Lattice coverings and Delaunay polytopes

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I. Lattices coverings

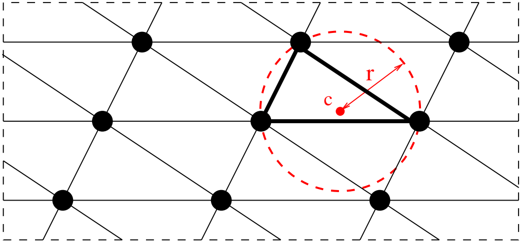
Lattice coverings

- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$.
- A covering is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius r and center x_i such that any $x \in \mathbb{R}^n$ belongs to at least one ball.

- If L is a lattice, the lattice covering is the covering defined by taking the minimal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a covering.

Empty sphere and Delaunay polytopes

- Def: A sphere $S(c, r)$ of center c and radius r in an n-dimensional lattice L is said to be an empty sphere if:
  (i) $\|v - c\| \geq r$ for all $v \in L$,
  (ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.
- Def: A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



- Delaunay polytopes define a tessellation of the Euclidean space $\mathbb{R}^n$

Lattice covering

- For a lattice L we define the covering radius $\mu(L)$ to be the smallest r such that the family of balls $v + B_n(0, r)$ for $v \in L$ cover $\mathbb{R}^n$.

- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \operatorname{vol}(B_n(0,1))}{\det(L)} \geq 1$$

with
  - $\mu(L)$ being the largest radius of Delaunay polytopes
  - or

$$\mu(L) = \max_{x \in \mathbb{R}^n} \min_{y \in L} \|x - y\|$$

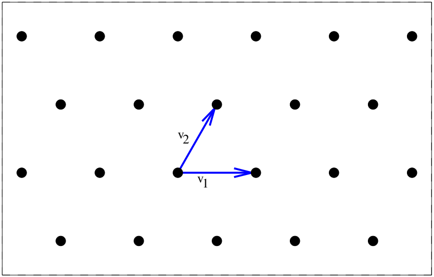
Covering minimization and maximization

- For a given lattice L the only general method for computing $\Theta(L)$ is to compute all Delaunay polytopes.
- The minimization problem is the problem of minimizing $\Theta(L)$ over all lattices L.
  The following is known:
  - For $n \leq 5$ the dual root lattice A_n^* is the best lattice covering.
  - For $n = 6$ there is a conjecturally best lattice covering discovered in F. Vallentin PhD thesis.
  - The Leech lattice Λ_{24} is conjectured to be optimal.
- The function Θ is unbounded from above but we will develop a theory for describing the local covering maxima.
  The following is known:
  - There is no local covering maxima for $n \leq 5$
  - For $n = 6$ there is exactly one covering maxima: E_6
  - For $n = 7$ there are exactly two covering maxima: E_7 and ER_7 (Erdahl & Rybnikov lattice)
  - There is an infinite series DS_n generalizing E_6 and E_7.

II. Gram matrix formalism

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices and $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices.
- Take a basis $(v_1, \dots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$$G_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Isometric lattices

- Take a basis $(v_1, \ldots, v_n)$ of a lattice L with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T\ V$.
  The matrix $G_{\mathbf{v}}$ is defined by $\frac{n(n+1)}{2}$ variables as opposed to n^2 for the basis V.
- If $M \in S^n_{>0}$, then there exists V such that $M = V^T\ V$ (Gram Schmidt orthonormalization)
- If $M = V_1^T\ V_1 = V_2^T\ V_2$, then $V_1 = OV_2$ with $O^T\ O = I_n$ (i.e. O corresponds to an isometry of $\mathbb{R}^n$).
- Also if L is a lattice of $\mathbb{R}^n$ with basis $\mathbf{v}$ and u an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Working with Gram matrices

- In practice all computations on lattices of $\mathbb{R}^n$ are best done with Gram matrices. For example computing

$$d(x) = \min_{y \in L} \|x - y\|$$

is equivalent to minimizing

$$\min_{y \in \mathbb{Z}^n} (v - y)^T A(x - y)$$

for some $v \in \mathbb{R}^n$ expressed from x.

- We have the determinant relation

$$\det\ L = \sqrt{\det\ G_{\mathbf{v}}}$$

- In general, Gram matrices are the only information taken into input by programs in lattice theory.
- They give a parameter space for lattices with a natural topology.

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice L then $V' = VP$ with $P \in GL_n(\mathbb{Z})$. This implies

$$G_{\mathbf{v}'} = {V'}^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in GL_n(\mathbb{Z})$ such that

$$A = P^T B P$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken & Souvignier wrote a program isom for testing arithmetic equivalence and a program autom for computing automorphism group of lattices.

III. The lattice covering problem

Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$\|w_i - c\| = r \text{ i.e. } {w_i}^T M w_i - 2{w_i}^T Mc + c^T Mc = r^2$$

- Substracting one obtains

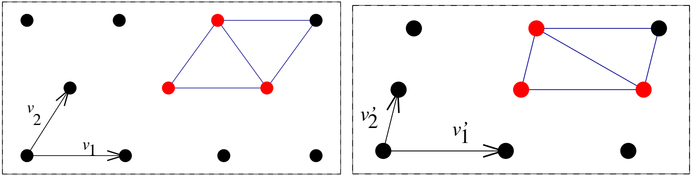
$$\left\{ {w_i}^T M w_i - {w_j}^T M w_j \right\} - 2 \left\{ {w_i}^T - {w_j}^T \right\} Mc = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into a linear inequality on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$, then one has

$$\|w - c\| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i {v_i}^T M v_i \geq 0$$

Iso-Delaunay domains

- Take a lattice L and select a basis $v_1, \ldots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



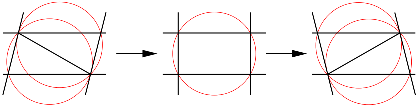
are part of the same iso-Delaunay domain.
- An iso-Delaunay domain is the assignment of Delaunay polytopes of the lattice.

Primitive iso-Delaunay

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding iso-Delaunay domain is of dimension $\frac{n(n+1)}{2}$, they are called primitive

Equivalence and enumeration

- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive iso-Delaunay domains.
- Voronoi proved that after this action, there is a finite number of primitive iso-Delaunay domains.
- Bistellar flipping creates one iso-Delaunay from a given iso-Delaunay domain and a facet of the domain. In dim. 2:

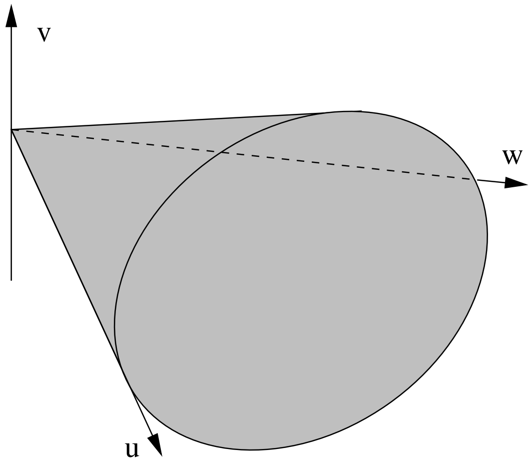


- Enumerating primitive iso-Delaunay domains is done classically:
  - Find one primitive iso-Delaunay domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.

  The algorithm is graph traversal and iteratively finds all the iso-Delaunay up to equivalence.

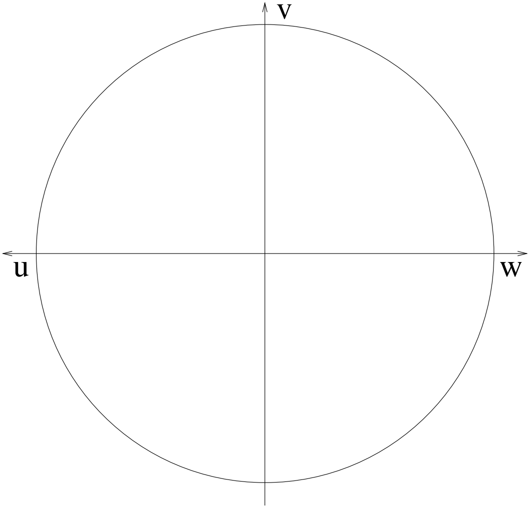
The partition of $S^2_{>0} \subset \mathbb{R}^3$ I

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



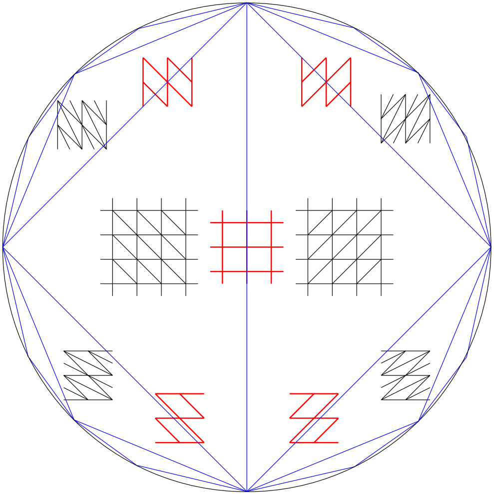
The partition of $S^2_{>0} \subset \mathbb{R}^3$ II

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{>0} \subset \mathbb{R}^3$ III

Primitive iso-Delaunay domains in $S^2_{>0}$:



IV. SDP
optimization

Radius of Delaunay polytope

- Fix a primitive iso-Delaunay domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \dots, D_m$.
- Thm: For every $D_i = Conv(0, v_1, \dots, v_n)$, the radius of the Delaunay polytope is at most 1 if and only if

$$\begin{pmatrix} 4 & \langle v_1, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_n, v_n\rangle \\ \langle v_1, v_1\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\ \langle v_2, v_2\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_n\rangle & \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \dots & \langle v_n, v_n\rangle \end{pmatrix} \in S^{n+1}_{\geq 0}$$

  by Delaunay, Dolbilin, Ryshkov & Shtogrin.
- The condition is a semidefinite condition.
- See for more details
  - A. Schürmann and F. Vallentin, *Computational approaches to lattice packing and covering problems*, Discrete & Computational Geometry **35** (2006) 73–116.
  - A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.

SDP optimization problem

- Fix a primitive iso-Delaunay domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \ldots, D_m$.
- Thm (Minkowski): The function $-\log\det(M)$ is strictly convex on $S^n_{>0}$.
- Solve the problem
  - M in the iso-Delaunay domain (linear inequalities),
  - the Delaunay D_i have radius at most 1 (semidefinite condition),
  - minimize $-\log\det(M)$ (strictly convex).
- Thm: Given an iso-Delaunay domain LT, there exist a unique lattice, which minimize the covering density over LT.
- The above problem is solved by the interior point methods implemented in MAXDET by Vandenberghe, Boyd & Wu. Unicity comes from the strict convexity of the objective function.

Solving the minimum covering problem

- The lattice covering problem is to find a lattice covering of minimal density.
- The solution of the SDP problem by interior point methods does not give exact solutions but approximate solutions available at any precision.
- The exact solution is expressible with algebraic integers once one knows which inequations are satisfied with equality.
- The method for solving the lattice covering problem in dimension n is thus:
  - Enumerate all iso-Delaunay domains LT up to equivalence
  - solve the SDP on all the domains
  - Take the one(s) of minimum covering density
- Pb: 222 primitive iso-Delaunay domains in dimension 5 (Baranovski, Ryshkov, Engel & Grishukhin) and at least 200 millions in dimension 6 (Engel).
  This is not practical at all

V. iso-Delaunay domains of $S^n_{>0}$-spaces

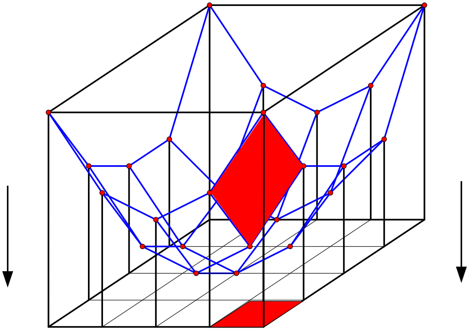
$S^n_{>0}$-spaces

- A $S^n_{>0}$-space is a vector space $\mathcal{SP}$ of S^n, which intersect $S^n_{>0}$.
- We want to describe the Delaunay decomposition of matrices $M \in S^n_{>0} \cap \mathcal{SP}$.
- Motivations:
  - The enumeration of iso-Delaunay is done up to dimension 5 but certainly not for higher dimension.
  - We hope to find some good covering by selecting judicious $\mathcal{SP}$. This is a search for best but unproven to be optimal coverings.
- A iso-Delaunay in $\mathcal{SP}$ is an open convex polyhedral set included in $S^n_{>0} \cap \mathcal{SP}$, for which every element has the same Delaunay decomposition.
- Typical choice of a space $\mathcal{S}P$ are the space of forms invariant under a finite integral matrix group G. In that case finiteness of the set of iso-Delaunay up to equivalence is proved.
- Dimension of the space $\mathcal{S}P$ is typically no larger than 4.

Lifted Delaunay decomposition

- The Delaunay polytopes of a lattice L correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertex-set:

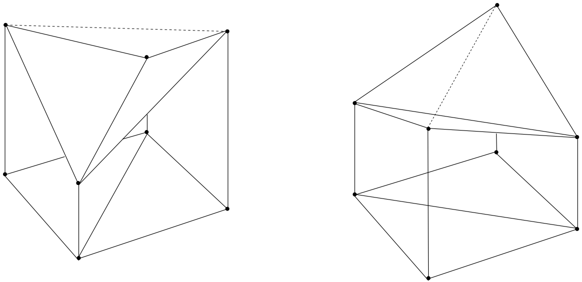
$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{n+1} .$$



- H. Edelsbrunner, N.R. Shah, *Incremental Topological Flipping Works for Regular Triangulations*, Algorithmica **15** (1996) 223–241.

Generalized bistellar flips

- The "glued" Delaunay form a Delaunay decomposition for a matrix M in the $(\mathcal{SP}, L)$-iso-Delaunay satisfying to $f(M) = 0$.
- The flipping break those Delaunays in a different way.
- Two triangulations of $\mathbb{Z}^2$ correspond in the lifting to:



- The polytope represented is called the repartitioning polytope. It has two partitions into Delaunay polytopes.
- The lower facets correspond to one tesselation, the upper facets to the other tesselation.

Enumeration technique

- Find a primitive $(\mathcal{SP}, L)$-iso-Delaunay domain, insert it to the list as undone.
- Iterate
  - For every undone primitive $(\mathcal{SP}, L)$-iso-Delaunay domain, compute the facets.
  - Eliminate redundant inequalities.
  - For every non-redundant inequality realize the flipping, i.e. compute the adjacent primitive $(\mathcal{SP}, L)$-iso-Delaunay domain. If it is new, then add to the list as undone.
- See for full details
  - M. Dutour Sikirić, F. Vallentin and A. Schürmann, *A generalization of Voronoi's reduction theory and applications*, Duke Math. J. 142 (2008), 127–164.

Best known lattice coverings

d	lattice / covering density Θ		
1	$\mathbb{Z}^1$ 1	13	L_{13}^c (DSV) 7.762108
2	A_2^* (Kershner) 1.209199	14	L_{14}^c (DSV) 8.825210
3	A_3^* (Bambah) 1.463505	15	L_{15}^c (DSV) 11.004951
4	A_4^* (Delaunay & Ryshkov) 1.765529	16	A_{16}^* (DSV) 15.310927
5	A_5^* (Ryshkov & Baranovski) 2.124286	17	A_{17}^9 (DSV) 12.357468
6	L_6^c (Vallentin) 2.464801	18	A_{18}^* 21.840949
7	L_7^c (Schürmann & Vallentin) 2.900024	19	A_{19}^{10} (DSV) 21.229200
8	L_8^c (Schürmann & Vallentin) 3.142202	20	A_{20}^7 (DSV) 20.366828
9	L_9^c (DSV) 4.268575	21	A_{21}^{11} (DSV) 27.773140
10	L_{10}^c (DSV) 5.154463	22	Λ_{22}^* (Smith) $\leq$ 27.8839
11	L_{11}^c (DSV) 5.505591	23	Λ_{23}^* (Smith, MDS) 15.3218
12	L_{12}^c (DSV) 7.465518	24	*Leech* 7.903536

- For $n \leq 5$ the results are definitive.
- The lattices A_n^r for r dividing $n+1$ are the Coxeter lattices. They are often good coverings and they are used for perturbations.
- For dimensions 10 and 12 we use laminations over Coxeter lattices of dimension 9 and 11.
- Leech lattice is conjecturally optimal (it is local optimal Schürmann & Vallentin)

VI. Quadratic functions and the Erdahl cone

The Erdahl cone

- Denote by $E_2(n)$ the vector space of degree 2 polynomial functions on $\mathbb{R}^n$. We write $f \in E_2(n)$ in the form

$$f(x) = a_f + b_f \cdot x + Q_f[x]$$

  with $a_f \in \mathbb{R}$, $b_f \in \mathbb{R}^n$ and Q_f a $n \times n$ symmetric matrix
- The Erdahl cone is defined as

$$Erdahl(n) = \{f \in E_2(n) \text{ such that } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}$$

- It is a convex cone, which is non-polyhedral since defined by an infinity of inequalities.
- The group acting on $Erdahl(n)$ is $\mathrm{AGL}_n(\mathbb{Z})$, i.e. the group of affine integral transformations

$$x \mapsto b + Px \text{ for } b \in \mathbb{Z}^n \text{ and } P \in \mathrm{GL}_n(\mathbb{Z})$$

Scalar product

- Def: If $f, g \in E_2(n)$, then:

$$\langle f, g\rangle = a_f a_g + \langle b_f, b_g\rangle + \langle Q_f, Q_g\rangle$$

- Def: For $v \in \mathbb{Z}^n$, define $ev_v(x) = (1 + v \cdot x)^2$.
- We have

$$\langle f, ev_v\rangle = f(v)$$

- Thus finding the rays of $Erdahl(n)$ is a dual description problem with an infinity of inequalities and infinite group acting on it.
- If $f \in Erdahl(n)$ then Q_f is positive semidefinite.
- Def: We also define

$$Erdahl_{>0}(n) = \{f \in Erdahl(n) \ : \ Q_f \text{ positive definite}\}$$

Relation with Delaunay polytope

- If D is a Delaunay polytope of a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ of empty sphere $S(c, r)$ then we define the function

$$\begin{array}{rcl} f_{D,\mathbf{v}} : \mathbb{Z}^n & \to & \mathbb{R} \\ x = (x_1, \dots, x_n) & \mapsto & \left\| \sum_{i=1}^n x_i v_i - c \right\|^2 - r^2 \end{array}$$

  Clearly $f_{D,\mathbf{v}} \in Erdahl_{>0}(n)$.
- The perfection rank of a Delaunay polytope is the dimension of the face it defines in $Erdahl(n)$.
- Def: If $f \in Erdahl(n)$ then

$$Z(f) = \{v \in \mathbb{Z}^n : f(v) = 0\}$$

- Thm: If $f \in Erdahl(n)$ then there exist a lattice L_f and a lattice L' containing a Delaunay polytope D_f such that

$$Z(f) = D_f + L_f$$

- We have $\dim L' + \dim L_f \leq n$. In case of equality $Z(f)$ is called a Delaunay polyhedra.

Perfect Delaunay polytopes/polyhedra

- Def: If D is a n-dimensional Delaunay polyhedra then we define
$$\text{Dom}_{\mathbf{v}}\ \ D = \sum_{v\mathbf{v} \in D} \mathbb{R}_{+} ev_{v}$$
- We have $\langle f_{D,\mathbf{v}}, \text{Dom}_{\mathbf{v}}\ \ D\rangle = 0$.
- Def: D is perfect if Dom D is of dimension $\binom{n+2}{2} - 1$ that is if the perfection rank is 1.
- This implies that f_D generates an extreme ray of $Erdahl(n)$ and f_D is rational.
- A perfect n-dimensional Delaunay polytope has at least $\binom{n+2}{2} - 1$ vertices. There is only one way to embed it as a Delaunay polytope of a lattice.
- Perfect Delaunay polytopes are remarkable and rare objects.

VII. Covering maxima, pessima and their characterization

Eutacticity

- If $f \in Erdahl_{>0}(n)$ then define μ_f and c_f such that

$$f(x) = Q_f[x - c_f] - \mu_f$$

Then define

$$u_f(x) = (1 + c_f \cdot x)^2 + \frac{\mu_f}{n} Q_f^{-1}[x]$$

- Def: $f \in Erdahl_{>0}(n)$ is eutactic if u_f is in the relative interior of Dom f.
- Def: Take a Delaunay polytope P for a quadratic form Q of center c_P and square radius μ_P. P is called eutactic if there are $\alpha_v > 0$ so that

$$\left\{\begin{array}{rcl} 1 & = & \sum_{v \in \text{vert } P} \alpha_v, \\ 0 & = & \sum_{v \in \text{vert } P} \alpha_v (v - c_P), \\ \frac{\mu_P}{n} Q^{-1} & = & \sum_{v \in \text{vert } P} \alpha_v (v - c_P)(v - c_P)^T. \end{array}\right.$$

Covering maxima

- A given lattice L is called a covering maxima if for any lattice L' near L we have $\Theta(L') < \Theta(L)$.
- Thm: For a lattice L the following are equivalent:
  - L is a covering maxima
  - Every Delaunay polytope of maximal circumradius of L is perfect and eutactic.
- The following are covering maxima:

name	# vertices	# orbits Delaunay polytopes
E_6	27	1
E_7	56	2
ER_7	35	4
O_{10}	160	6
BW_{16}	512	4
O_{23}	94208	5
Λ_{23}	47104	709

- Thm: For any $n \geq 6$ there exist one lattice $L(DS_n)$ which is a covering maxima.
  There is only one perfect Delaunay polytope $P(DS_n)$ of maximal radius in $L(DS_n)$.

The infinite series

- For n even $P(DS_n)$ is defined as the lamination over D_{n-1} of
  - one vertex
  - the half cube $\frac{1}{2}H_{n-1}$
  - the cross polytope CP_{n-1}

  For $n = 6$, it is E_6.
- For n odd as the lamination over D_{n-1} of
  - the cross polytope CP_{n-1}
  - the half cube $\frac{1}{2}H_{n-1}$
  - the cross polytope CP_{n-1}

  For $n = 7$, it is E_7.
- Conj: The lattice DS_n has the following properties:
  - $L(DS_n)$ has the maximum covering density among all n-dim. covering maxima
  - Among all perfect Delaunay polytopes, $P(DS_n)$ has
    - maximum number of vertices
    - maximum volume

  If true this would imply Minkowski conjecture by results of
  - U. Shapira and B. Weiss, *Stable Lattices and the Diagonal Group*, preprint

Pessimum and Morse function property

- For a lattice L let us denote $D_{crit}(L)$ the space of direction d of deformation of L such that Θ increases in the direction d.
- Def: A lattice L is said to be a covering pessimum if the space D_{crit} is of measures 0.
- Thm: If the Delaunay polytopes of maximum circumradius of a lattice L are eutactic and are not simplices then L is a pessimum.

name	# vertices	# orbits Delaunay polytopes
$\mathbb{Z}^n$	2^n	1
D_4	8	1
D_n ($n \geq 5$)	2^{n-1}	2
E_6^*	9	1
E_7^*	16	1
E_8	16	2
K_{12}	81	4

- Thm: The covering density function $Q \mapsto \Theta(Q)$ is a topological Morse function if and only if $n \leq 3$.

VIII. Enumeration of perfect Delaunay polytopes

Perfect Delaunay polytope

- There is a finite number of them in each dimension n. Known results:

dim.	perfect Delaunay	authors
1	$[0,1]$ in $\mathbb{Z}$	
2	$\emptyset$	
3	$\emptyset$	
4	$\emptyset$	
5	$\emptyset$	$\uparrow$ (Deza, Laurent & Grishukhin)
6	2_{21} in E_6	(Deza & Dutour)
7	3_{21} in E_7	
	and ER_7 in $L(ER_7)$	(Dutour Sikirić)
8	≥ 27	(Dutour Sikirić & Rybnikov)
9	≥ 100000	(Dutour Sikirić)

- Thm: There exist perfect Delaunay polytopes D such that $\mathbb{Z}D \neq \mathbb{Z}^n$ (dimension $n \geq 13$, Rybnikov & Dutour Sikirić).
- Thm: There exist lattices with several perfect Delaunay polytopes (dimension 15 and 23, Rybnikov & Dutour Sikirić).
- Thm: For $n \geq 6$ there exist a perfect Delaunay polytope with exactly $\binom{n+2}{2} - 1$ vertices (Erdahl & Rybnikov) ER_n.

Extreme rays of *Erdahl*(n)

- Def: If $f \in Erdahl_{>0}(n)$ then we define

$$\text{Dom } f = \sum_{v \in Z(f)} \mathbb{R}_+ ev_v$$

- We have $\langle f, \text{Dom } f \rangle = 0$.
- Thm (Erdahl): The extreme rays of $Erdahl(n)$ are:
  (a) The constant function 1.
  (b) The functions
  $$(a_1x_1 + \cdots + a_nx_n + \beta)^2$$
  with $(a_1, \ldots, a_n)$ not collinear to an integral vector.
  (c) The functions f such that $Z(f)$ is a perfect Delaunay polyhedra.
- Note that if $f \in Erdahl(n)$ with $Z(f)$ a Delaunay polyhedra, then there exist a lattice L' of dimension $k \leq n$, a Delaunay polytope D of L', a basis $\mathbf{v}'$ of L' and a function $\phi \in \text{AGL}_n(\mathbb{Z})$ such that

$$f \circ \phi(x_1, \ldots, x_n) = f_{D,\mathbf{v}'}(x_1, \ldots, x_k)$$

Delaunay polyhedra retract

- For a function $f \in Erdahl(n)$ a proper decomposition is a pair (g, h) with $f = g + h$, $g \in Erdahl(n)$ and $h(x) \geq 0$ for $x \in \mathbb{R}^n$.
- Lem: For a proper decomposition we have

$$Vect\, Z(f) + \text{Ker } Q_f \subset \text{Ker } Q_h$$

  and there exist a proper decomposition with equality.
- Fix an integral complement L' of $Vect\, Z(f) + \text{Ker } Q_f$. A proper decomposition is called extremal if $\det Q_h|_{L'}$ is maximal among all proper decompositions.
- Thm: For $f \in Erdahl(n)$, there exist a unique extremal decomposition. For it we have that $Z(g)$ is a Delaunay polyhedra.
- Conj: The decomposition depends continuously on $f \in Erdahl(n)$.
- On the other hand in a neighborhood of $f \in Erdahl(n)$ we can have an infinity of Delaunay polyhedra.

Enumeration of perfect Delaunay polyhedra

- From a given n-dimensional Delaunay polyhedron P of form f we can define the local cone

$$Loc(f) = \{g \in E_2(n) \text{ s.t. } g(x) \geq 0 \text{ for } x \in Z(f)\}.$$

  We set the define the degeneracy $d(P) = dimL_f$.
- Thm: For a Delaunay polyhedron P let $(P_i)_{i \in I}$ the set of Delaunay polyhedra of degeneracy $d(P) - 1$ and perfection rank $r(P) - 1$. P_i and P_j are adjacent if $P_i \cap P_j$ is of perfection rank $r(P) - 2$. The obtained graph is connected.
- Thm: In a fixed dimension n there exist an algorithm for enumerating the perfect Delaunay polytopes of dimension n. The algorithm is iterative. It relies on dual description. If the degeneracy rank is $d > 0$ then we find a sub Delaunay polyhedron of degeneracy $d - 1$, finds its facets and do the liftings. This requires knowing the facets of CUT_{n+1}.
- Thm: In dimension 7 there is only 3_{21} and ER_7.