Kalman filtering in oceanography

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(based on Hoteit Ibrahim thesis)

Problem

- An evolution equation with some uncertainties:
  - incomplete modelization of the system
  - incorrectness of the numerical model.
- Some measurement with some uncertainty.
- ➠ combine this to get good estimate on the “true” state of the system.

History

- (1949) Kolmogorov - Wiener, spectral decomposition (under assumption of linear autonomous system)
- (1960) Kalman - Bucy, sequential filtering for linear system.
- (1994) Evensen, Ensemble Kalman filtering

I. Original Kalman filtering

Gaussian variables

- A random variable X is Gaussian if its probability density is proportional to

$$exp(-\frac{1}{2\sigma}(x-m)^2)$$

- Gaussian random variable are characterized by

$$E(X) = m \quad \text{and} \quad E(X^2) = \sigma + m^2$$

- The Gaussian Random variable appearing in Oceanography are of course multi-dimensional. They are characterized by a vector and a covariance matrix.

Stochastic Differential equations

- Stochastic differential equation:

$$dX_t = F(t)X_t dt + C(t)dU_t$$

with dU_t a “white noise”, i.e. is a Brownian motion, Gaussian process. It is the model equation.

- Stochastic differential equation:

$$dZ_t = G(t)X_t dt + D(t)dV_t$$

with dV_t another independent “white noise”. It is the measurement equation.

At every given t, X_t, Z_t will be Gaussian variables.

Kalman solution

- The stochastic equations are considered to be Ito equations (the alternatice is Stratonovich equations).
- Kalman-Bucy theory allows to compute
  - the expectancy,
  - covariance matrix
- But they are solution of a nonlinear Riccati ordinary differential equation.

Discrete version

- Evolution equation:

$$X^t(t_k) = A_k X^t(t_{k-1}) + \eta(t_k)$$

with $\eta(t_k)$ being Gaussian of covariance matrix $Q(t_k)$

- Measurement equation

$$Y_k^0 = H_k X^t(t_k) + \epsilon_k$$

with ϵ_k being Gaussian of covariance matrix $R(t_k)$

All X^t terms appearing are Gaussian.

The Equations

- **Initialisation step**

$$\begin{aligned} X^a(t_0) &= m_0 \\ P^a(t_0) &= P_0 \end{aligned}$$

- **Prevision step**

$$\begin{aligned} X^f(t_k) &= A_k X^a(t_{k-1}) \\ P^f(t_k) &= A_k P^a(t_{k-1}) A_k^T + Q(t_k) \end{aligned}$$

- **Correcting step**

$$\begin{aligned} X^a(t_k) &= X^f(t_k) + K_k \{ Y_k^0 - H_k X^f(t_k) \} \\ P^a(t_k) &= (I - K_k H_k) P^f(t_k) \end{aligned}$$

with $K_k = P^f(t_k) H_k^T \{ H_k P^f(t_k) H_k^T + R_k \}^{-1}$.

Derivation of the equations

Denote X^t and Y^t the true state and observation of the system at instant t_k

- We want to minimize the expectancy

$$
\begin{aligned}
&= E[(X^a - X^t)(X^a - X^t)^T] \\
&= E[(X^f - X^t + K(Y^0 - HX^f))(\dots)^T] \\
&= E[((I - KH)(X^f - X^t) + K(Y^0 - Y^t))(\dots)^T] \\
&= (I - KH)E[(X^f - X^t)(X^f - X^t)^T](I - KH)^T \\
&\quad +KE[(Y^0 - Y^t)(Y^0 - Y^t)^T]K^T \\
&= (I - KH)P^f(I - KH)^T + KRK^T
\end{aligned}
$$

- Minimization is obtained by $K = P^f(t_k)H_k^T\{H_k P^f(t_k)H_k^T + R_k\}^{-1}$.

II. Extensions and restrictions

Extended Kalman filtering

- Equations for nonlinear systems:

$$
\begin{aligned}
X^t(t_k) &= F_k(X^t(t_{k_1})) + \eta(t_k) \\
Y_k^0 &= H_k X^t(t_k) + \epsilon_k
\end{aligned}
$$

- Kalman filtering applies only to linear systems.
- nonlinear $\Rightarrow$ Gaussianity no longer preserved.
- Extended Kalman filtering is a linearization of the equations.
- Consequences and results:
  - We lose optimality of Kalman filtering
  - It works relatively well for weakly nonlinear systems.
  - It does not work well for strongly nonlinear systems.

Issues for oceanography

- In order to apply Extended Kalman filtering, one needs:
  - the variance matrices of measurement.
  - have an estimation of “what we miss”
  - have an estimation of the computer errors.
- It is not possible to invert matrices of size $1.10^6 \times 1.10^6$.

➠ Kalman filtering or Extended Kalman filtering cannot be applied in Oceanography.

The SEEK filter

SEEK: Singular Evolutiv Extended Kalman Filter

- Idea is that the phase state has a variety called attractor.
  - normal directions to the attractor are dissipative: errors are corrected themselves.
  - tangent directions to the attractor are hyperbolic: errors are amplified.
- SEEK: we consider the error only in a chosen tangent space of dimension r.
- The computational cost is $(r + 1)$ times the cost of the model.

The SEIK filter

SEIK: Singular Extended Interpolated Kalman Filter

- Take a cloud of $r + 1$-points around the initial state and make them evolve.
- Compute the initial state and its covariance matrix by

$$P^a(t_k) = \frac{1}{r+1} \sum_{i=1}^{r+1} [X_i^a(t_k) - X^a(t_k)][\ldots]^T$$

- cost is identical to the one of SEEK, performance are superior; one possible reason: linearization implies a bigger error than doing averaging.

The subspace in SEIK and SEEK

- In order to make those filters run, one needs to select a basis.
- Hoteit uses
  - Empirical Orthogonal Functions.
  - Localized EOF around areas of interest.
- Empirical orthogonal functions consist of having N states and selecting an orthogonal basis of r states ($r < N$) that encapsulates the main trends.
- localized EOF consist of selecting some areas, and localizing the functions on those areas, then doing classical EOF.
- “mixed” basis consist of localized EOF + some global EOF. In SESEEK he makes evolve only the global EOF.

Computationnal issues

- SEEK and SEIK are possible to use in Oceanography
- their cost remain high.
- Further ideas are needed...

SFEK and SAEK filters

- If we fix the basis of the space in which computations are done at the beginning, we obtain the SFEK filter.
- In linear autonomous case, the matrix P^a converges to a fixed matrix.
  - ➠ The idea is to fix the subspace considered, SAEK.
- Hoteit indicates some poor behaviour, due to the fact that Ocean is nonlinear.

SIEIK

Singular Intermittemt Extended Interpolated Kalman Filter

- SFEK and SAEK are problematic, since the base is evolving.
- The idea is to make the base evolves by intermittence:
  - if the model stays calm, we keep the same basis,
  - if some significant modification happen, we update the basis.
- It makes a faster filter and it works almost as good as SEEK.

III. Ensemble Kalman filtering

The idea

- The idea is to replace the formalism of average and covariance matrix by a population of states ($O(100)$ states in Evensen experiments)
- ➠ the system can handle nonlinear situations better.
- Its cost is similar to the one of SEEK, SEIK.
- The overall idea is to do Markov Chain Monte Carlo simulation to make evolve the net of points.

The error terms

- The uncertainty on measurement is handled by the ensemble.
- What about the uncertainty of the model? It is still assumed to be a linear white noise of the form

$$\psi_k = f(\psi_{k-1}) + q_k$$

- Despite this the EnKF performs very well and manages to work even on Lorenz model.

Thank

You