G_{288},\mathbb{Z})_{(3)} = 0 & H_9(G_{288},\mathbb{Z})_{(3)} = 0 \\
H_5(G_{288},\mathbb{Z})_{(3)} = 0 & H_{10}(G_{288},\mathbb{Z})_{(3)} = 5\mathbb{Z}_3
\end{array}
$$

So, by doing the same computation as for $A_5$ we can prove that the mapping $H_n(G_{288},\mathbb{Z})_{(3)} \to H_n(\mathrm{PSL}_4(\mathbb{Z}),\mathbb{Z})_{(3)}$ is a surjection for $n \leq 5$.