Oceanic models and data assimilation

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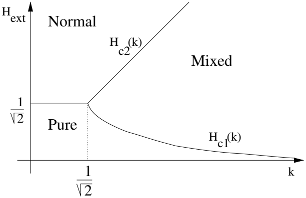
I. Previous Scientific Activities

Thesis

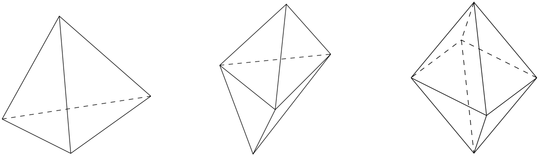
- A metal is supraconductor if when put at a very low temperature the electrical resistance vanish and the magnetic field is expelled.
- There are two types of supraconductors:
  - Type I: Pure and Normal state
  - Type II: Pure, Normal and Mixed states.
- The Ginzburg Landau model uses a quantum phase function ϕ, a vector potential $\overrightarrow{A}$ and a parameter κ.



➠ M. Dutour, *Phase diagram for Abrikosov lattice*, Journal of Mathematical Physics **42-10** (2001) 4915–4926.

Simplicial complexes with short links

- A simplicial complex is a family of simplices (generalized triangles).
- It has short links if every $n-2$ dimensional face is contained in 3 or 4 faces.
- They are classified in term of partitions of $\{1, \ldots, n+1\}$:

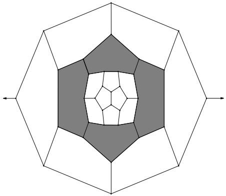


$\{1,2,3\}$ $\{1,2\},\{3\}$ $\{1\},\{2\},\{3\}$

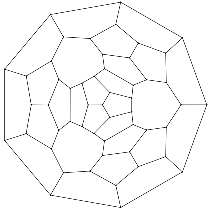
➠ M. Deza, M. Dutour and M. Shtogrin, *On simplicial and cubical complexes with short links*, Israel Journal of Mathematics **144** (2004) 109–124.

Face-regular maps

- A plane graph is one, whose edges do not self-intersect.
- A $(\{a, b\}, 3)$-plane graph is one, whose vertices have degree 3 and whose faces have size a or b.
- A $(\{a, b\}, 3)$-plane graph is called bR_j if every face of size b is adjacent to j faces of size b:



$(\{5, 8\}, 3)$-plane graph $8R_2$

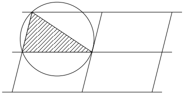


$(\{5, 9\}, 3)$-plane graph $9R_0$

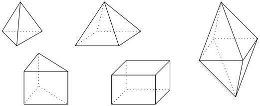
➡ M. Deza and M. Dutour Sikirić, *Polycycles and two-faced maps*, book in preparation for Cambridge University Press.

Lattice Delaunay polytopes

- If $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ is a n-dimensional lattice, then a Delaunay polytope is the convex hull of vertices on an empty sphere:



A 2-dimensional Delaunay polytope



Their classification in dimension 3

- Classification results

dim.	# types	Authors
2	2	Dirichlet (1860)
3	5	Fedorov (1885)
4	19	Erdahl, Ryshkov (1987)
5	138	Kononenko (1997)
6	6241	Dutour (2002)

➠ M. Dutour, *The six-dimensional Delaunay polytopes*, European Journal of Combinatorics **24-4** (2004) 535–548.

Extreme Delaunay polytopes

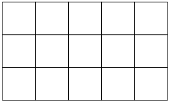
- A Delaunay polytope is called extreme if the only affine transformations preserving its property of being Delaunay are the isometries.

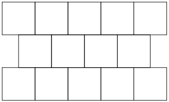
dim.	# polytopes	names
1	1	interval $[0,1]$
2,3,4,5	0	
6	1	Schlafli polytope
7	≥ 2	Gosset polytope, Rybnikov polytope
8	≥ 27	
9	≥ 1000	

- M. Deza and M. Dutour, *The hypermetric cone on seven vertices*, Experimental Mathematics **12-4** (2004) 433–440.
- M. Dutour, *Adjacency method for extreme Delaunay polytopes*, Proceedings of "Third Voronoï Conference of the Number Theory and Spatial Tesselations", 94–101.
- M. Dutour, R. Erdahl and K. Rybnikov, *Perfect Delaunay Polytopes in Low Dimension*, submitted.

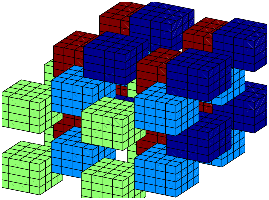
Cube packings

- We consider 2-periodic packings by cubes $z + [0, 1]^d$ into $\mathbb{R}^d$.
- All 2-dimensional cube packings are extendible to cube tilings:





- In dimension 3, there exist a non-extendible cube packing



- We find new non-extendible cube packings in dimension 4, 5, 6.

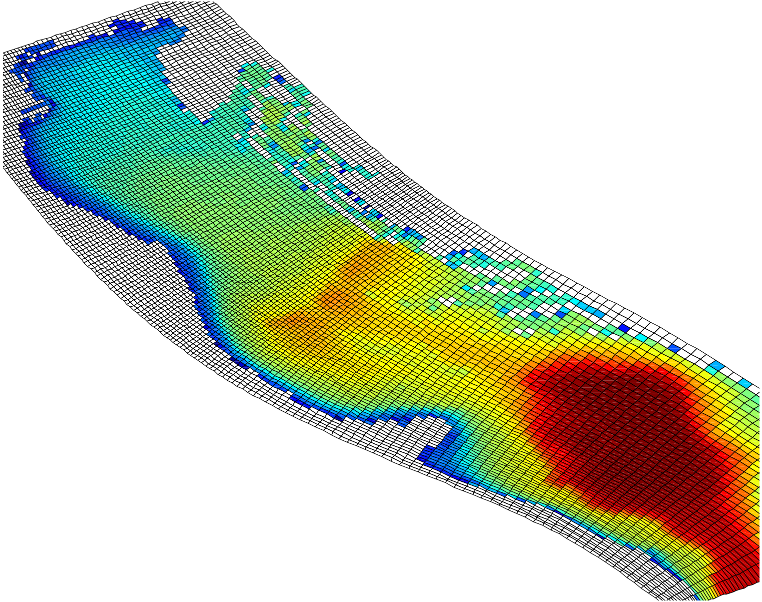
➠ M. Dutour, Y. Itoh and A. Poyarkov, *Cube packings, second moment and holes*, European Journal of Combinatorics **28-3** (2007) 715–725.

II. The Adriatic Sea

Describing the state

- We want to determine currents, temperature, salinity of the Adriatic sea.
- The Adriatic sea has many specific features:
  - The bathymetry varies a lot from 1200m to 50m.
  - The island structure on the Croatian side is quite complex.
  - The tides are the highest in the Mediteranean sea.
  - Violent events of Bura, cools it and create a complex eddy structure.
  - Po river has a large volume and influence.
  - Dense water is formed in its northern part.
- The knowledge of the physical processes allow for further analysis: Oxygen levels, biological processes, etc.

A grid and the bathymetry



Available measurements

- Satelite sea surface temperature are available every few hours provided that no cloud is present.
- *In situ* measurements are available (sparsely in time and space).
- Sea level gauges are available.
- ADCP (Acoustic Doppler Current Profiler) measures of currents.
- Output of meteorological models are available to get forcing data.

III. Possible Modelizations

Computational limits

- Fluid mechanics processes is usually discretized in boxes or triangles.
- Suppose one divides lengthes by 2. What happens to the time run?
  - We get a factor 4 from horizontal discretization.
  - We get a factor 2 for vertical discretization.
  - We get a factor 2 for time discretization.
- Computer speed multiplies by 2 every year at best.
- So, at best 4 years to divide cell sizes by 2 with identical programs and physics.
- Actually smaller sizes imply different physical processes.
- At present we deal with a grid length of 2km.

The models used

- ROMS from Rutgers university (mainly authored by H. Arango):
  - finite difference model,
  - high order advection schemes,
  - biology, ability to couple with other models.
- SWAN from Delft university:
  - finite difference model,
  - it is a spectral model for forecasting waves.
- TRUXTON is a finite element model for tide analysis.

Comparison between finite element, finite difference models

Advantages of finite element models:

- We can adapt the grid with respect to the problem, i.e., we can put larger grid-spacing in places of higher depth.
- We don't need to put land points contrary to finite difference models.

Advantages of finite difference models

- Programming is simpler, for example for parallel processing the block decomposition is straighforward.
- We have higher order advection schemes.

Model stability 1

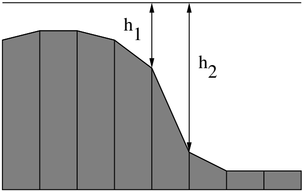
- If wave propagate at speed c then the space discretization Δx and the time discretization Δt should satisfy the CFL criterion:

$$\frac{\Delta x}{\Delta t} > c$$

  This is an approximate condition, true for a specific setting (uniform depth, linear equations, etc.).
- The characteristic wave speed is the wave speed and it is $\sqrt{gh}$ with h the bathymetry. With a maximum bathymetry of $1200m$ this puts strong requirement.
- If the T/S balance of the initial state is wrong, then the stability is compromised.
- The vertical levels have to be spaced reasonably equally. A common error is to concentrate them on the surface.
- Storm surge are also a possible source of instabilities.

Model stability 2

- Wrong open boundary condition:
  - Radiation boundary condition creates blowups.
  - Flather boundary condition seems not to.
- Compiler problem: Intel Fortran Compiler with "-O3" was blowing up after some time, while "-O2" was not.
- The steepness factor is $r = \max \frac{|h_1 - h_2|}{h_1 + h_2}$, where the maximum is taken over adjacent cells. In order to be stable, we need to have $r < 0.4$. Two solutions:
  - increase the horizontal resolution,
  - modify the bathymetry.



Linear programming for the bathymetry

- Linear programming: $(f_i)_{1\leq i\leq m}$ is a set of linear functions on $\mathbb{R}^n$, $(b_i)_{1\leq i\leq m}$ a set of m reals, g is linear on $\mathbb{R}^n$, then the linear programming problem is:

$$\begin{array}{rl} \textbf{maximize} & g(x) \\ \textbf{subject to} & f_i(x) \leq b_i. \end{array}$$

  There exist programs for solving those for very large n and m (10000 variables is not a problem).
- We write the bathymetry as $h = h^{real} + \delta$. We want to have

$$\frac{|h_1 - h_2|}{h_1 + h_2} \leq r \text{ and minimize } \sum_i |\delta_i|.$$

  For a fixed r we have a linear program.
- This method does perturbation to the bathymetry only when it is needed.
- The sum of total perturbation is typically 4 times less than what would come from an averaging operation.

Optimal analysis of initial state

- We have various climatological observations on the state of the Adriatic and we want to find the best initial state.
- In order to do this, we used the OAFE computer program.
- The solution to the OAFE problem is mathematically:

  $$x = C_{uu}E^*\{EC_{uu}E^* + C_{nn}\}^{-1}d$$

  with C_{nn} the matrix of observation noise (diagonal), C_{uu} the matrix of state noise and E the observation operator.
- If we have D measurements, then we have a $D \times D$ matrix to invert in order to get the best state x from the observations d.
- Even storing this matrix in memory is problematic if D is large.
- Actually, what we really need is to solve

  $$\{EC_{uu}E^* + C_{nn}\}y = d.$$

The conjugate gradient algorithm

- The method solves the equation $Ay = d$ for A a symmetric positive definite matrix.
- It iteratively finds better and better solutions.
- The solution at step n belong to the Krylov space:

$$Vect(d, Ad, \dots, A^{n-1}d).$$

- We no longer need to store the matrix A, what we need is to be able to compute Ad from d.
- In practice we have a 100-fold improvement in memory and similarly in speed.

170 meas.	13 iter.	1000 meas.	32 iter.
2100 meas.	50 iter.	7200 meas.	130 iter.

To get solutions y with $||d - Ay|| \leq 1.10^{-5}||d||$.

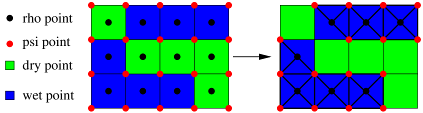
Model coupling

- It is impossible to have single model doing everything.
- A paradigm is to split the computation between different models, which exchange their data after a number of specified time steps with the Model Coupling Toolkit. For example:
  - SWAN requires to know the sea level and the currents to make its computations,
  - ROMS requires wave spectral information for the bottom boundary layer friction.
- Based on Arango and Warner version, work has been done on having those two models coupled.

IV. Tide assimilation

Adriatic tides

- The Adriatic has the highest tides of the Mediteranean sea (50cm). They are induced by the open boundary at the Ottranto strait.
- We want to find those boundary conditions from the sea level data for ROMS.
- We use an iteration scheme using ROMS (finite difference) forward iteration and TRUXTON (finite element) backward iteration. The transformation:



➠ I. Janeković and M. Dutour Sikirić, *Improving tidal open boundary conditions for the Adriatic Sea numerical model*, European Geosciences Union conference

Details

- Measurement is obtained from 20 ADCP moorings.
- ROMS is run for periods of 200 days.
- Another way to get the tidal information is to run a finite element model like QUODDY as a forward model and then to do interpolation.
- Numerical results:

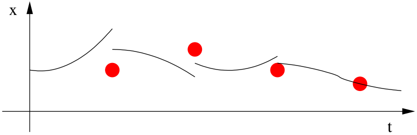
Tide	interpolation	Direct	Gain
O1	0.96	0.37	157%
K1	2.65	0.90	194%
M2	1.00	0.87	15%
S2	1.15	0.47	146%

  RMS amplitude in cm with respect to the 20 stations.
- The open boundary causes instability and we needed to extend the grid by adding a spunge layer.

V. Assimilation Methods

The idea of filtering and assimilation

- In the limit of the models, we are limited by:
  - our limited knowledge of the initial state,
  - the limited precision on the forcing data.
- So even if we could solve exactly the Initial Value Problem posed, it would not correspond to the reality.
- On the other hand measurement on the state of the model are continuously available.
- Sequential filtering is to use them to improve the state of the model:



Probabilistic model

- We define $\mathcal{S}$ to be the set of possible states of the model.
- At at time t_0 we know the state of the model with some imprecision, i.e., we have a probability density:

$$p_{t_0}(s) \text{ on } \mathcal{S}.$$

- Every measurement m has some imprecision, and give a probability density:

$$p_m(s) \text{ on } \mathcal{S}.$$

- According to the Bayesian theorem, the right probability density after the measurement is proportional to:

$$\alpha p_{t_0}(s) p_m(s) \text{ on } \mathcal{S} \text{ with } \alpha \text{ constant.}$$

- We cannot store probability distribution, at best we can store clouds of points.

Gaussian probabilities

- If we assume Gaussian probabilities, then we simply need to store the average $X(t_0)$ of the model and the Covariance matrix $P^a(t_0)$. Forecast state at step t_1:

$$P^f(t_1) = A_k P^a(t_0) A_k^T + Q \quad \text{and} \quad X^f(t_1) = A_k X^a(t_0)$$

  with A_k the matrix of the (linear) model and Q its covariance.
- If we assume Gaussian hypothesis for the measurement, then we have an average and a covariance matrix for it.
- The analysis step of Kalman filtering is then:

$$P^a(t_1) = (1 - KH)P^f(t_1) \text{ and } X^a(t_1) = X^f(t_1) + K(Y(t_1) - HX^f(t_1))$$

  with K the Kalman factor, H the measurement operator and $Y(t_1)$ the measurement at t_1.
- If we have $n = 1.10^6$ variables for the state, then we have n^2 variables for the covariance matrix. So, we cannot store this matrix.

Subspace methods

- Since we cannot store the full covariance matrix, we consider a subspace $S_m \subset \mathcal{S}$ of dimension m.
- We take a basis $X_1, \ldots, X_m$ of S_m.
- The memory requirements are now:
  - storing the m states,
  - storing the $m \times m$ matrix of the covariance matrix.
- The difficulties are:
  - choosing the right space S_m (one possible way is by Empirical Orthogonal Functions),
  - ensuring the stability of the model,
  - being able to improve the solution.

Projects

- SUNTANS it is a finite element model with a non-hydrostatic pressure gradient. It might be needed for a description of the dense water formation in the Adriatic.
- WRF is a finite difference atmospheric model and it would be good to be able to run it concurently with ROMS so as to get good forecasts.

Collaborations

Group for satellite oceanography, Institut Rudjer Bošković:

- Milivoj Kuzmić
- Ivica Janeković
- Igor Tomažić

Other collaborations:

- Michel Deza, École Normale Supérieure, France
- Patrick Fowler, Sheffield University, UK
- Viatcheslav Grishukhin, CEMI Ran, Russia
- Yoshiaki Itoh, Institute of Statistical Mathematics, Japan
- Konstantin Rybnikov, Lowell State University, USA
- Mikhail Shtogrin, Steklov Institute, Russia
- Frank Vallentin, CWI Amsterdam, Netherland
- Sergey Shpectorov, Bowling Green State University, USA

THANK

YOU