Torus Cube Tilings

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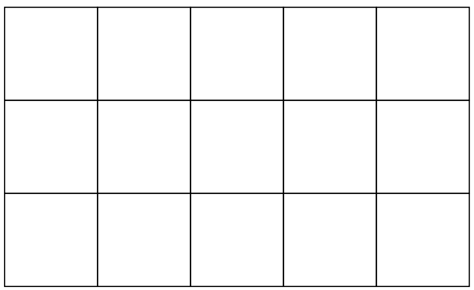
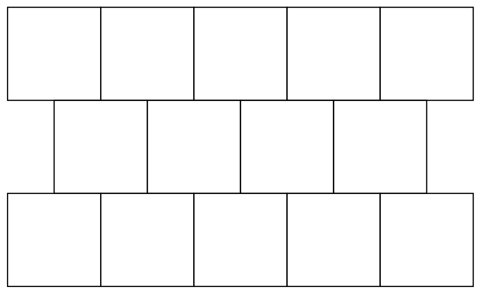
I. Torus tilings and packings

Torus Cube Tilings

- A special cube tiling is a $4\mathbb{Z}^d$-periodic tiling of the space $\mathbb{R}^d$ by integral translates of the hypercube $[0,2]^d$.
- A general cube tiling is a tiling of $\mathbb{R}^d$ by translate of the cube $[0,2]^d$.
- Special cube tilings can be lifted to the torus.
- There is only one special cube tiling in dimension 1.
- There are two special cube tilings in dimension 2:





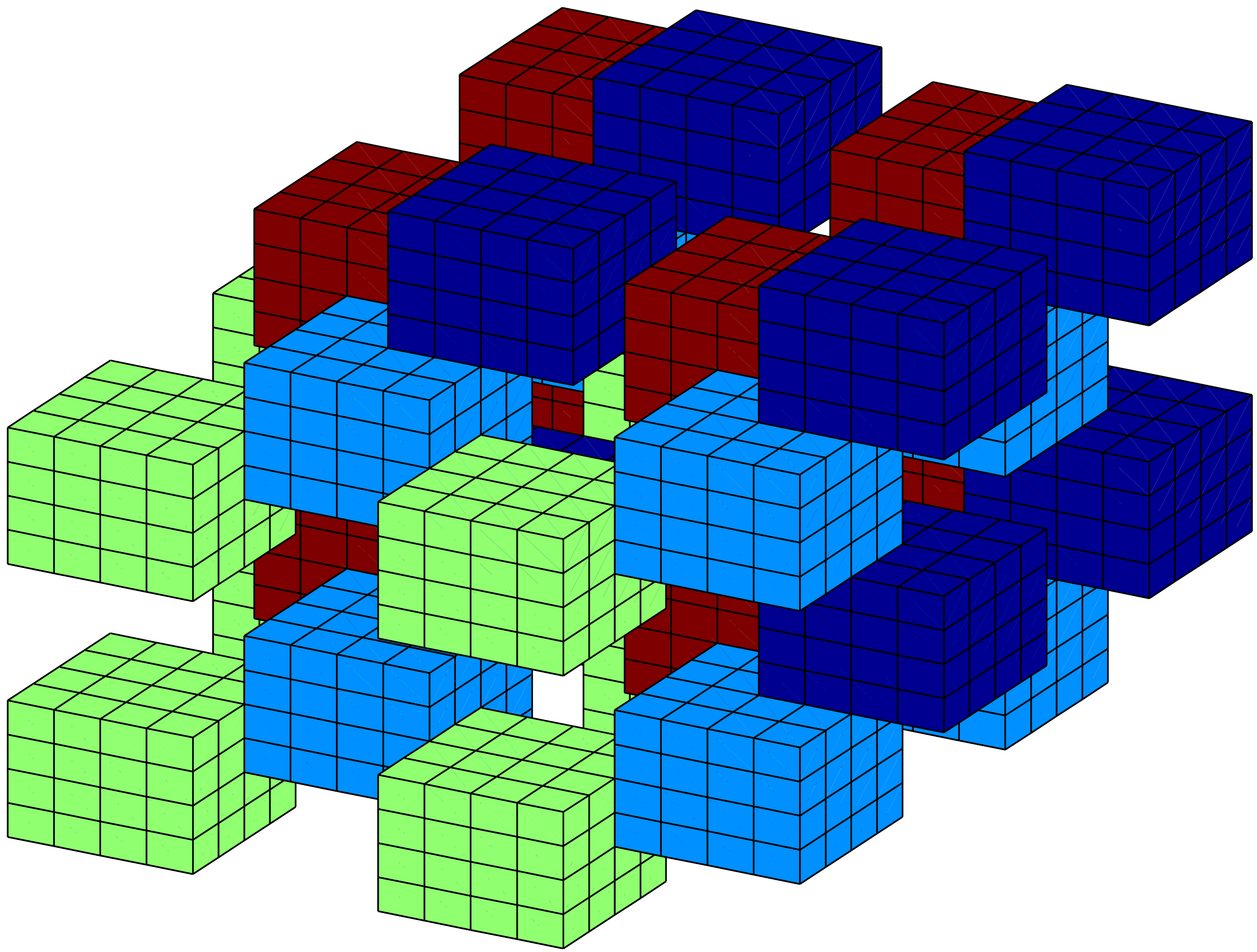
Keller conjecture

- Conjecture: for any general cube tiling, there exist at least one face-to-face adjacency.
- This conjecture was proved by Perron (1940) for dimension $n \leq 6$.
- Szabo (1986): if there is a counter-example to the conjecture, then there is a counter-example, which is special.
- Lagarias & Shor (1992) have constructed counter-example to the Keller conjecture in dimension $d \geq 10$
- Mackey (2002) has constructed a counter-example in dimension $d \geq 8$.

Cube packings

- A cube packing is a packing of $\mathbb{R}^d$ by integral translates of cubes $[0,2]^d$, which is $4\mathbb{Z}^d$-periodic.
- If we cannot extend a cube packing by adding another cube, then it is called non-extendible.
- Non-extendible cube packing does not exist in dimension 1 and 2.
- There is an unique non-extendible cube packing in dimension 3.
- We are interested in the values of N, for which there is a non-extendible cube packing with N cubes.

non-extendible cube packing



II. Algorithms of generation

Clique formalism

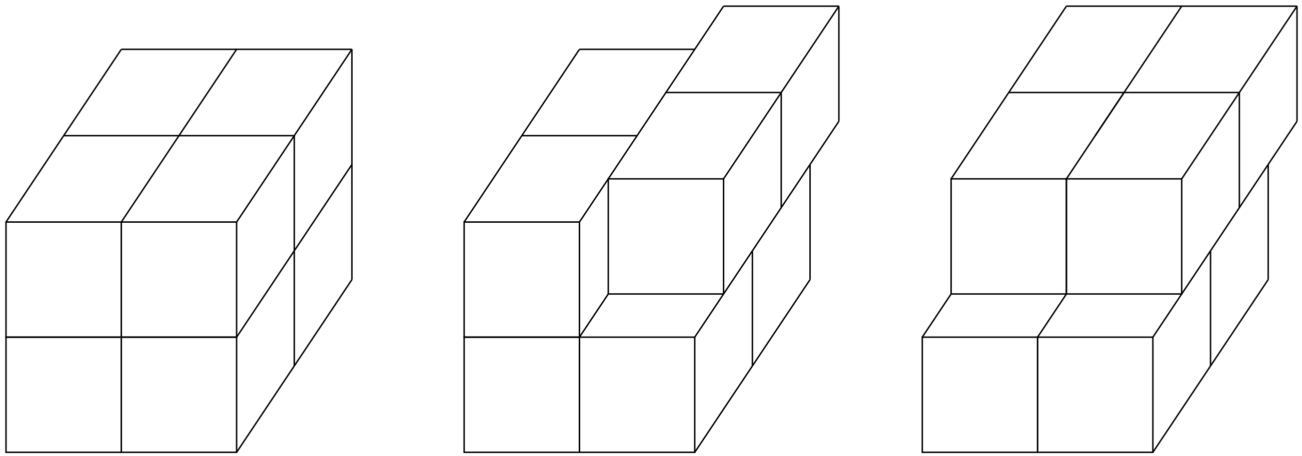
- Associate to every cube C its center $c \in \{0,1,2,3\}^d$
- Two cubes with centers x and x' are non-overlapping if and only if there exist a coordinate i, such that $|x_i - x'_i| = 2$.
- The graph G_d is the graph with vertex set $\{0,1,2,3\}^d$ and two vertices being adjacent if and only if the corresponding cubes do not overlap.
- A clique S in a graph is a set of vertices such that any two vertices in S are adjacent.
- Cube tilings correspond to cliques of size 2^d in the graph G_d.
- All problems about those cube tilings are finite, since G_d is finite, but the number of possibilities is huge.

GAP enumeration

- The graph G_d has a symmetry group of size $d!8^d$, which acts on the 4^d elements $\{0,1,2,3\}^d$
- Cliques are associated to subsets of those 4^d elements. GAP has extremely efficient techniques (backtrack search) for checking if two subsets are equivalent under a permutation group.
- We set $L_1 = \{\{v\}\}$ and iterate i from 2 to 2^d:
  - For every subset in L_{i-1}, consider all vertices, which are adjacent to all element in L_{i-1}.
  - Test if they are isomorphic to existing elements in L_i and if not, insert them into L_i.

Results for $d \leq 4$

- In dimension 3, there is a single non-extendible cube tiling and there are 9 types of cube tilings.



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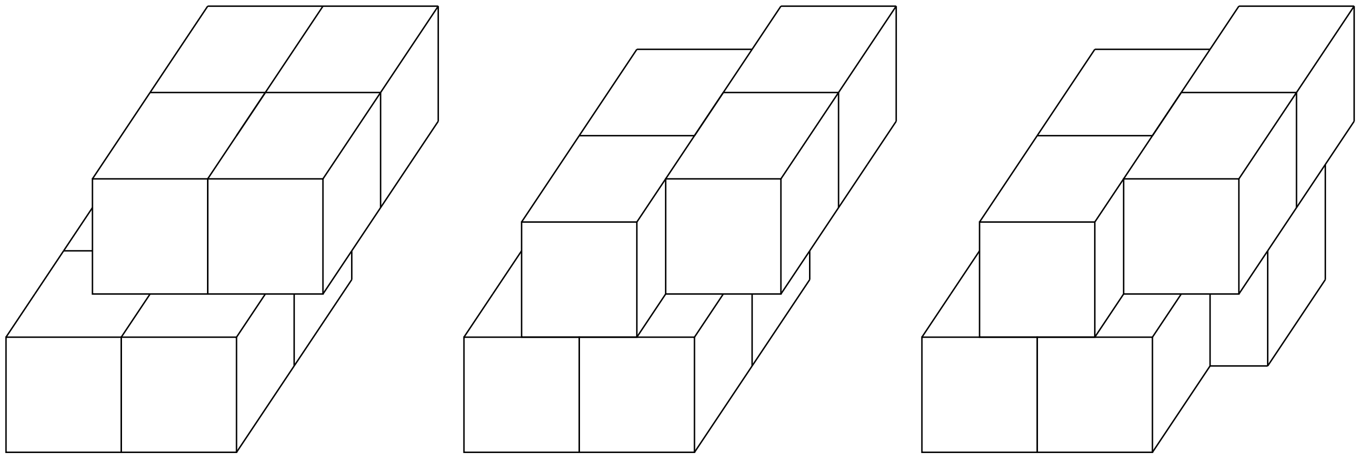

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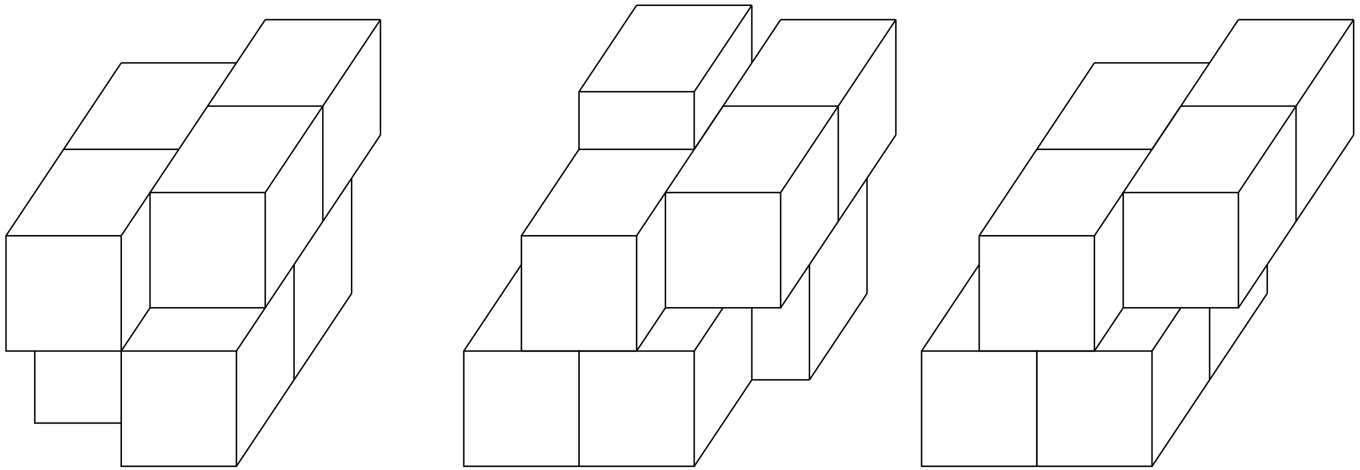
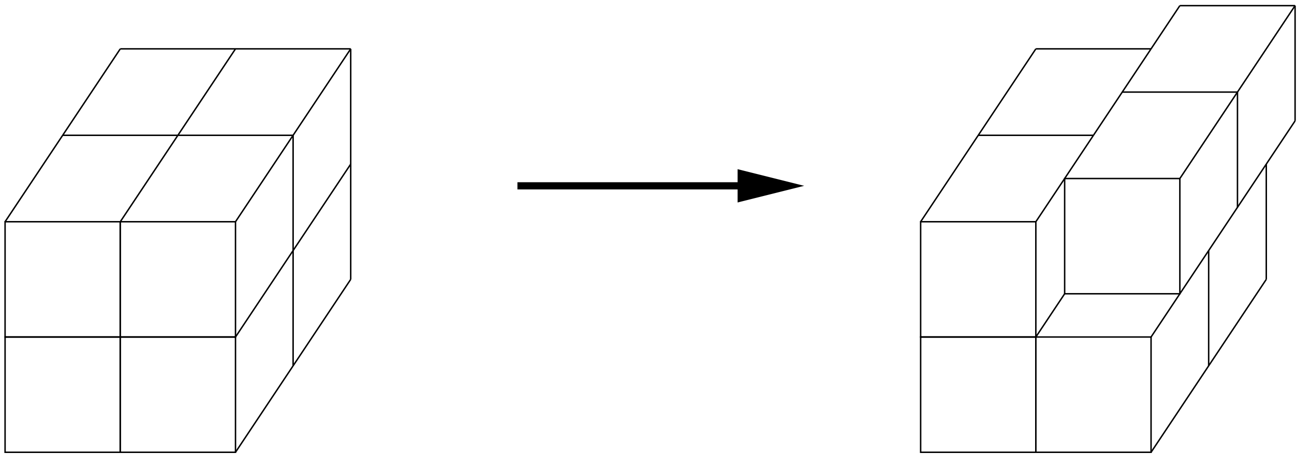


- In dimension 4, the repartition is as follows:

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
nb	0	0	0	0	0	0	0	38	6	24	0	71	0	0	0	744

Flipping algorithm

- 3-dimensional cube tilings suggest a flipping algorithm:



- This algorithm uses face-to-face adjacencies to generate new tilings.
- We know that this flipping algorithm will not work in dimension ≥ 8, since there are cube tilings with no face-to-face adjacencies.
- This algorithm works in dimension 3 and 4.

Random generation

- Another possibility is to take cubes at random and add them if and only if they do not overlap, until there is no space left any-more.
- Practical implementation:
  - Generate elements at random and add them if they do not overlap.
  - After the number of failures in the random generation, reach a certain threshold, enumerate all possible cubes, which do not overlap.
  - Then, choose element at random in this list and update the list by removing cubes, which will overlap.
  - Finish, when there is no choice left.

Greedy and Metropolis algorithm

- If one wants to generate non-extendible cube packings with low density, then some other strategies are possible:
- Greedy algorithm In the choice of a cube at random, select the one, which covers the most important part of the space.
- Metropolis algorithm Take an existing non-extendible cube packing with low density and then:
  - Remove some elements chosen at random in this cube packing.
  - Do a random packing procedure on the left holes.
  - If the density of the obtained non-extendible cube packing is too high, then discard it; otherwise, take it as basis for future computations.

III. non-extendible cube packings

Extension of cube packings

- Theorem A cube packing with N cubes, $N \geq 2^d - 3$ tiles is extendible to a cube tiling.
- Given
  - a cube packing with $2^d - \delta$ cubes of coordinates x^k, $1 \leq k \leq 2^d - \delta$,
  - a coordinate j and
  - a value $\alpha \in \{0, 1, 2, 3\}$.

  The induced cube packing is the cube packing of $\mathbb{R}^{d-1}$ obtained by taking all vectors x^k with $x_j^k = \alpha, \alpha + 1 \pmod 4$ and removing the j-th coordinate.
  Such cube packings have at least $2^{d-1} - \delta$ tiles.
- The proof is by induction, using induced cube packings.

Extension of theorem

- The complement of an non-extendible cube packing $\mathcal{CP}$ is the set $\mathbb{R}^d - \mathcal{CP}$.
- Conjecture: If $\mathcal{CP}$ is an non-extendible cube packing with $2^d - 4$ tiles, then its complement is of the same shape, as the one in dimension 3.
- Conjecture: If $\mathcal{CP}$ is a cube packing with $2^d - 5$ cubes, then it is extendible by at least one cube.
- A complement of a cube tiling is called irreducible if it is not the union of cube tilings on different layers.
- Conjecture: For a given δ, there is a finite number of irreducible complements of volume $2^d\delta$.

Low density cube packings

- Denote by $f(d)$ the smallest number of cubes of non-extendible cube packing.
- $f(3) = 4$ and $f(4) = 8$.
- The following inequality holds:

$$f(n+m) \leq f(n)f(m) \ .$$

  The cube packing realizing this is constructed by "product" of two cube packings of $\mathbb{R}^n$ and $\mathbb{R}^m$
- So, $f(6) \leq f(3) \times f(3) = 16$.
- But no random algorithm manage to find such a packing!

Covering sets

- A covering set $\mathcal{CS}$ is a set of cubes (possibly, overlapping), such that we cannot add a cube without overlapping with at least one cube in $\mathcal{CS}$.
- Let us denote by $h(d)$ the smallest number of cubes in covering sets.
- $h(d) \leq f(d)$.
- $N < h(d)$ if and only for every set of N cubes (possibly overlapping), there exist a cube, which do not overlap with them.
- Theorem. One has the relation

$$h(d+1) \geq \left\lfloor \frac{4h(d)-1}{3} \right\rfloor + 1.$$

Proof of theorem

- Lemma. If N satisfies the inequality $\lfloor \frac{3N}{4} \rfloor < h(d)$, then one has $h(d+1) > N$.
- Take N vectors in $\{0,1,2,3\}^{d+1}$ and consider its last coordinate. At least $\lceil \frac{N}{4} \rceil$ vectors satisfy to $x_{d+1} = t$ for some $t \in \{0,1,2,3\}$.
- In illustrated proof below, one has $d = 3$ and $N = 5$.

Proof of theorem

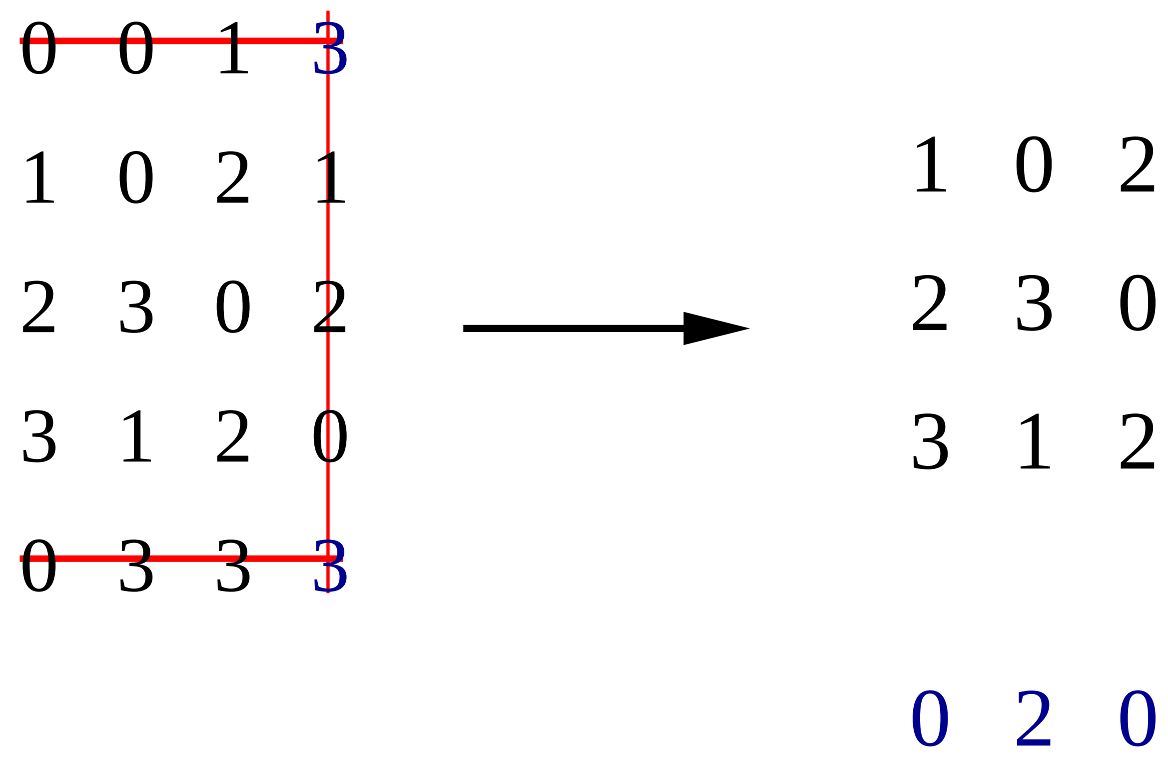
0	0	1	3
1	0	2	1
2	3	0	2
3	1	2	0
0	3	3	3

→

~~0~~	~~0~~	~~1~~	~~3~~
1	0	2	1
2	3	0	2
3	1	2	0
~~0~~	~~3~~	~~3~~	~~3~~

Eliminate those vectors and, for the remaining vector, their last coordinate

Proof of theorem



We have $\lfloor \frac{3N}{4} \rfloor$ ($< h(d)$) remaining vectors; so, one can find a not overlapped cube.

Proof of theorem

				0	0	1	3
1	0	2		1	0	2	1
2	3	0	→	2	3	0	2
3	1	2		3	1	2	0
				0	3	3	3
0	2	0		0	2	0	1

By the basic assumption, setting the last coordinate to $t + 2 \pmod 4$ gives a cube, which does not overlap with the preceding ones.

Proof of theorem

- Set $N = \left\lfloor \frac{4h(d)-1}{3} \right\rfloor$.
- Then it holds:

$$\left\lfloor \frac{3N}{4} \right\rfloor = \left\lfloor \frac{3\left\lfloor \frac{4h(d)-1}{3} \right\rfloor}{4} \right\rfloor \leq \left\lfloor \frac{4h(d)-1}{4} \right\rfloor < h(d).$$

- So, by the Lemma, $h(d+1) > N$, i.e.:

$$h(d+1) \geq \left\lfloor \frac{4h(d)-1}{3} \right\rfloor + 1.$$

Values of $f(d)$

- $h(4) = 7$, while $f(4) = 8$.
- Above theorem yields

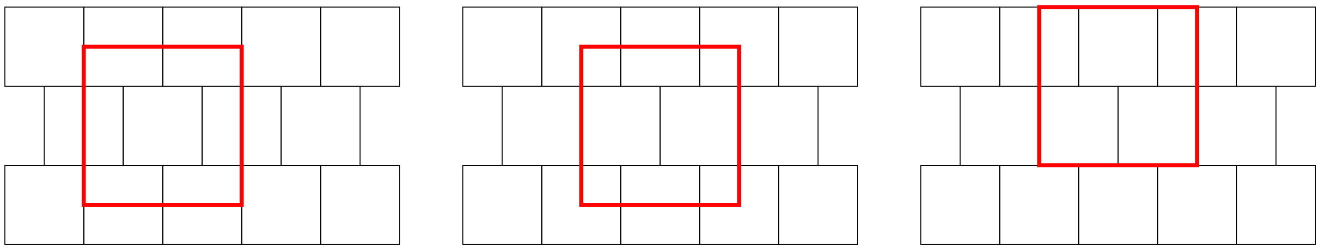
$$f(5) \geq 10 \text{ and } f(6) \geq 14 .$$

- We found (by random method) many non-extendible packings with 12 cubes, but not with less than 12 cubes.
- Could it be that $f(5) = 12$?
- It seems also likely, that $f(6) = 16$ and that the best cube packing in dimension 6 is unique.
- An interesting question if to estimate the asymptotic behavior of $h(d)$ or $f(d)$.

IV. Second moment

The counting function

- Take a cube packing $\mathcal{CP}$ and $z \in \mathbb{Z}^d$
  - C_z is the cube $z + [0,4]^d$ of corner z
  - N_z the number of cubes inside C_z.



$N_z(\mathcal{CP}) = 1$ $\quad$ $N_z(\mathcal{CP}) = 2$ $\quad$ $N_z(\mathcal{CP}) = 3$

- N_z is $2\mathbb{Z}^d$-periodic. Denote by $E(.)$ the averaging operator.
- First moment $\mathcal{M}_1 = E(N_z) = (\frac{3^d}{4^d})N$.

Maximal second moment

- We want to find minimal values of the second moment, i.e. $\mathcal{M}_2 = E(N_z^2)$.
- One defines the space

$$\mathcal{G} = \left\{ \begin{array}{c} f : \{0,1,2,3\}^d \to \mathbb{R}. \\ \forall x \in \{0,1,2,3\}^d \text{ one has } \sum_{x+\{0,1\}^d} f(x) = 1 \\ \text{and } f(x) \geq 0 \end{array} \right\}$$

Cube tilings correspond to $(0,1)$-vectors of $\mathcal{G}$.

- We will prove that $\mathcal{M}_2 = E(N_z(f)^2)$, with $f \in \mathcal{G}$, is maximal for f associated to a regular cube tiling.

Maximal second moment

- Given $f \in \mathcal{G}$, define:

$$M_i(f)(x) = \begin{cases} f(x) + f(x+e_i) & \text{if} \quad x_i = 0 \text{ or } 2 \\ 0 & \text{if} \quad x_i = 1 \text{ or } 3\,. \end{cases}$$

  $M_i(f)$ belongs to $\mathcal{G}$.
- One proves $E(\{N_z(M_i(f))^2) \geq E(N_z(f)^2)$ for every $f \in \mathcal{G}$.
- If $f \in \mathcal{G}$, then $M_d \ldots M_1(f)$ is the function of the regular cube tiling.
- So, the second moment $\mathcal{M}_2$ is maximal for regular cube tiling.

Lower bound

- Theorem. If $\mathcal{CP}$ is a cube packing with N cubes, then its second moment $\mathcal{M}_2$ satisfies the inequality:

$$\mathcal{M}_1 + N(N-1)2^d + 2^d d\{q(q-1) + rq\} \leq \mathcal{M}_2$$

with $N = 4q + r$, $0 \leq r \leq 3$ and

$$\mathcal{M}_1 = (\frac{3^d}{4^d})N \ .$$

- For $d = 3$ and $N = 4$, the cube packing minimizing the second moment is the non-extendible one.

Proof of lower bound

- Take a cube packing $A^1, \dots, A^N$.
- If C_j for $1 \le j \le 4^d$ is the collection of all 4×4-cubes, then every A^i is contained in 3^d cubes C_j.
- Denote by n_j, for $1 \le j \le 4^d$, the number of cubes A^i contained in C_j.

$$\mathcal{M}_1 = \frac{1}{4^d} \sum_j n_j = (\frac{3^d}{4^d}) N \text{ and } \mathcal{M}_2 = \frac{1}{4^d} \sum_j n_j^2.$$

- Denote by t_{ij} the number of cubes C_k, containing A^i and A^j. One has:

$$\sum_{i<j} t_{ij} = \sum_j \frac{n_j(n_j - 1)}{2} .$$

Proof of lower bound

- Denote μ_{ij} the number of equal coordinates of A^i and A^j.
- Then one has:

$$t_{ij} = (\frac{3}{2})^{\mu_{ij}} 2^d \geq 2^d + 2^{d-1} \mu_{ij} \ .$$

with equality for $\mu_{ij} = 0$ or 1.

- So, one gets:

$$\sum_{i<j} t_{ij} \geq N(N-1) 2^{d-1} + 2^{d-1} \sum_{i<j} \mu_{ij} \ .$$

Proof of lower bound

- Denote R_k the number of equal pairs in column k. One has:

$$\sum_{i<j} \mu_{ij} = \sum_{k=1}^{d} R_k \ .$$

- Fix k; if d_u is the number of vectors of value u in column k, then it holds:

$$R_k = \sum_{u=0}^{3} \frac{d_u(d_u - 1)}{2}, \quad d_u \geq 0 \ \text{ and } \ \sum_{u=0}^{3} d_u = N \ .$$

- Writing $N = 4q + r$ and minimizing over d_u, one gets:

$$R_k \geq 2q(q-1) + rq \ .$$

THANK YOU