Polycycles and face-regular two-maps

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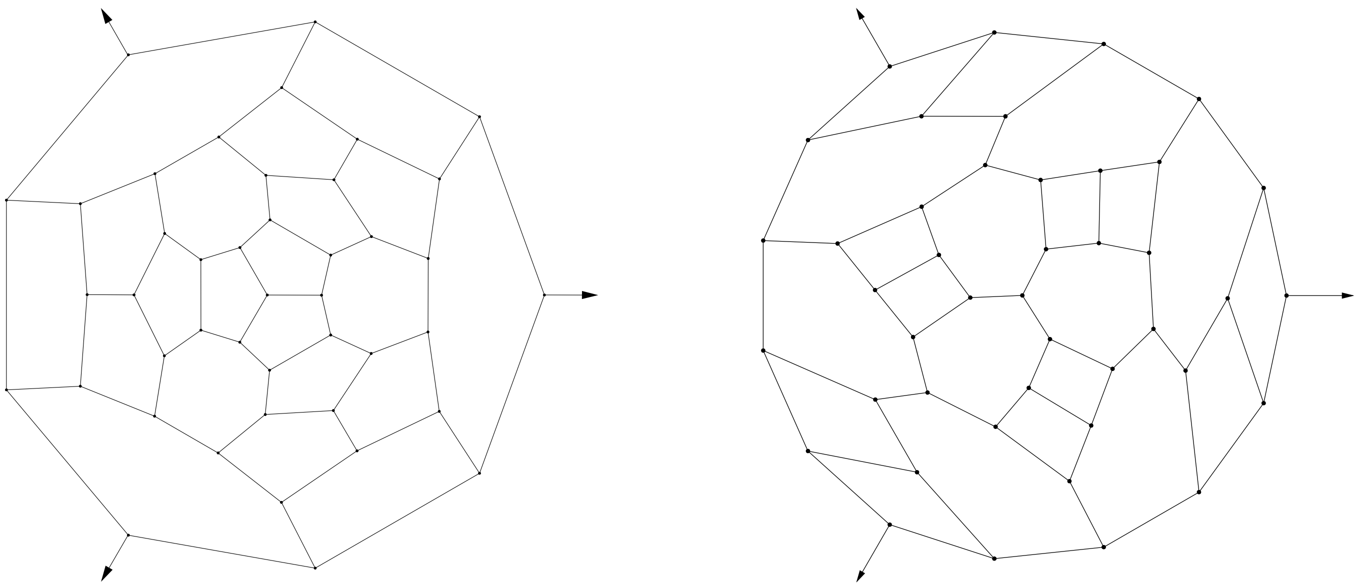
I. Strictly face-regular two-maps

Definition

A strictly face-regular two-map is

- a 3-connected 3-valent map (on sphere or torus), whose faces have size p or q ((p,q)-sphere or (p,q)-torus)
- pR_i holds: any p-gonal face is adjacent to i p-gons
- qR_j holds: any q-gonal face is adjacent to j q-gons



$(5,7)$-sphere $5R_3$, $7R_1$ $(4,7)$-sphere $4R_1$, $7R_4$

Euler formula

- If e_{p-q} denote the number of edges separating p- and q-gon, then one has:

$$e_{p-q} = (p-i)f_p = (q-j)f_q$$

- Euler formula $V - E + F = 2 - 2g$ with g being the genus, can be rewritten as

$$(6-p)f_p + (6-q)f_q = 6(2-2g)$$

- This implies

$$e_{p-q}\{\frac{6-p}{p-i} + \frac{6-q}{q-j}\} = e_{p-q}\alpha(p,q,i,j) = 12(1-g)$$

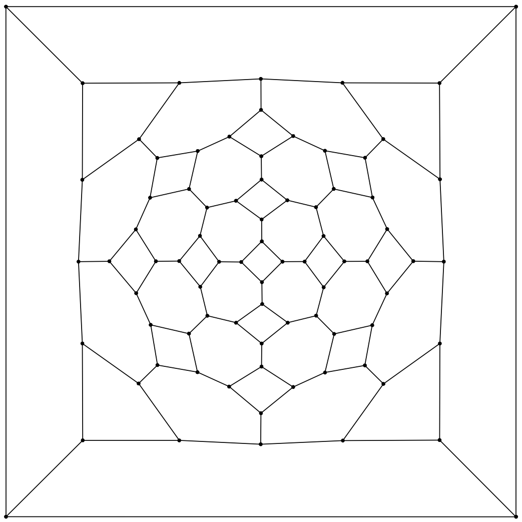
A classification

- If $\alpha(p,q,i,j) > 0$, then $g = 0$, the map exists only on sphere and the number of vertices depends only on $\alpha(p,q,i,j)$.
- If $\alpha(p,q,i,j) = 0$, then $g = 1$, the map exists only on torus.
- If $\alpha(p,q,i,j) < 0$, then $g > 1$, the map exists only on surfaces of higher genus and the number of vertices is determined by the genus and $\alpha(p,q,i,j)$.

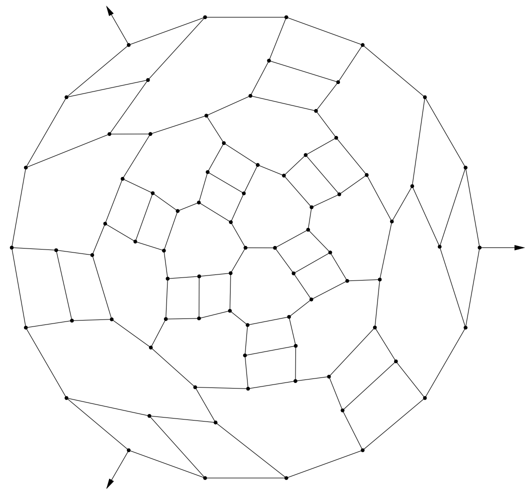
Detailed classification:

- On sphere: 55 sporadic examples + two infinite series: $Prism_q$ and $Barrel_q$
- On torus: 7 sporadic examples + 16 infinite cases.

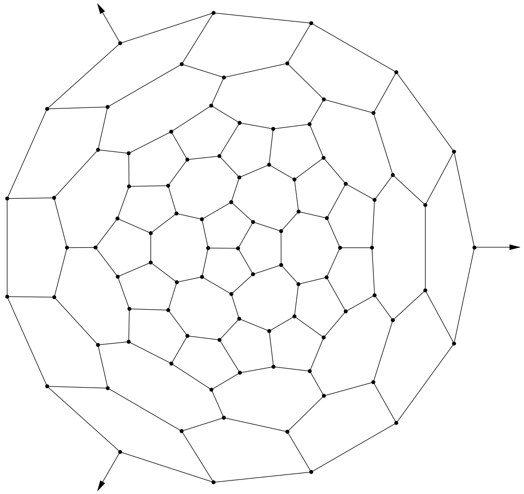
Some sporadic spheres



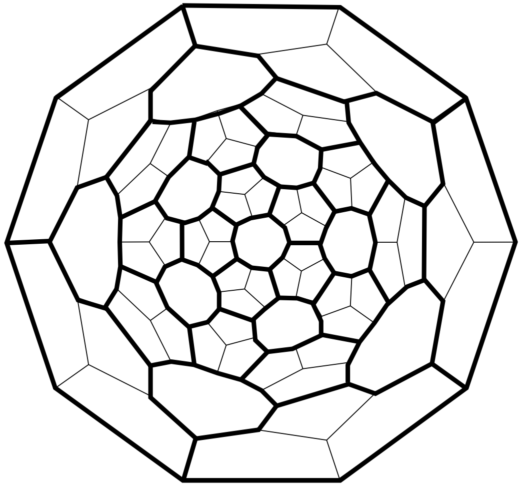
$(4, 7)$-sphere $4R_0$, $7R_4$



$(4, 8)$-sphere $4R_1$, $8R_4$

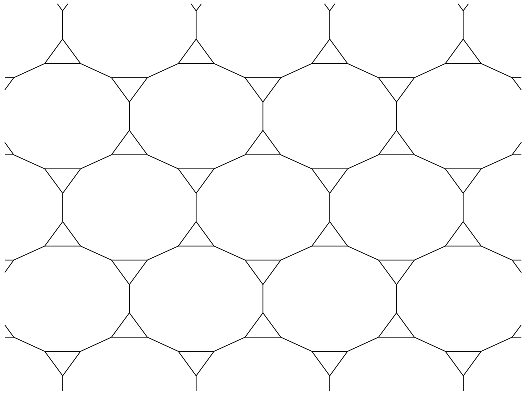


$(5, 7)$-sphere $5R_2$, $7R_2$

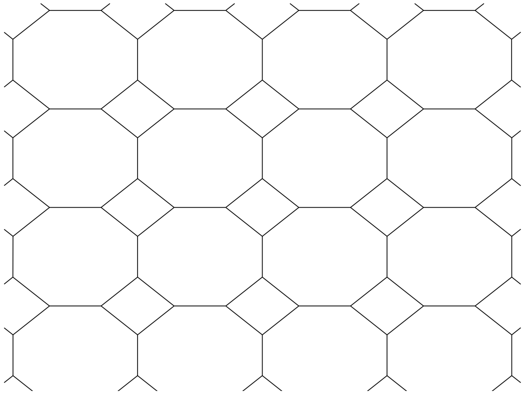


$(5, 10)$-sphere $5R_3$, $10R_0$

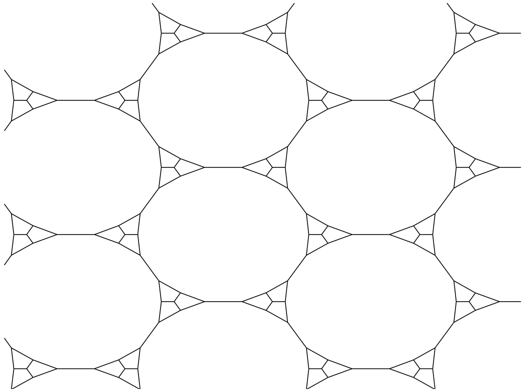
The parameters with sporadic tori



$(3,12)$-torus $3R_0$, $12R_6$

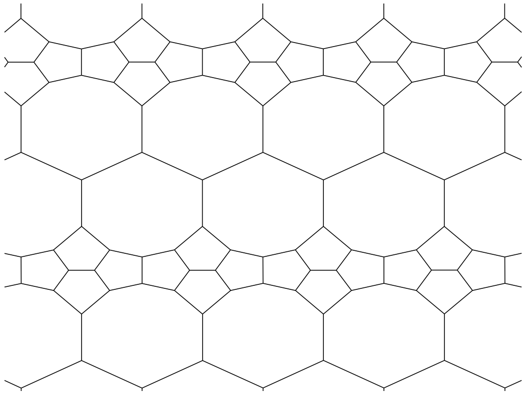


$(4,8)$-torus $4R_0$, $8R_4$

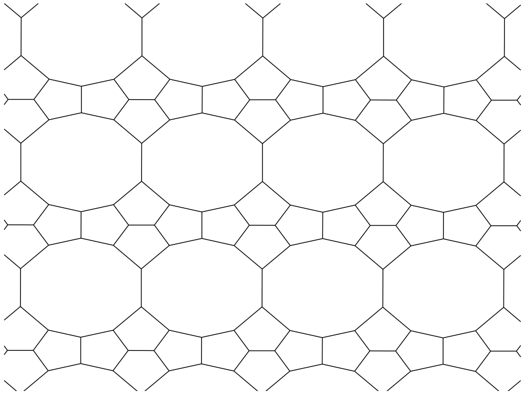


$(4,18)$-torus $4R_2$, $18R_6$

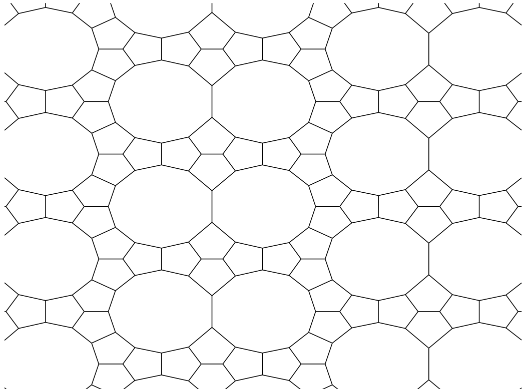
The parameters with sporadic tori



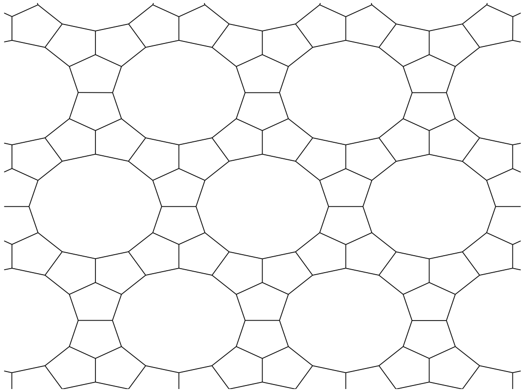
$(5, 8)$-torus $5R_3$, $8R_4$



$(5, 10)$-torus $5R_3$, $10R_2$



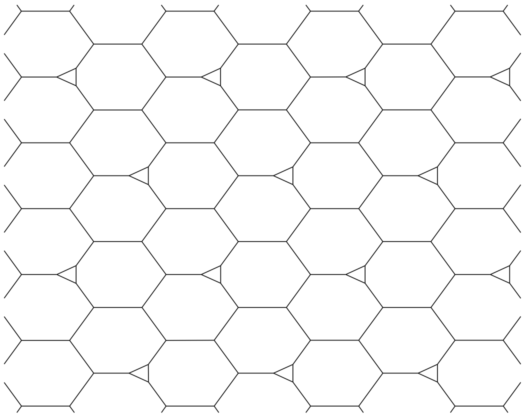
$(5, 11)$-torus $5R_3$, $11R_1$



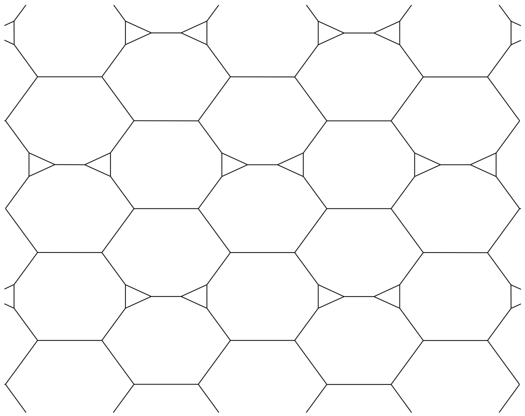
$(5, 12)$-torus $5R_3$, $12R_0$

$(3, q)$-tori $3R_0$, qR_6 $(7 \leq q \leq 12)$

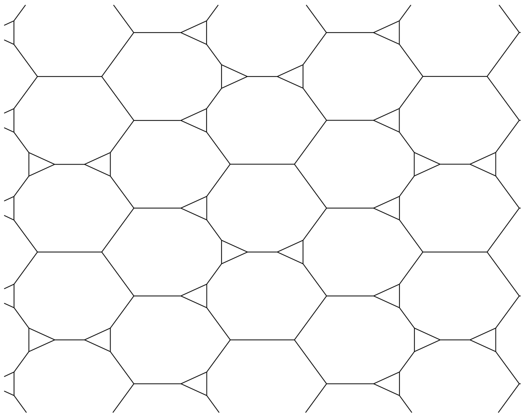
- They are obtained by truncating a 3-valent tesselation of the torus by 6-gons on the vertices from a set S_q, such that every face is incident to exactly $q - 6$ vertices in S_q.
- There is an infinity of possibilities, except for $q = 12$.



$(3, 7)$-torus $3R_0$, $7R_6$



$(3, 8)$-torus $3R_0$, $8R_6$

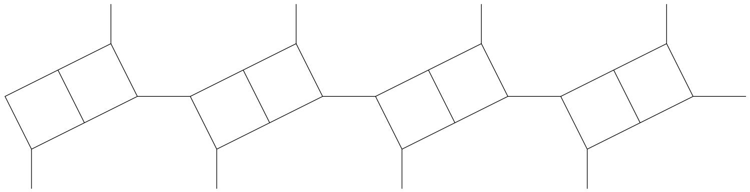


$(3, 9)$-torus $3R_0$, $9R_6$

- $(4, q)$-tori $4R_2$, qR_6 ($4 \leq \frac{q}{2} \leq 9$) are obtained (from 6 above) by 4-triakon (dividing 3-gon into triple of 4-gons)

$(4,10)$-tori $4R_1$, $10R_4$

- Take the symbols



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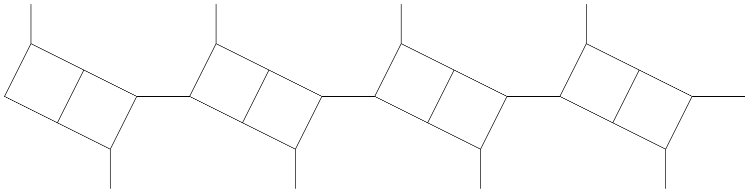
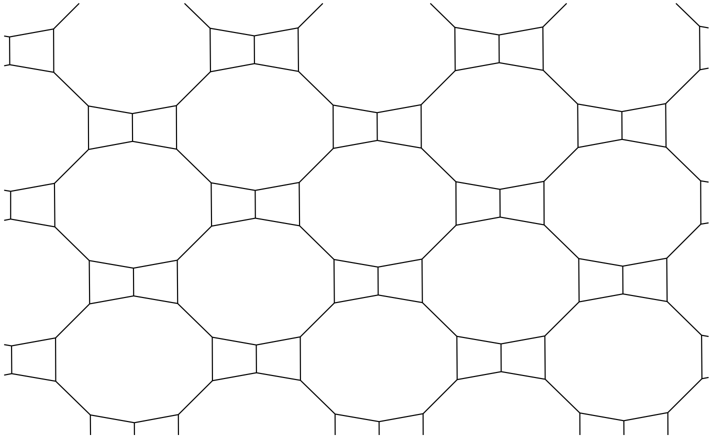
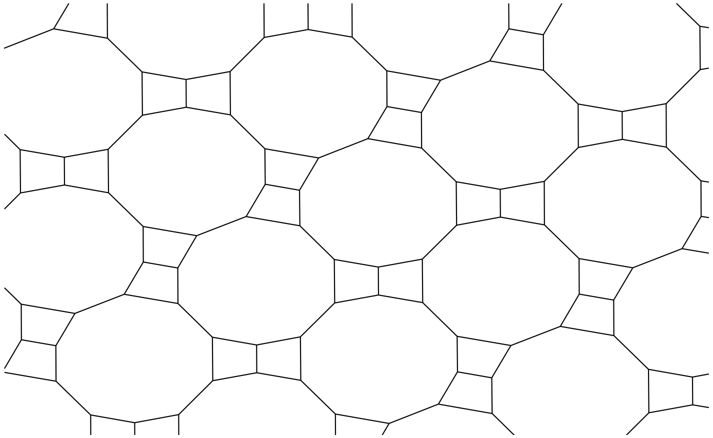


v

- The torus correspond to words of the form $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being equal to u or v.



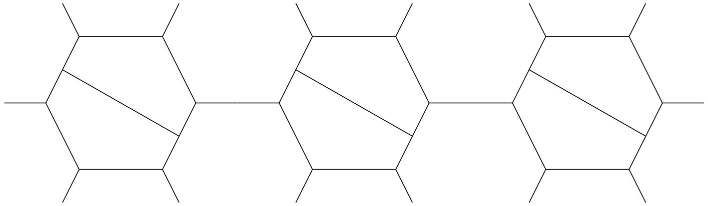
$(u)^\infty$



$(uv)^\infty$

$(5,7)$-tori $5R_1$, $7R_3$

- Take the symbols



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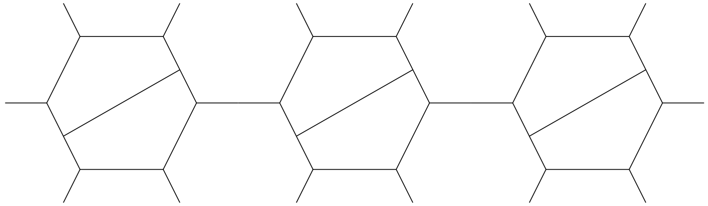
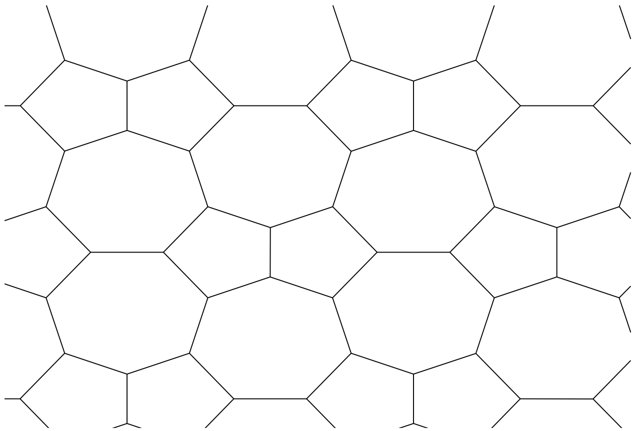
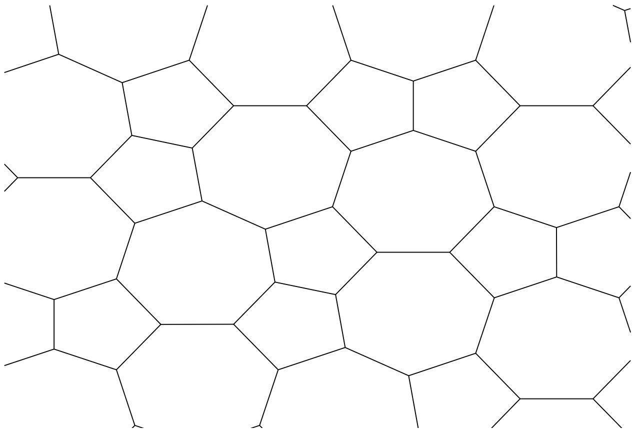


v

- The torus correspond to words of the form $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being equal to u or v.



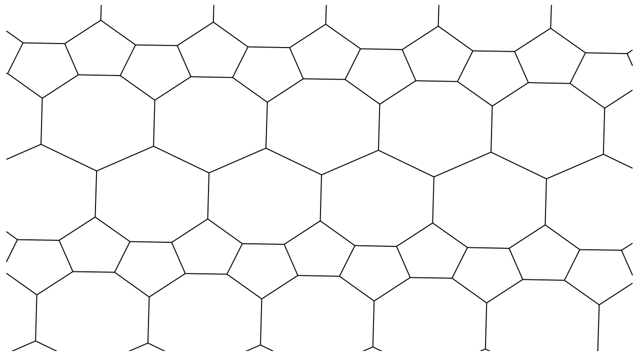
$(u)^\infty$



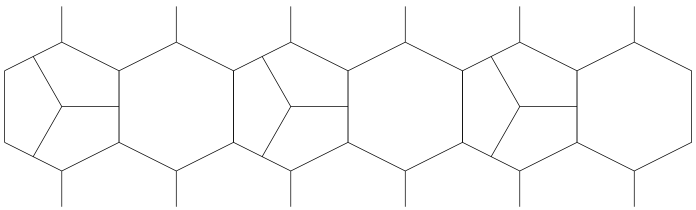
$(uv)^\infty$

$(5,7)$-tori $5R_2$, $7R_4$

- If 5-gons forming infinite lines, then one possibility:



- Take the symbols



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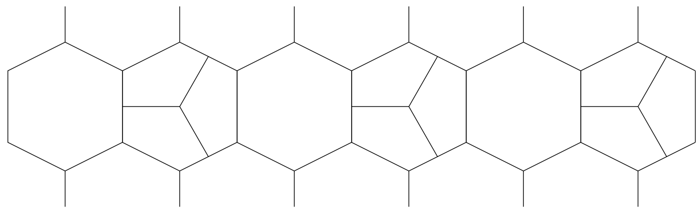
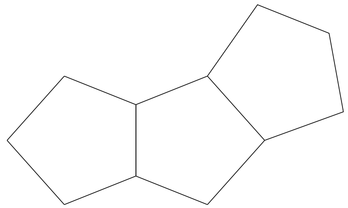


v

- Other tori correspond to words of the form $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being equal to u or v.

$(5,8)$-tori $5R_2$, $8R_2$

- 5-gons and 8-gons are organized in infinite lines.
- Only two configurations for 5-gons locally:



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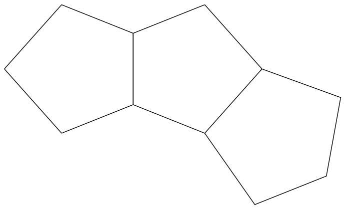
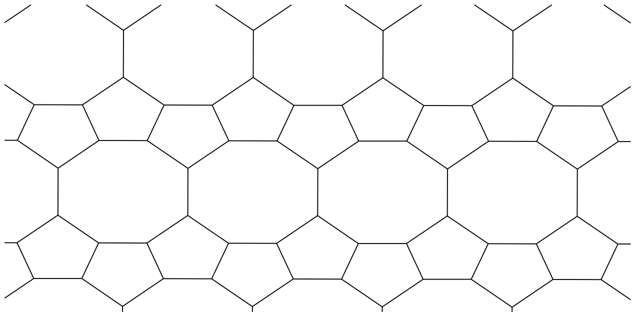
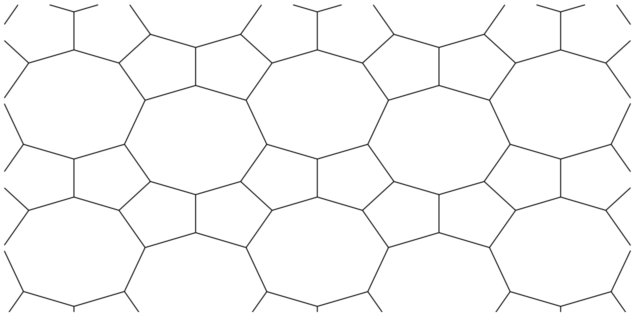


v

- Words of the form $(\alpha_0 \dots \alpha_n)^\infty$ with α_i being equal to uv or vu.



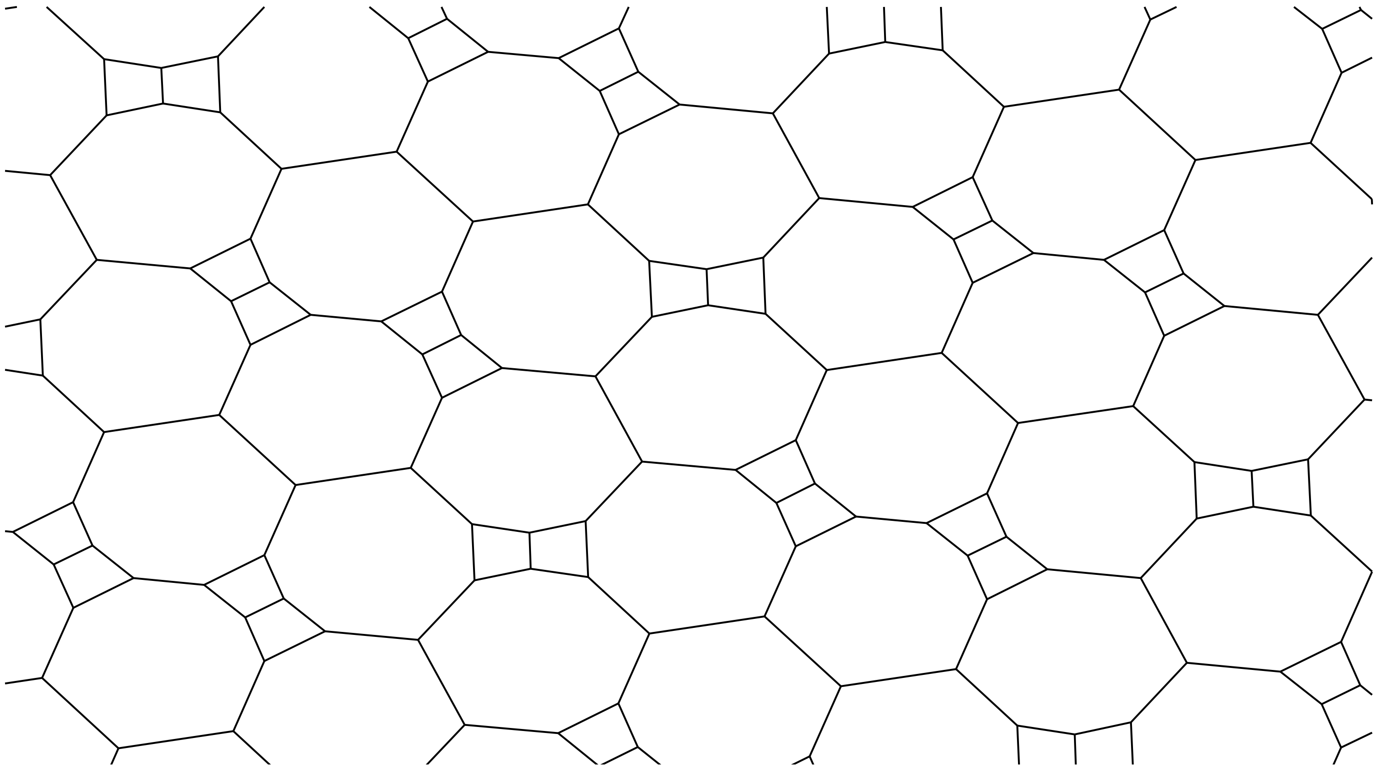
$(uv)^\infty$



$(uvvu)^\infty$

$(4,8)$-tori $4R_1$, $8R_5$

- They are in one-to-one correspondence with perfect matchings PM of a 6-regular triangulation of the torus, such that every vertex is contained in a triangle, whose edge, opposite to this vertex, belongs to PM.



$(4,8)$-tori $4R_1$, $8R_5$

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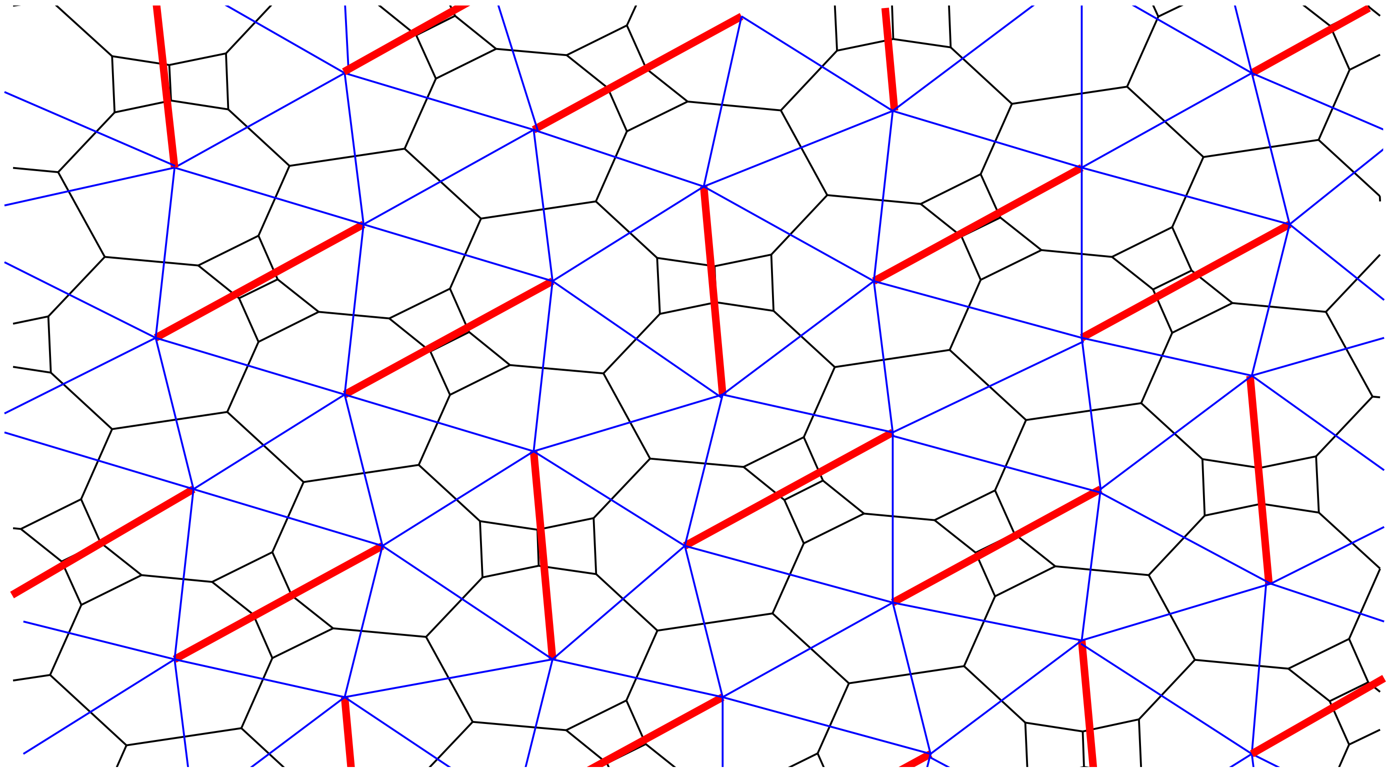
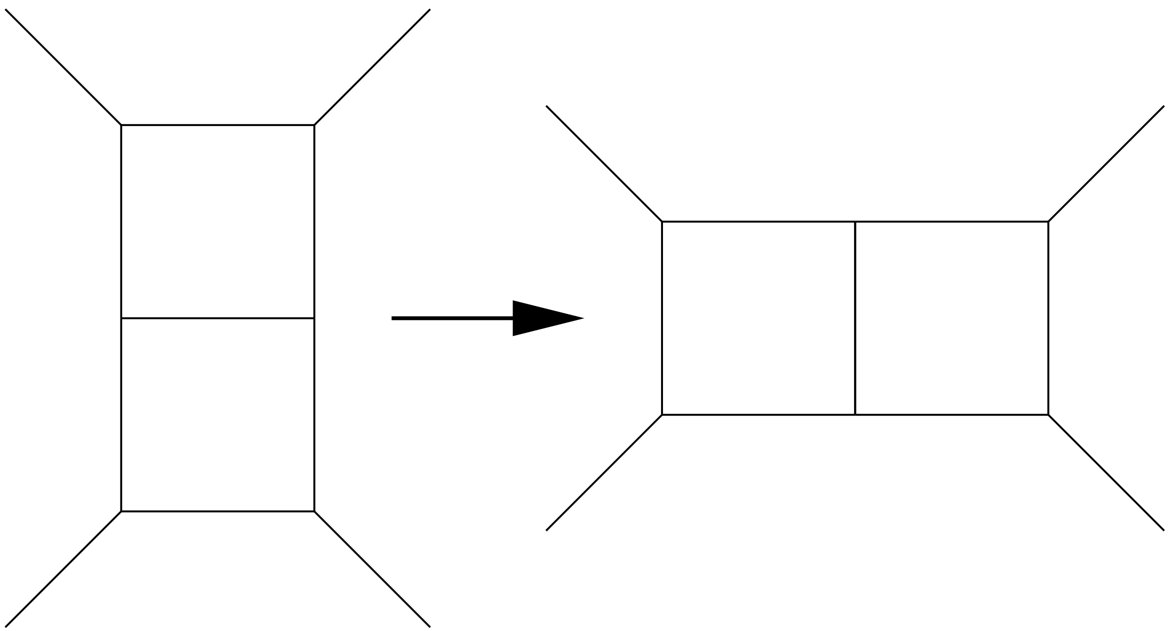


$(4,7)$-tori $4R_0, 7R_5$

- Given a $(4,8)$-torus, which is $4R_1$ and $8R_5$, the removal of edges between two 4-gons produces a $(4,7)$-torus, which is $4R_0$ and $7R_5$.
- Any such $(4,7)$-torus can be obtained in this way from two $(4,8)$-tori T_1 and T_2, which are $4R_1$ and $8R_5$.
- T_1 and T_2 are obtained from each other by the transformation



Our research program

- We investigated the cases of 3-regular spheres and tori being pR_i or qR_j.
- Such maps with $q = 6$ should be on sphere only.
  - All $(3, 6)$-spheres are $3R_0$.
  - There are infinities of $(4, 6)$-spheres $4R_i$ for $i = 0$, 1, 2; there are 9 $(4, 6)$-spheres $6R_j$.
  - There are infinities of $(5, 6)$-spheres $5R_i$ for $i = 0$, 1, 2; there are two spheres $5R_3$ and 26 spheres $6R_j$.
- For a (p, q)-polyhedron, which is qR_j, one has $j \leq 5$.
- For a 3-connected (p, q)-torus, which is qR_j, one has $j \leq 6$.

Representations of (p, q)-maps

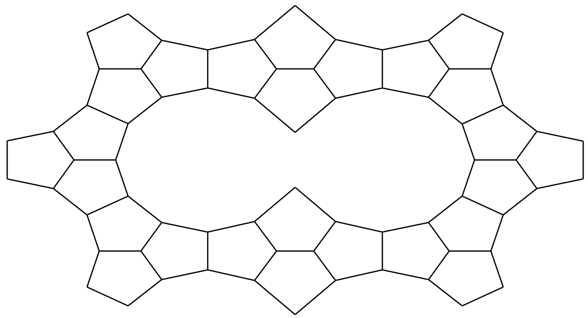
- Steinitz theorem: Any 3-connected planar graph is the skeleton of a polyhedron.
- Torus case:
  - A (p, q)-torus has a fundamental group isomorphic to $\mathbb{Z}^2$, its universal cover is a periodic (p, q)-plane.
  - A periodic (p, q)-plane is the universal cover of an infinity of (p, q)-tori.
  - Take a (p, q)-torus T and its corresponding (p, q)-plane P. If all translation preserving P arise from the fundamental group of T, then T is called minimal.
  - Any (p, q)-plane is the universal cover of a unique minimal torus.

II. $(p, 3)$-polycycles

$(p,3)$-polycycles

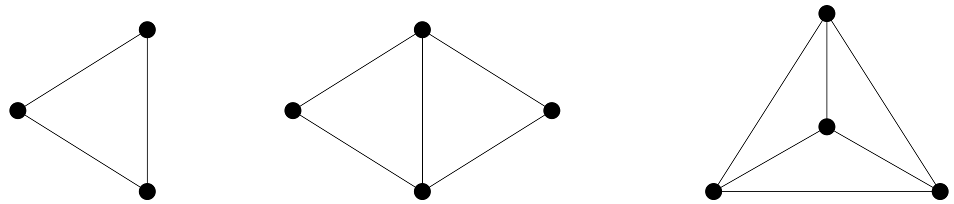
A generalized $(p,3)$-polycycle is a 2-connected plane graph with faces partitioned in two families F_1 and F_2, so that:

- all elements of F_1 (proper faces) are (combinatorial) p-gons;
- all elements of F_2 (holes, the exterior face is amongst them) are pairwisely disjoint;
- all vertices have valency 3 or 2 and any 2-valent vertex lies on a boundary of a hole.

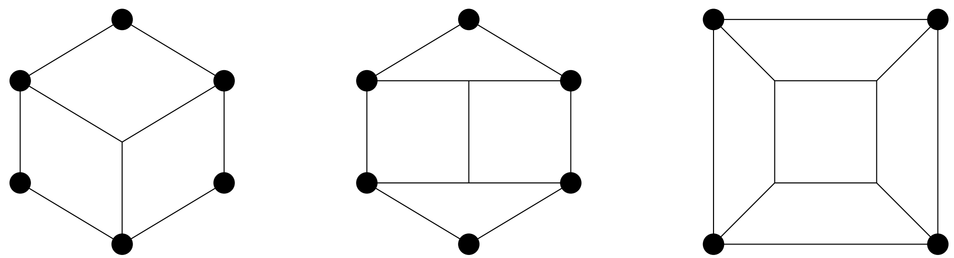


$(3,3)$ and $(4,3)$-polycycles

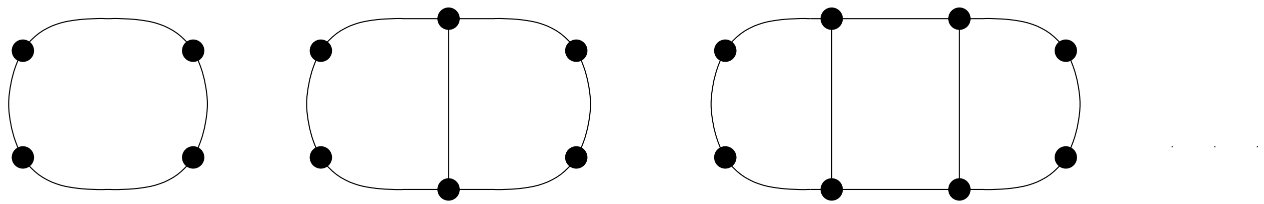
(i) Any $(3,3)$-polycycle is one of the following 3 cases:



(ii) Any $(4,3)$-polycycle belongs to the following 3 cases:



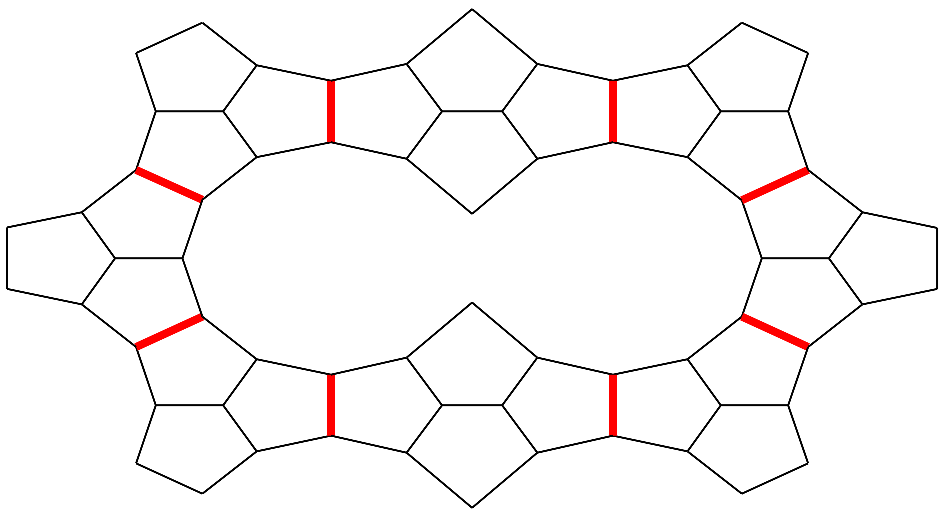
or belong to the following infinite family of $(4,3)$-polycycles:



This classification is very useful for classifying $(4,q)$-maps.

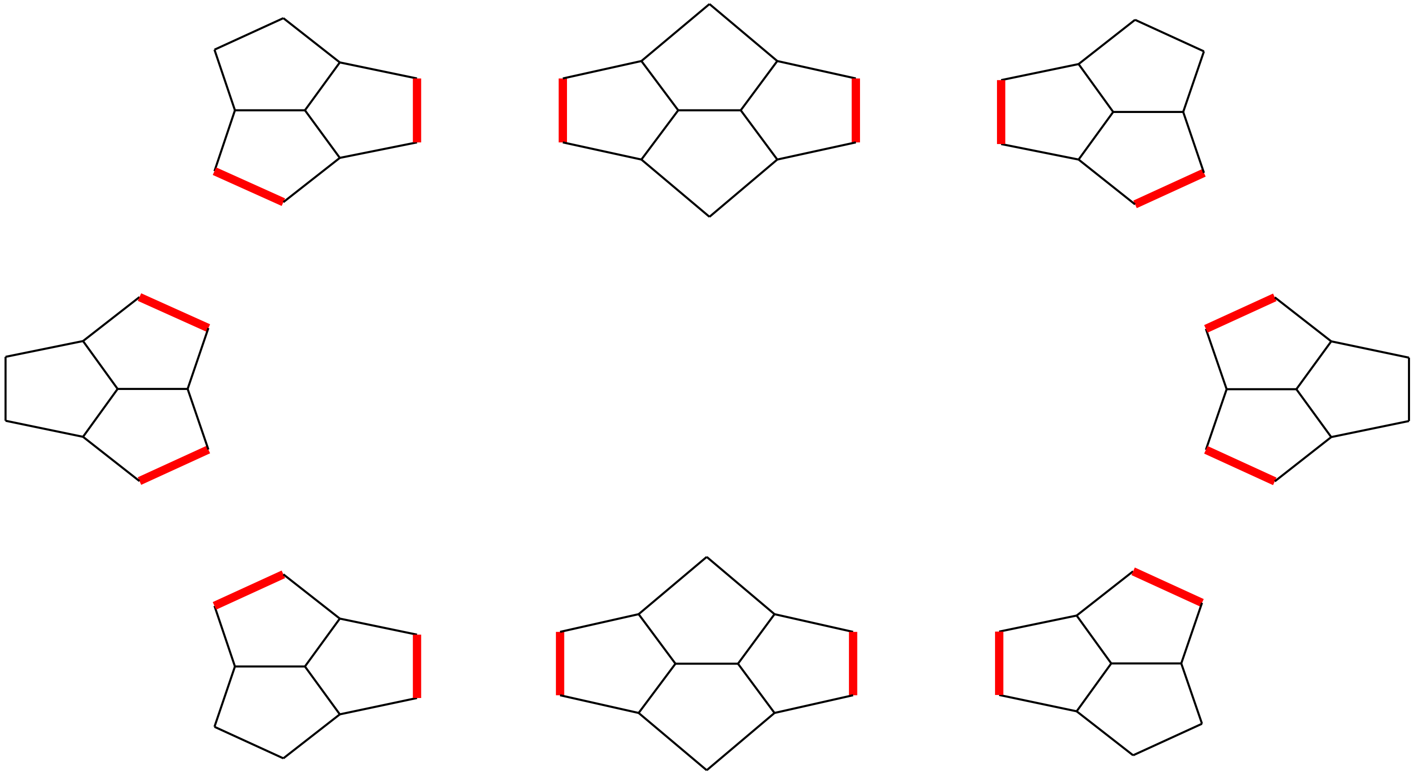
Polycycle decomposition

A bridge is an edge going from a hole to a hole (possibly, the same).



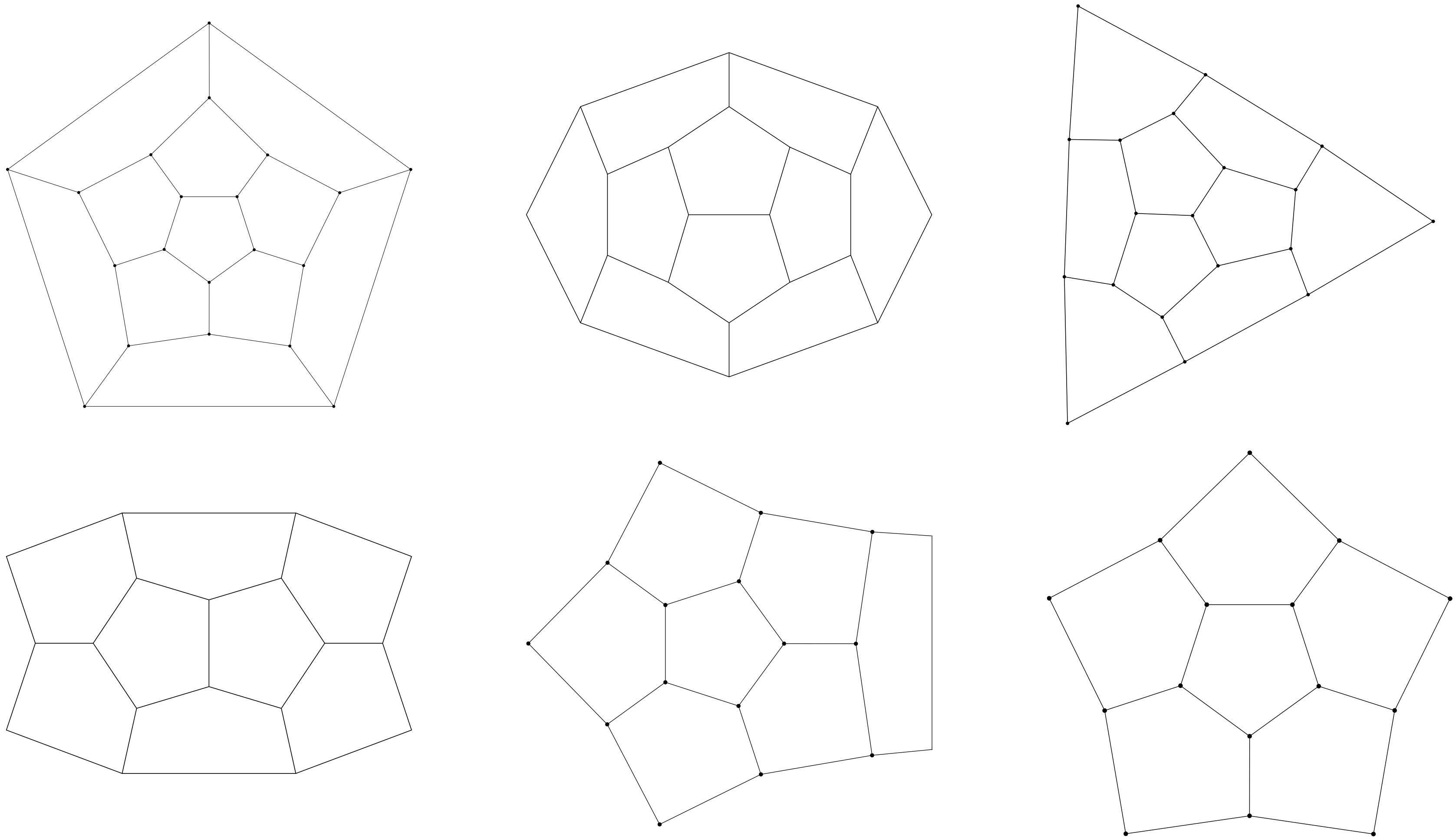
Polycycle decomposition

Any generalized $(p,3)$-polycycle is uniquely decomposable along its bridges.

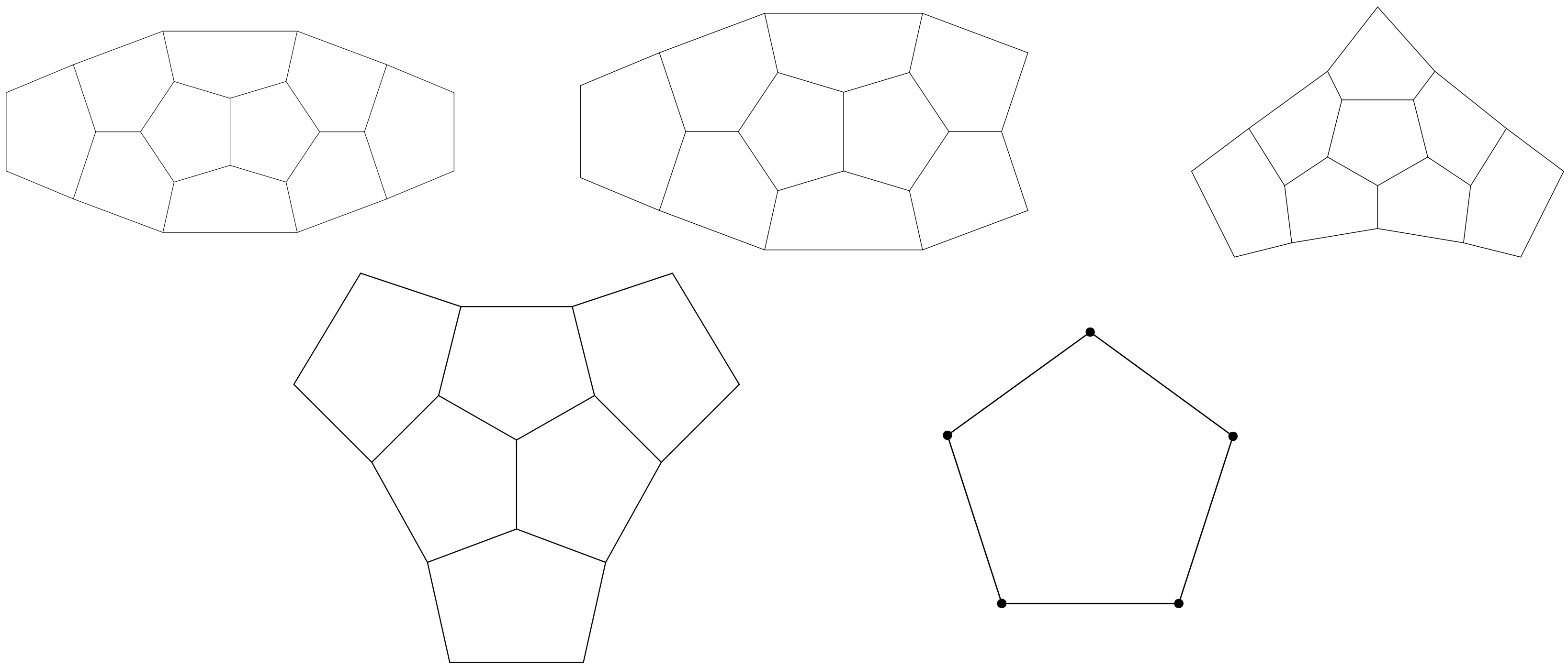


Polycycle decomposition

The set of non-decomposable $(5,3)$-polycycles has been classified:

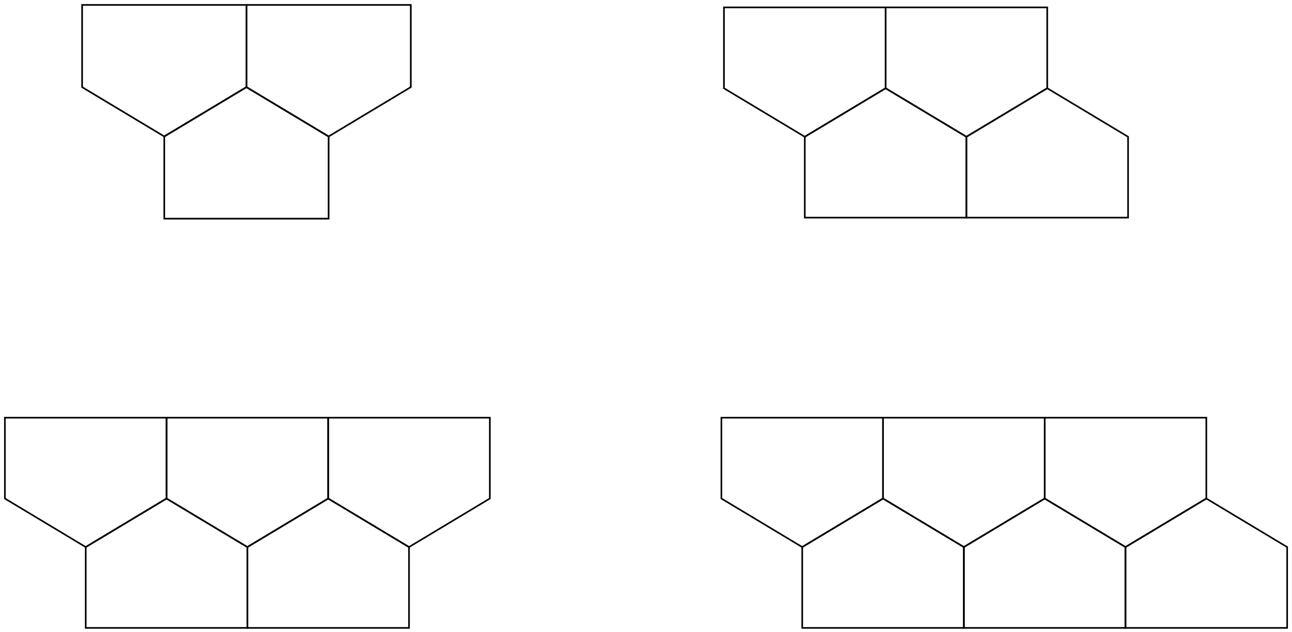


Polycycle decomposition



Polycycle decomposition

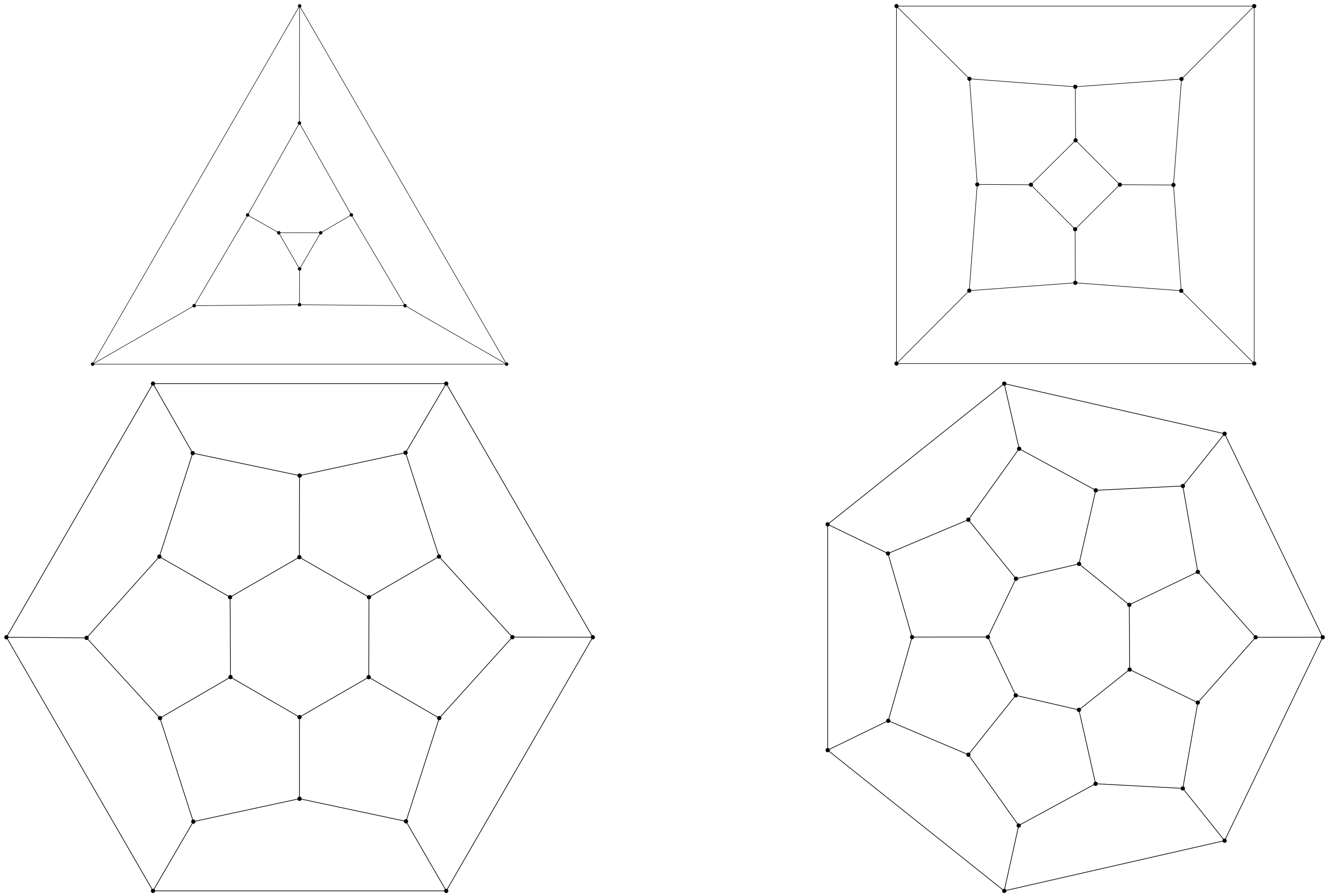
The infinite series of non-decomposable $(5,3)$-polycycles $E_n, n \geq 1$:



The only non-decomposable infinite $(5,3)$-polycycle are $E_{\mathbb{Z}^+}$ and $E_{\mathbb{Z}}$

Polycycle decomposition

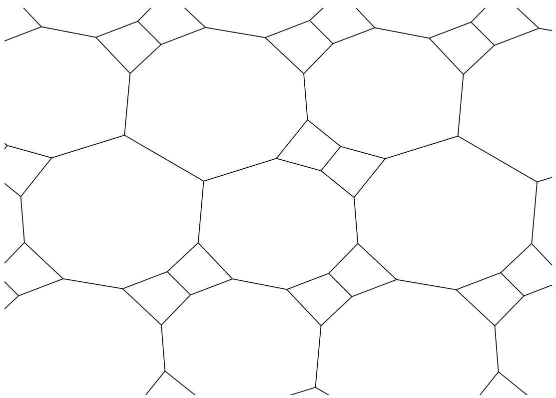
The infinite series of non-decomposable generalized $(5,3)$-polycycles $Barrel_q$, $q \geq 3$, $q \neq 5$:



III. pR_i-maps

$4R_0$- and $4R_1$-cases

- $4R_0$-maps exist only for $q = 7$ or 8.
  - For $q = 7$: infinity of spheres and minimal tori.
  - For $q = 8$, the only case is strictly face-regular $(4, 8)$-torus $4R_0$, $8R_4$.
- $4R_1$-maps exist only for $7 \leq q \leq 10$
  - For $q = 7$ and 8: infinity of spheres and minimal tori.
  - For $q = 9$, no sphere is known, but such tori exist:



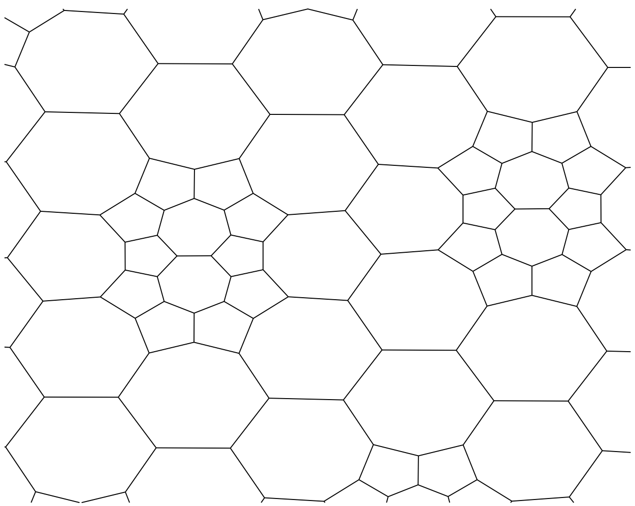
  - For $q = 10$, only tori exist and they are $10R_4$.

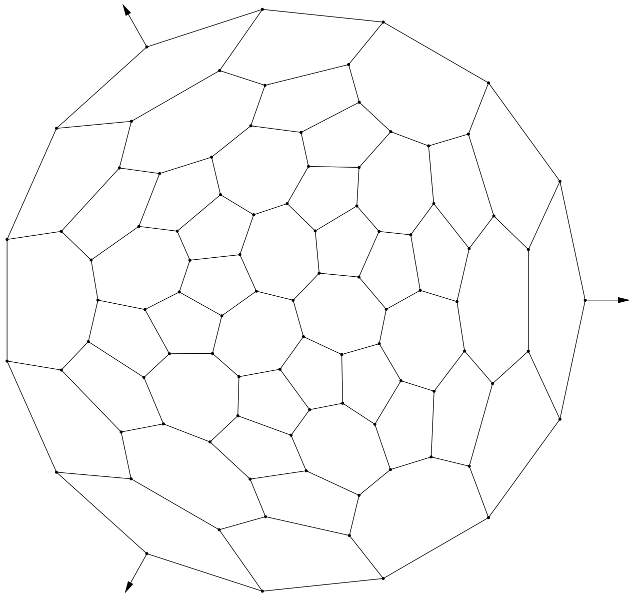
$4R_2$-case

- $Prism_q$ is always $4R_2$; so, we consider different maps.
- 4-gons are organized in triples.
- One has $7 \leq q \leq 16$ or $q = 18$
  - For $q = 14, 16, 18$, they exist only on torus and are qR_6
  - Infinity of spheres is found for $7 \leq q \leq 10$, $q = 12$ and $q = 15$.
  - Uncertain cases are $q = 11$ and 13. Possible spheres have at least 92 and 116 vertices.

$5R_1$- and $5R_2$-cases

- $5R_1$-maps are only $(5,7)$-tori and they are $7R_3$.
- $5R_2$-maps exist only for $q = 7$ and 8.
  - For $q = 7$, there is an infinity of spheres (Hajduk & Sotak) and tori.

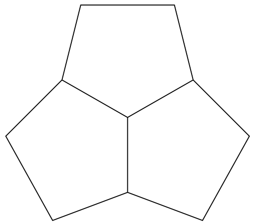


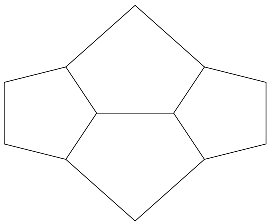


  - For $q = 8$, they exist only on torus and are also $8R_2$.

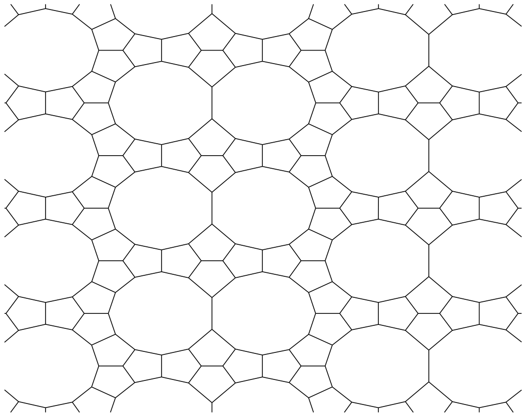
$5R_3$-case

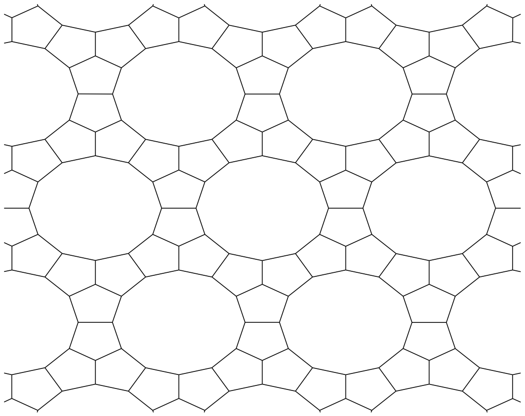
- Possible only for $6 \leq q \leq 12$. The set of 5-gons is decomposed along the bridges into polycycles E_1 and E_2:





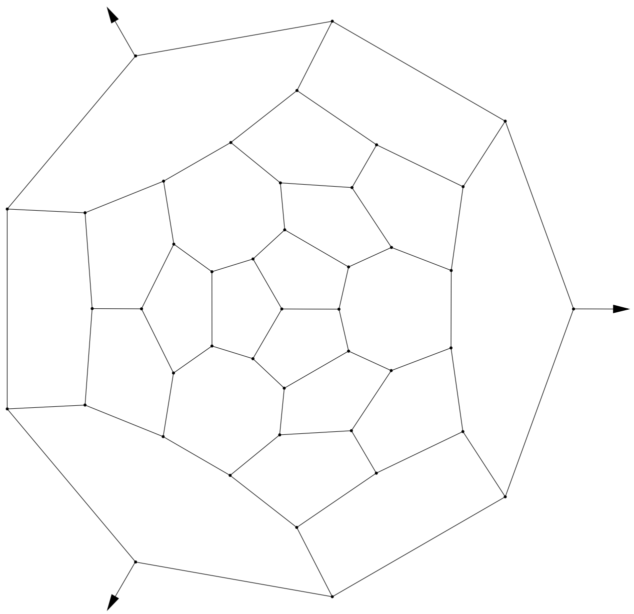
- For $q = 12$, they exist only on torus and are $12R_0$
- For $q = 11$, they exist only on torus and are $11R_1$

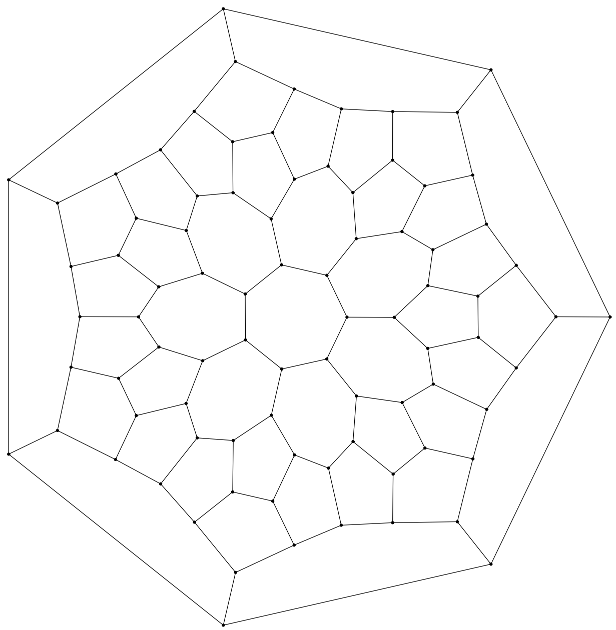




$5R_3$-case

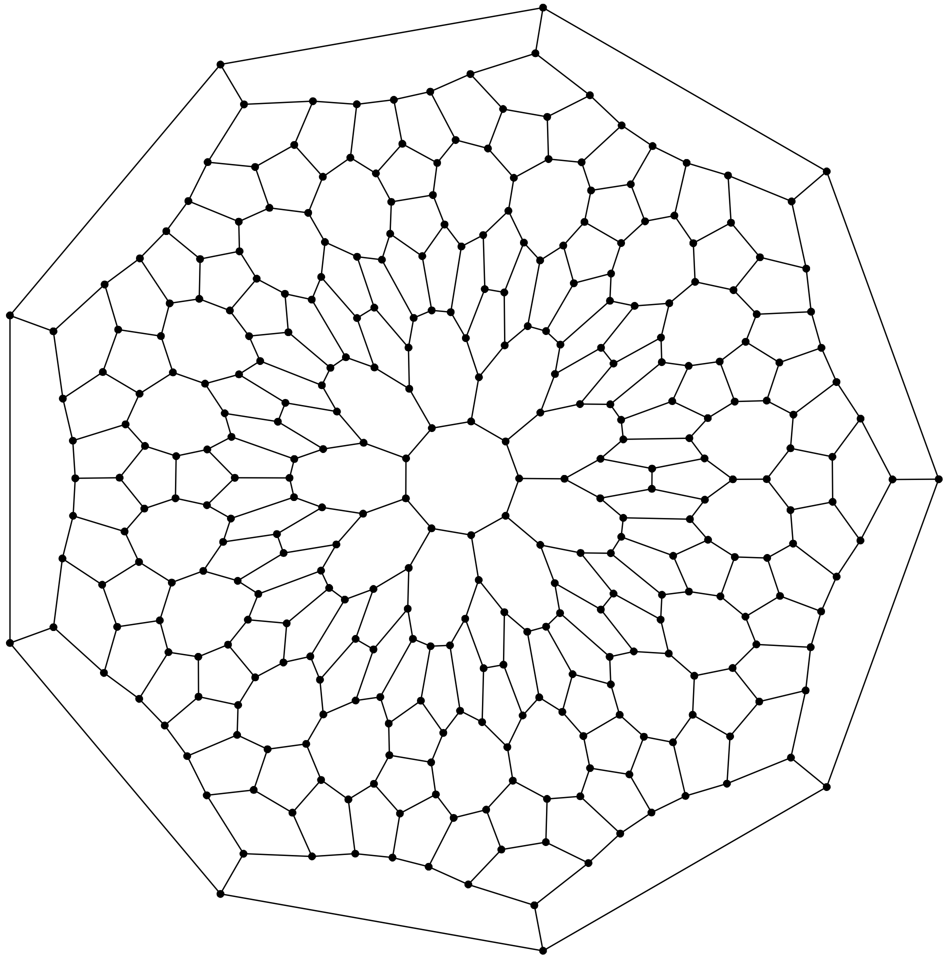
- For $q = 7$, they exist only on sphere and are:





$5R_3$-case

- For $q = 9$, it exist only on sphere and is:



$5R_3$-case

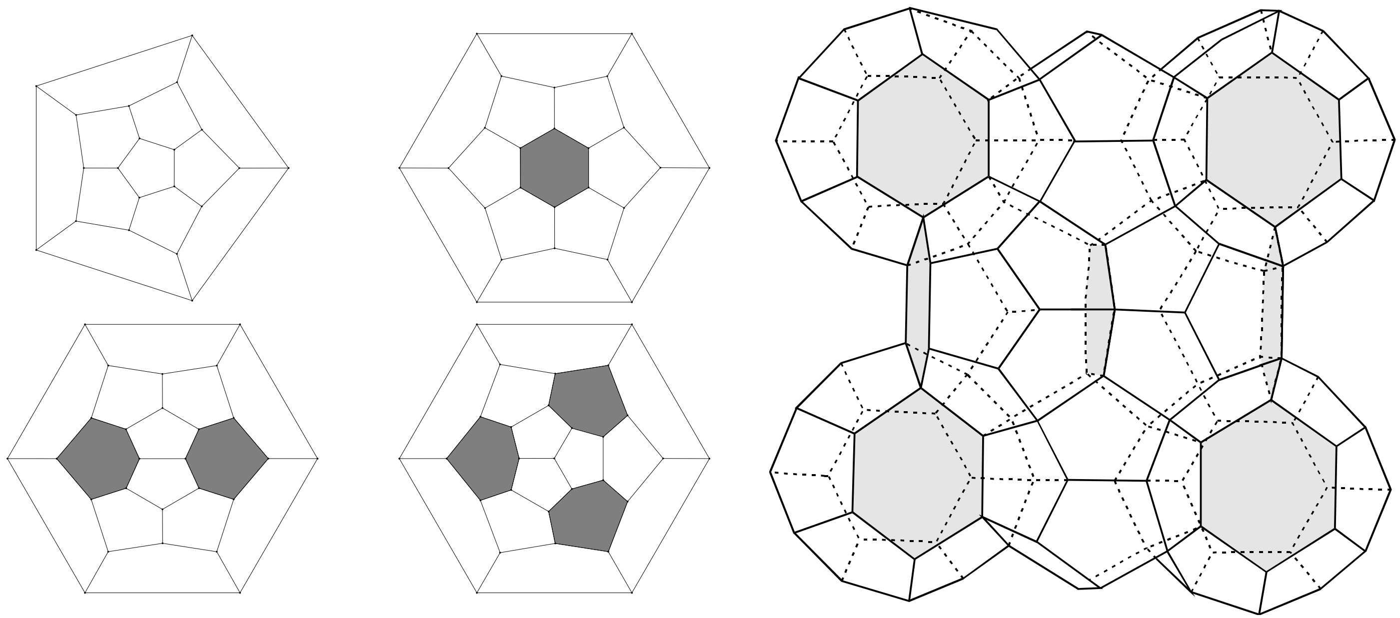
- For $q = 8$, an infinity of $(5, 8)$-spheres is known. Two tori are known, one being $8R_4$, the other not.
- For $q = 10$, some spheres are known with 140, 740 and 7940 vertices. Infinitness of spheres and existence of tori, which are not $10R_2$, are undecided.

III. Frank-Kasper maps,

i.e. qR_0-maps

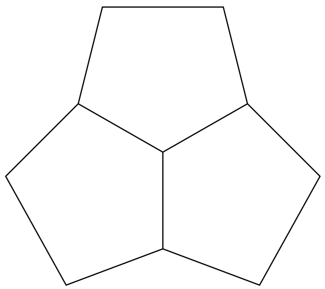
Frank-Kasper polyhedra

- A Frank-Kasper polyhedron is a $(5, 6)$-sphere which is $6R_0$. Exactly 4 cases exists.
- A space fullerene is a face-to-face tiling of the Euclidean space E^3 by Frank-Kasper polyhedra. They appear in crystallography of alloys, bubble structures, clathrate hydrates and zeolites.

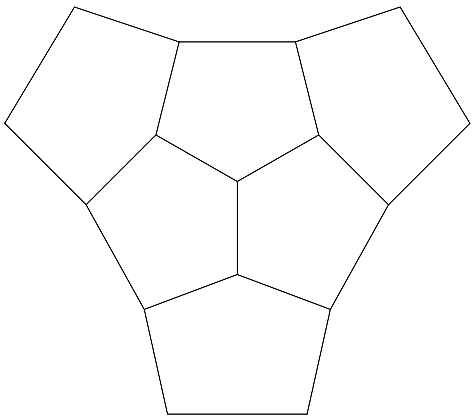


Polycycle decomposition

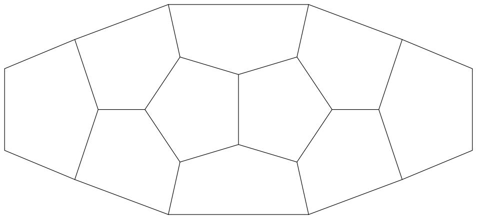
- We consider $(5, q)$-spheres and tori, which are qR_0
- The set of 5-gonal faces of Frank-Kasper maps is decomposable along the bridges into the following non-decomposable $(5, 3)$-polycycles:



E_1



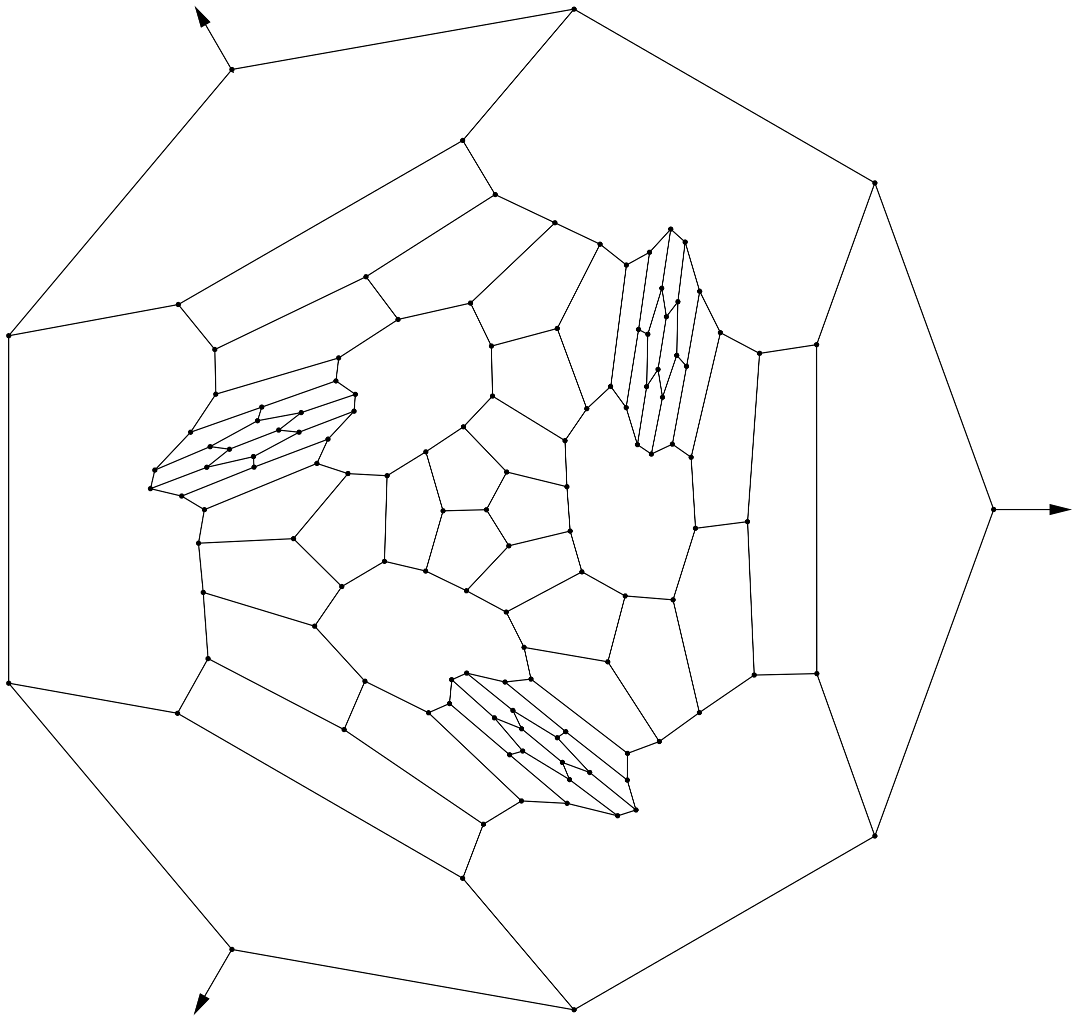
C_3



C_1

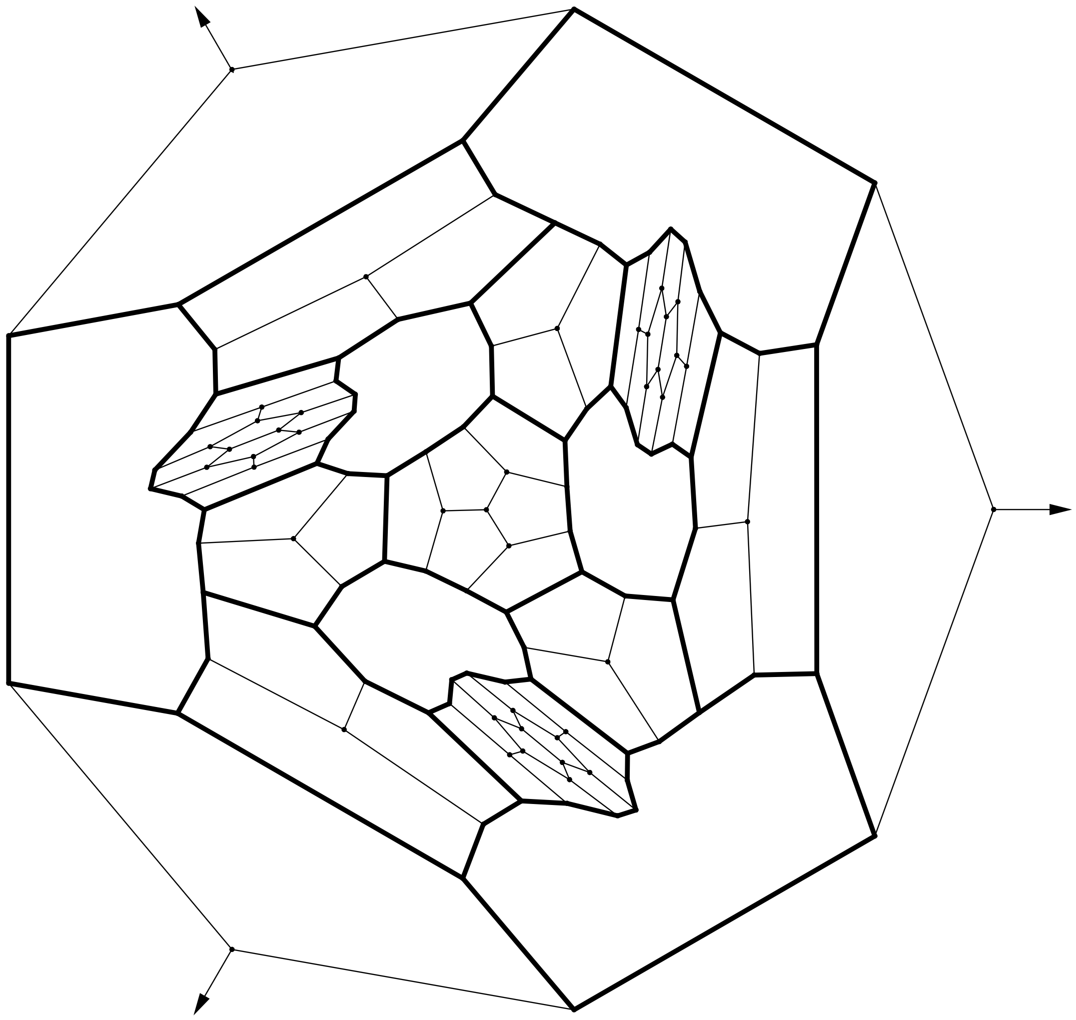
- The major skeleton $Maj(G)$ of a Frank-Kasper map is a 3-valent map, whose vertex-set consists of polycycles E_1 and C_3.

Polycycle decomposition



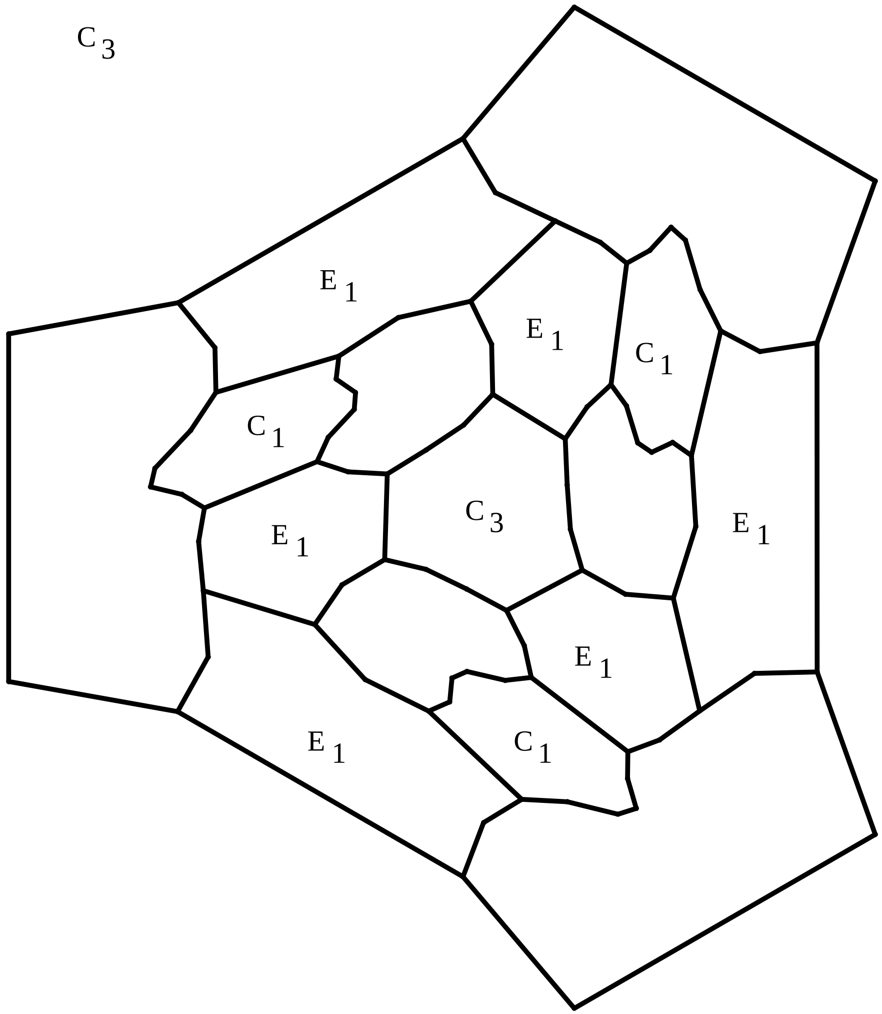
A Frank-Kasper $(5, 14)$-sphere

Polycycle decomposition



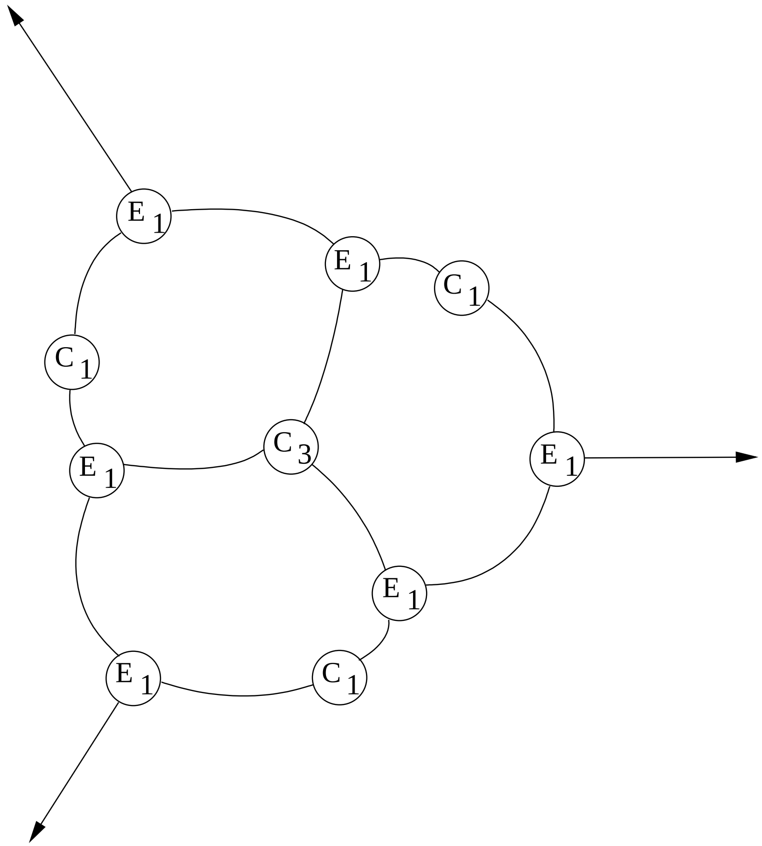
The polycycle decomposition

Polycycle decomposition



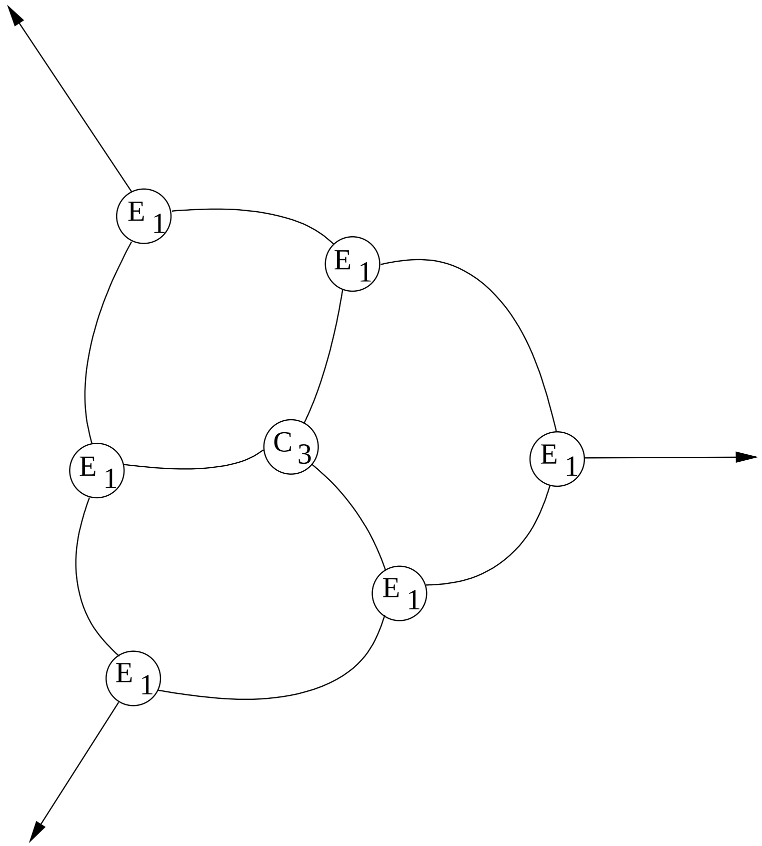
Their names

Polycycle decomposition



The graph of polycycles.

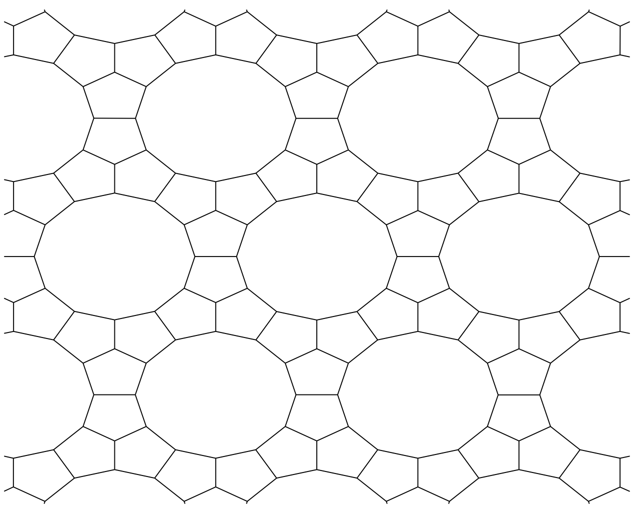
Polycycle decomposition



$Maj(G)$: eliminate C_1, so as to get a 3-valent map

Results

- For a Frank-Kasper $(5, q)$-map, the gonality of faces of the 3-valent map $Maj(G)$ is at most $\lfloor \frac{q}{2} \rfloor$.
- If $q < 12$, then there is no $(5, q)$-torus qR_0 and there is a finite number of $(5, q)$-spheres qR_0.
  For $q = 12$:
  - There is a unique $(5, 12)$-torus $12R_0$
  - The $(5, 12)$-spheres $12R_0$ are classified.



- Conjecture: there is an infinity of $(5, q)$-spheres qR_0 for any $q > 12$.

IV. qR_1-maps

Euler formula

If P is a (p, q)-map, which is qR_1 (q-gons in isolated pairs), then:

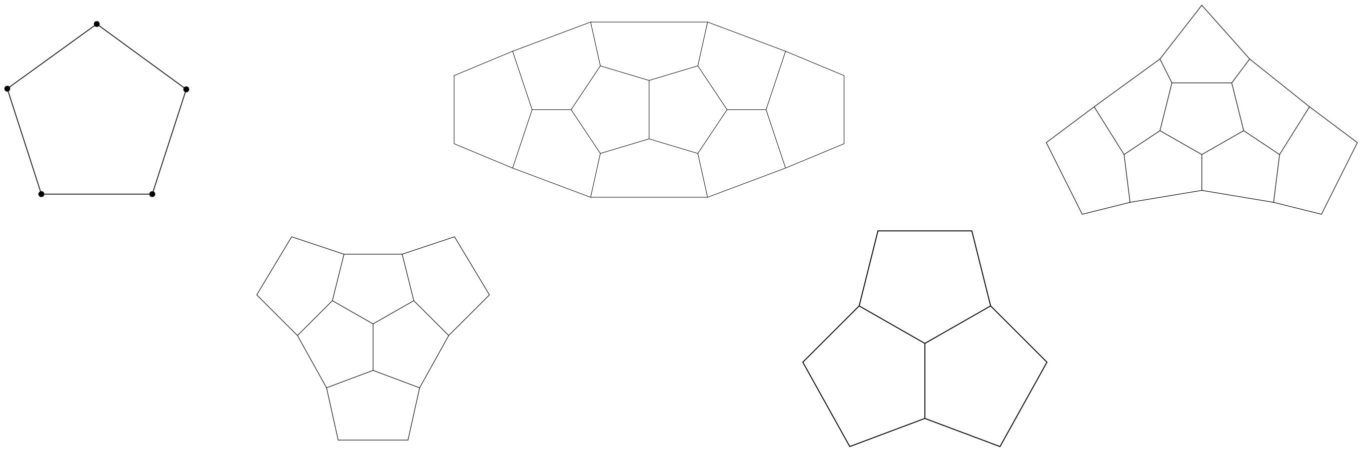
$$\begin{cases} (6-p)x_3 + (2(p-q) + (6-p)(q-1))f_q = 4p & \text{on sphere,} \\ (6-p)x_3 + (2(p-q) + (6-p)(q-1))f_q = 0 & \text{on torus.} \end{cases}$$

with x_3 being the number of vertices included in three p-gonal faces.

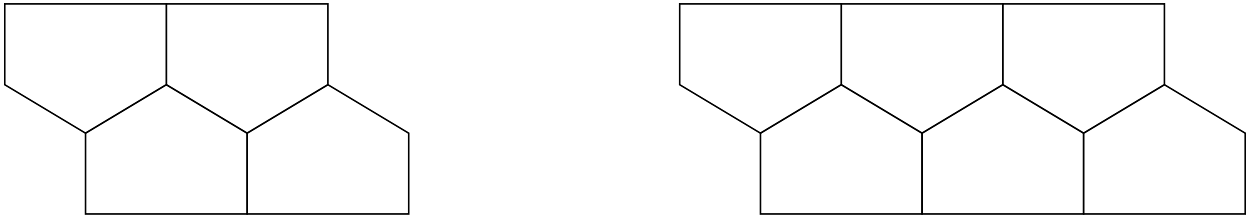
- For $(4, q)$-maps this yields finiteness on sphere and non-existence on torus.
- For $(5, q)$-maps this implies finiteness on sphere for $q \leq 8$ and non-existence on torus

Polycycle decomposition

- There is no $(4, q)$-sphere qR_1.
- $(5, q)$-map qR_1, the non-decomposable $(5, 3)$-polycycles, appearing in the decomposition are:



and the infinite serie E_{2n} (see cases $n = 1, 2$ below):



$(5, 9)$-maps $9R_1$

- In the case $q = 9$, Euler formula implies that the number of vertices, included in three 5-gons, is bounded (for sphere) or zero (for torus).
- All non-decomposable $(5, 3)$-polycycles (except the single 5-gon) contain such vertices. This implies finiteness on sphere and non-existence on torus.
- While finiteness of $(5, q)$-spheres qR_1 is proved for $q = 8$ and $q = 9$, the actual work of enumeration is not finished.

$(5, 10)$-tori $10R_1$ and beyond

- Using Euler formula and polycycle decomposition, one can see that the only appearing polycycles are:

- $(5, 10)$-torus, which is $10R_1$, corresponds, in a one-to-one fashion, to a perfect matching PM on a 6-regular triangulation of the torus, such that every vertex is contained in a triangle, whose edge, opposite to this vertex, belongs to PM.
- For any $q \geq 10$, there is a $(5, q)$-torus, which is qR_1.
- Conjecture: there exists an infinity of $(5, q)$-spheres qR_1 if and only if $q \geq 10$.

V. qR_2-maps

Euler formula

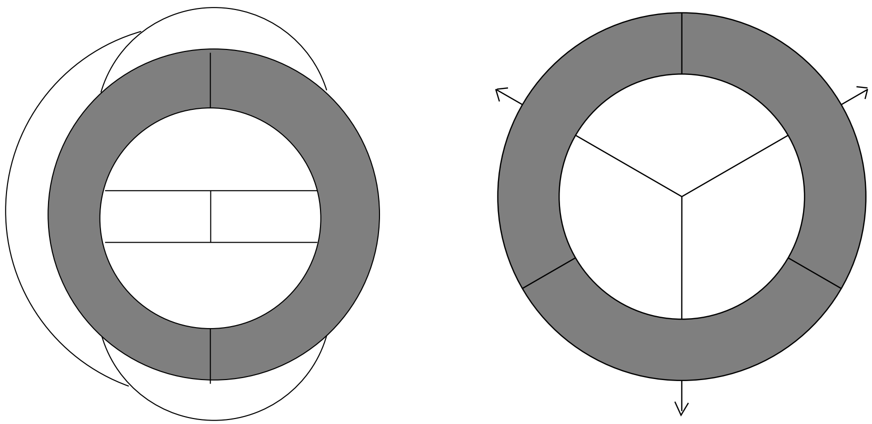
- The q-gons of a qR_2-map are organized in rings, including triples, i.e. 3-rings.
- One has the Euler formula

$$\begin{cases} (4-(4-p)(4-q))f_q + (6-p)(x_0+x_3) = 4p & \text{on sphere} \\ (4-(4-p)(4-q))f_q + (6-p)(x_0+x_3) = 0 & \text{on torus.} \end{cases}$$

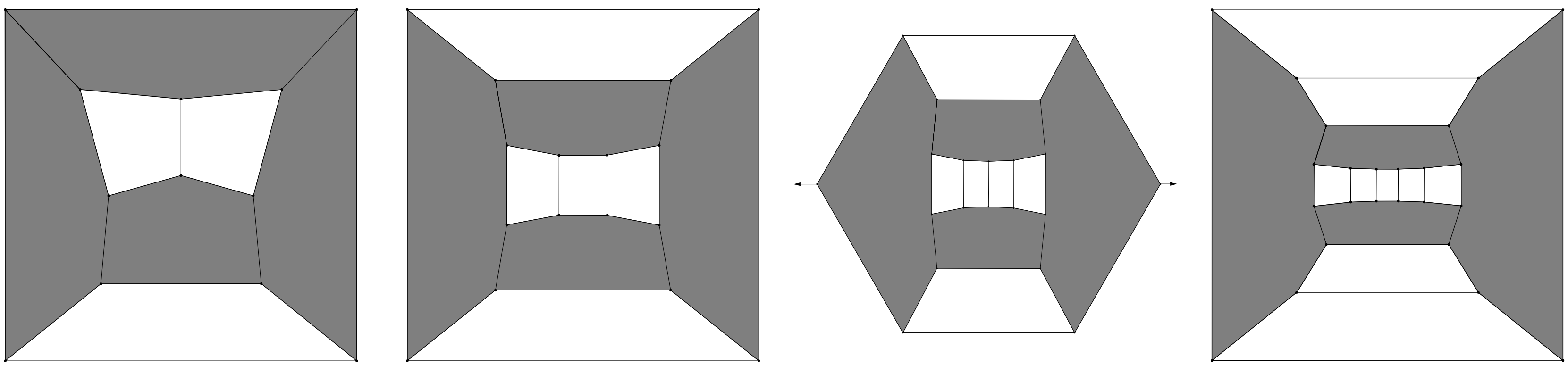
  - x_0 is the number of vertices incident to 3 p-gonal faces and
  - x_3 the number of vertices incident to 3 q-gonal faces.
- It implies the finiteness for $(4,q)$, $(5,6)$, $(5,7)$.

All $(4, q)$-maps qR_2

- two possibilities (for $q = 8, 6$):

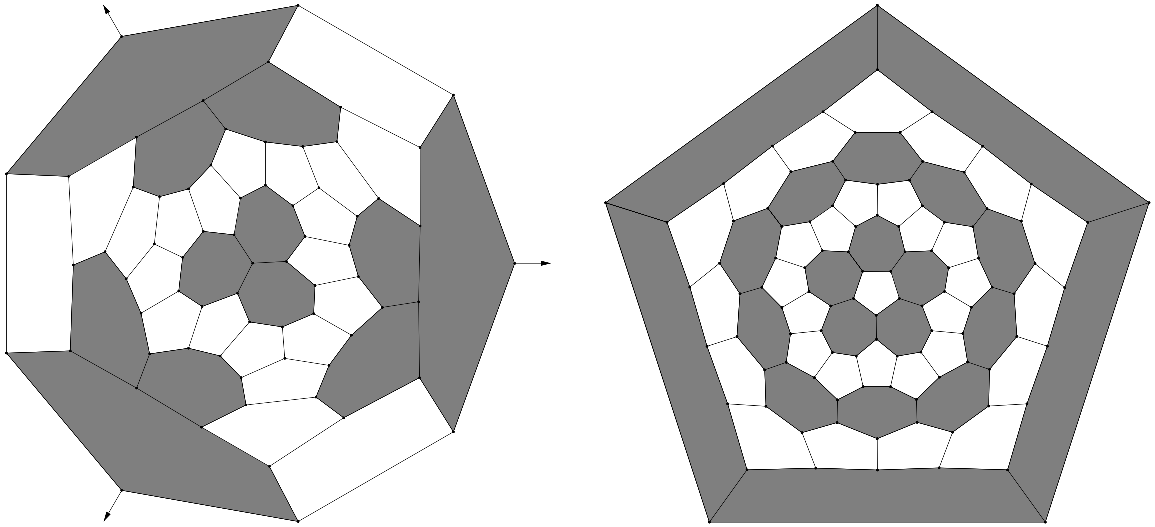


- and the infinite series



$(5, q)$-maps qR_2

- For $q = 7$, 26 spheres and no tori. Two examples:

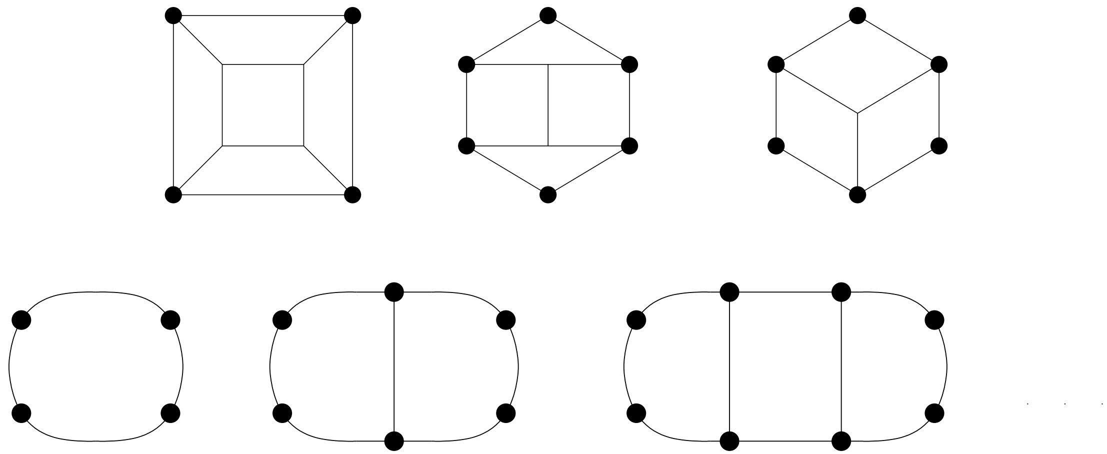


- For $q \geq 8$, there is an infinity of $(5, q)$-spheres and minimal $(5, q)$-tori, which are qR_2.
- a $(5, 8)$-map is $8R_2$ if and only if it is $5R_2$

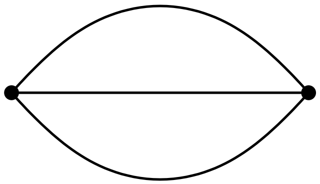
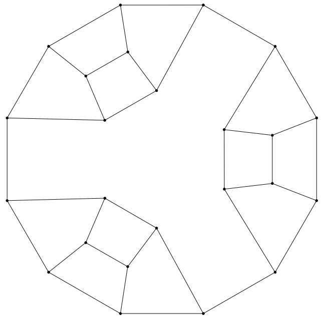
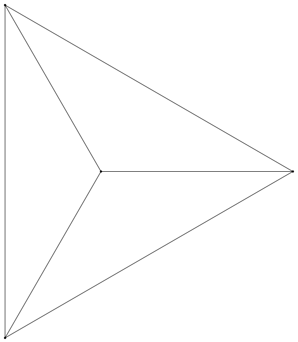
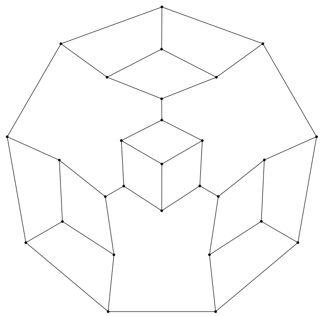
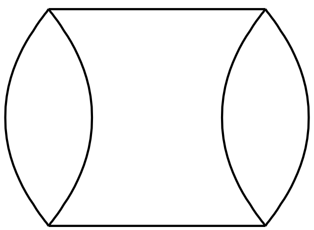
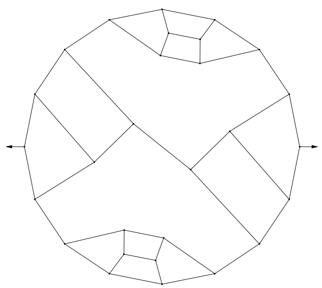
III. qR_3-maps

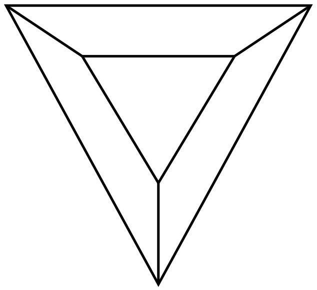
Classification for $(4, q)$-case

- The $(4,3)$-polycycles, appearing in the decomposition, are:

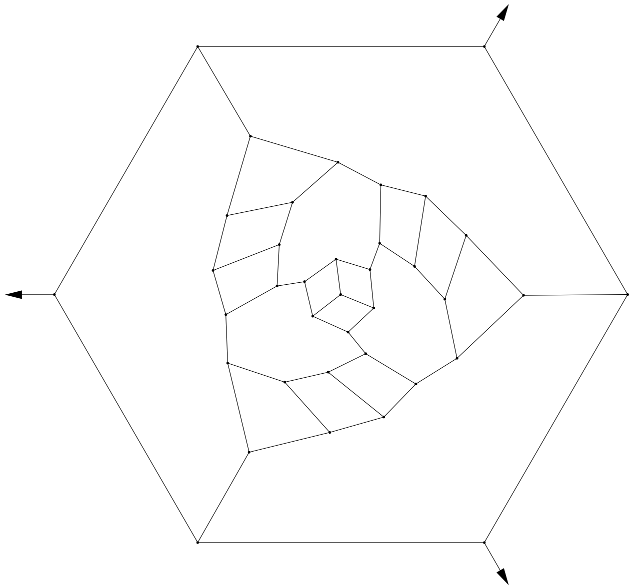


- Consider the graph, whose vertices are q-gonal faces of a $(4,q)$-sphere qR_3 (same adjacency).
  - It is a 3-valent map
  - Its faces are 2-, 3- or 4-gons.
  - It has at most 8 vertices.

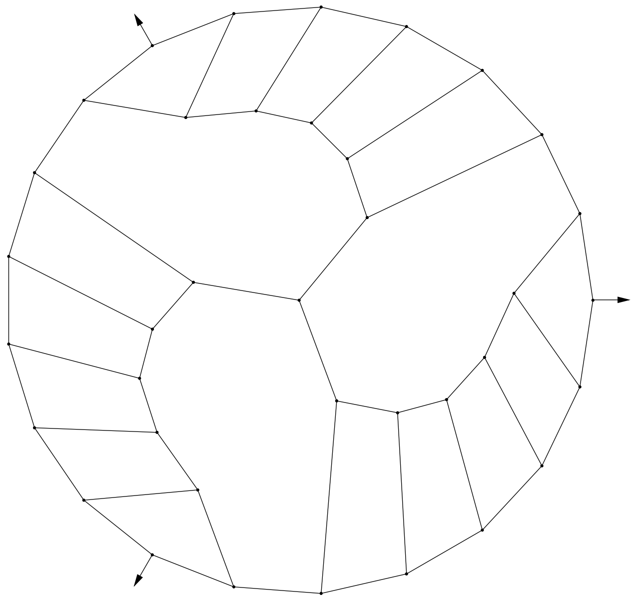
-  yields 
-  yields 
-  yields  (one infinite series)



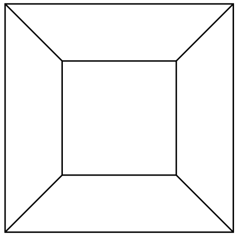
yields

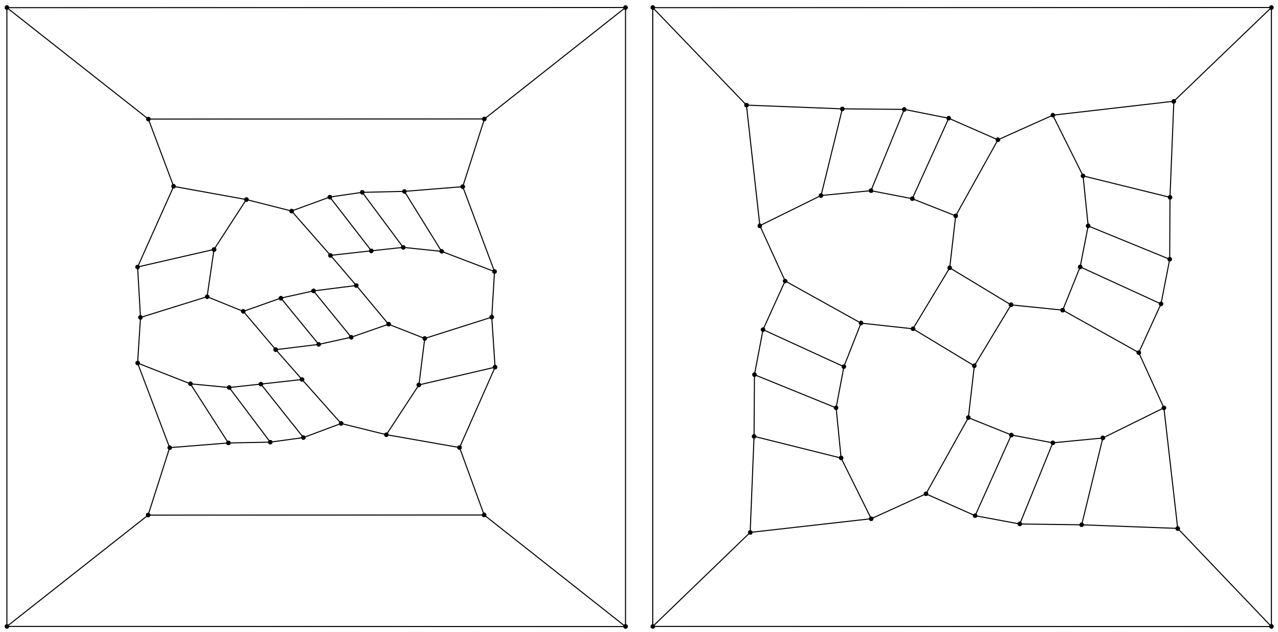


and



(two infinite series)

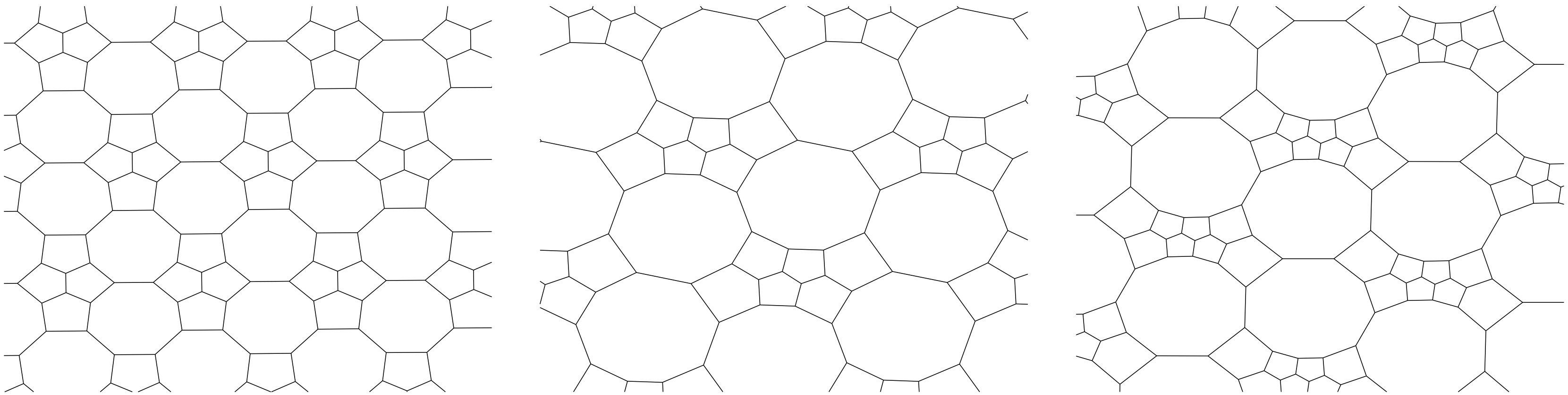
-  yields



(a family $K_{b,q}$ with $1 \leq b \leq q-5$)

$(5, q)$-maps qR_3

- A $(5, 7)$-torus is $7R_3$ if and only if it is $5R_1$.
- A $(5, 7)$-sphere, which is $7R_3$, has $x_0 + x_3 = 20$ with x_i being the number of vertices contained in i 5-gonal faces.
- For all $q \geq 7$, $(5, q)$-tori, qR_3 are known:



- Conj. For any $q \geq 7$ there is an infinity of $(5, q)$-spheres.

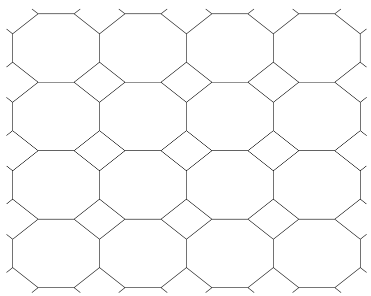
III. qR_4-maps

Classification of $(4,8)$-maps $8R_4$

- For $(4,8)$-maps, which are $8R_4$, one has

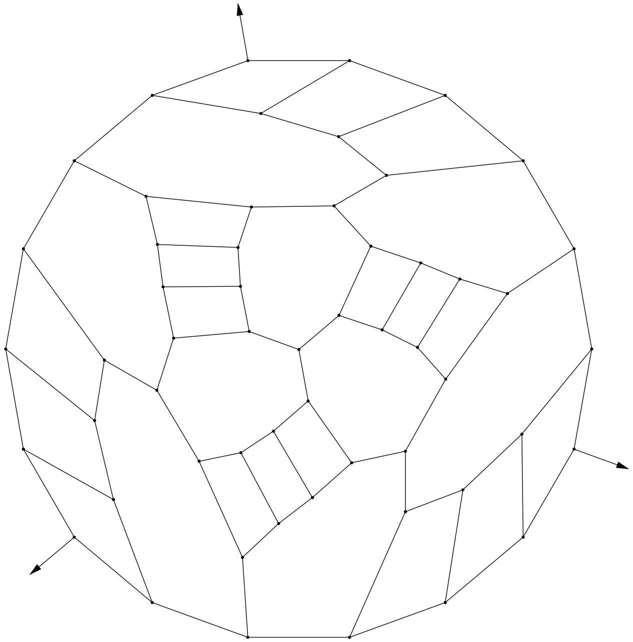
$$\begin{cases} x_0 + x_3 & = & 8(1-g) \\ e_{4-4} & = & 12(1-g) \end{cases}$$

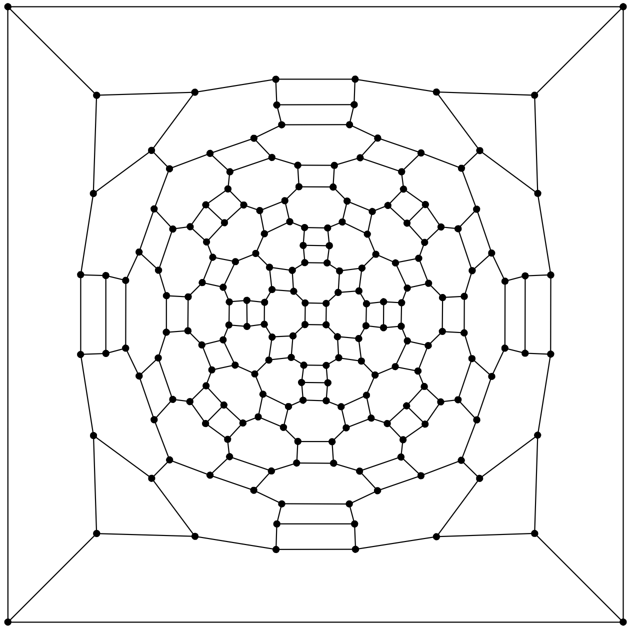
  with g being the genus (0 for sphere and 1 for torus) and x_i the number of vertices contained in i 4-gonal faces.
- There exists a unique $(4,8)$-torus $8R_4$:



- We use for the complicated case of $(4,8)$-sphere $8R_4$ an exhaustive computer enumeration method.

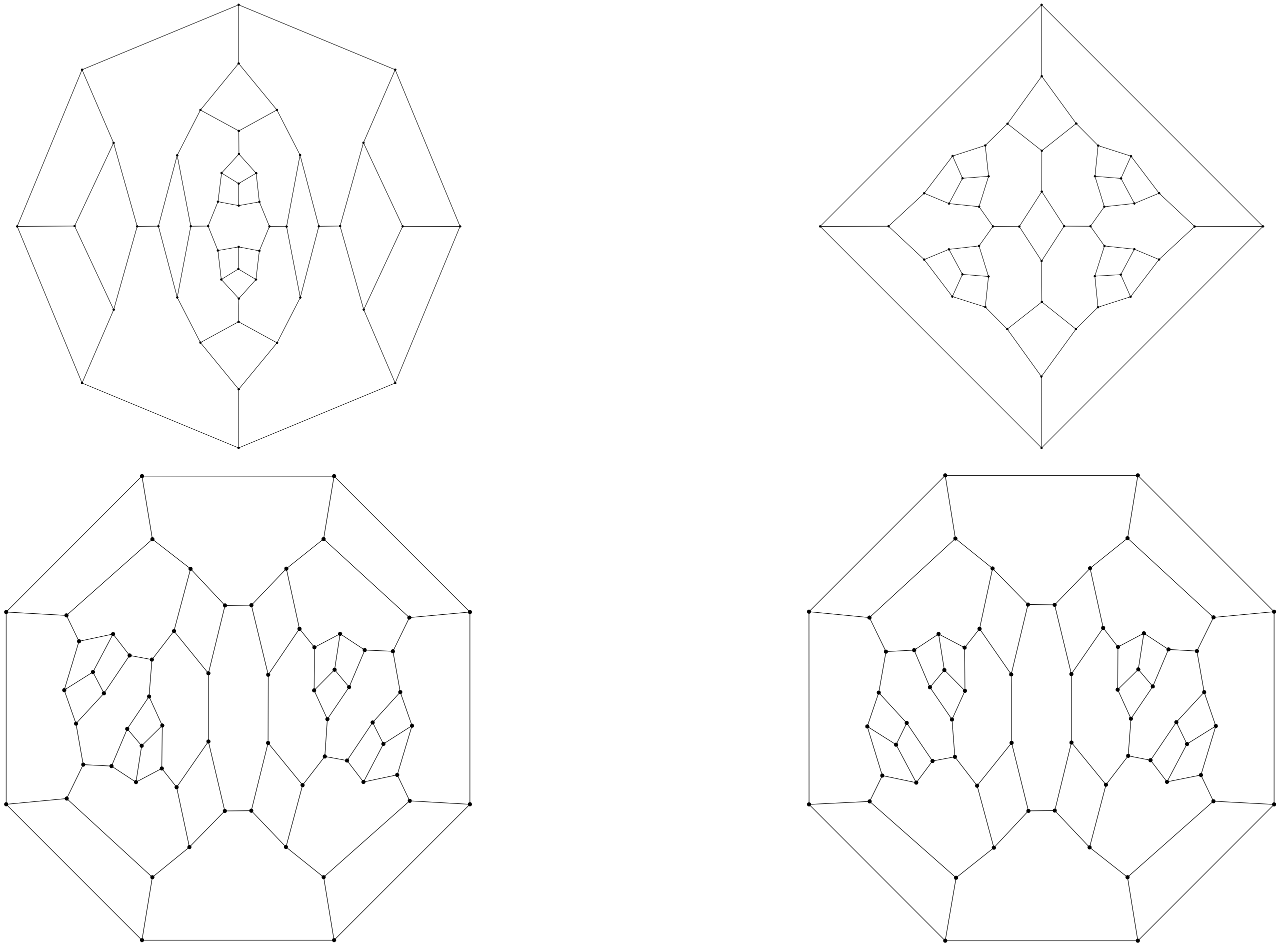
Classification of $(4, 8)$-maps $8R_4$





Two examples amongst 78 sporadic spheres.

Classification of $(4,8)$-maps $8R_4$

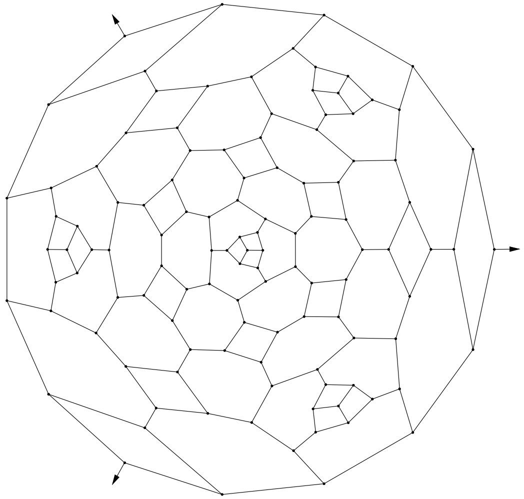


One infinite series amongst 12 infinite series.

III. qR_5-maps

$(4, q)$-case

- $(4, q)$-tori, which are qR_5, are known for any $q \geq 7$.
- For $q = 7$, they are $4R_0$.
- $(4, 7)$-spheres $7R_5$ satisfy to $e_{4-4} = 12$. Is there an infinity of such spheres?



$(5, q)$-case

- The smallest $(5, q)$-spheres qR_5 for $q = 7$, 8, 9 are: