Cones of metrics, quasimetrics and hemimetrics

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I. Metric and cut polytopes

Distances and metrics

Given a set X of points a distance on X is a function $d : X \times X \mapsto \mathbb{R}$ such that

- $d(x,x) = 0$ for all $x \in X$
- $d(x,y) = d(y,x)$ for all $x, y \in X$

The distance becomes a metric if in addition it satisfies the triangle inequality

$$d(x,y) \leq d(x,z) + d(z,y) \text{ for all } x, y, z \in X.$$

We have following sets:

- We call $\text{MET}(K_n)$ the set formed by all the metrics on n points $\{1, \dots, n\}$. It is a convex polyhedral cone.
- We call $\text{METP}(K_n)$ the set formed by all $d \in \text{MET}(K_n)$ satisfying in addition the perimeter inequalities

$$d(x,y) + d(x,z) + d(z,y) \leq 2 \text{ for all } x, y, z \in X.$$

Cuts and the cut cone/polytope

For a subset $S \subset \{1, \ldots, n\}$ we define a cut metric to be

$$\delta_S(x, y) = \begin{cases} 1 & \text{if } |S \cap \{x, y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We have $\delta_S = \delta_{\{1,\ldots,n\}-S}$ and $\delta_\emptyset = 0$.

- We define $\mathrm{CUT}(K_n)$ to be the cone spanned by the δ_S. It has $2^{n-1} - 1$ extreme rays.
- We define $\mathrm{CUTP}(K_n)$ to be the convex hull of the δ_S. It has 2^{n-1} vertices.

The cut cone corresponds to the metrics that are embeddable in $\mathbb{R}^n$ for the L^1 norm.

Symmetries

For a subset $S \subset \{1, \dots, n\}$ we define a switching operation on the distances on $\{1, \dots, n\}$. For d a distance and $1 \leq x, y \leq n$ we write

$$F_S(d)(x,y) = \begin{cases} 1 - d(x,y) & \text{if } |S \cap \{x,y\}| = 1 \\ d(x,y) & \text{otherwise.} \end{cases}$$

The following holds:

- We have $F_S(\delta_T) = \delta_{S \Delta T}$ so F_S acts on $\mathrm{CUTP}(K_n)$.
- The switchings also act on $\mathrm{METP}(K_n)$.
- The switchings and the symmetries on n points define a symmetry group of order $2^{n-1} n!$.
- For $n \neq 4$ this is all the symmetries there is for $\mathrm{METP}(K_n)$ and $\mathrm{CUTP}(K_n)$.

Vertices/Facets of METP(K_n) and CUTP(K_n)

P	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
CUTP(K_n), e	4(1)	8(1)	16(1)	32(1)	64(1)	128(1)
CUTP(K_n), f	4(1)	16(1)	56(2)	368(3)	116,764(11)	217,093,472(147)
CUT(K_n), e	3(1)	7(2)	15(2)	31(3)	63(3)	127(4)
CUT(K_n), f	3(1)	12(1)	40(2)	210(4)	38,780(36)	49,604,520(2,169)
MET(K_n), e	3(1)	7(2)	25(3)	296(7)	55,226(46)	119,269,588(3,918)
MET(K_n), f	3(1)	12(1)	30(1)	60(1)	105(1)	168(1)
METP(K_n), e	4(1)	8(1)	32(2)	554(3)	275,840(13)	1,550,825,600(533)
METP(K_n), f	4(1)	16(1)	40(1)	80(1)	140(1)	224(1)

Computation of facets of CUTP(K_8), vertices of METP(K_8) was done in

- M. Deza, M. Dutour Sikirić, *Enumeration of the facets of cut polytopes over some highly symmetric graphs*, International Transactions in Operational Research 23-5 (2016) 853–860
- A. Deza, K. Fukuda, T. Mizutani, C. Vo, *On the face lattice of the metric polytope*, Lecture Notes in Comput. Sci., 2866, 118–128

II. Algorithms for dual description problems

Program comparisons

We consider a polytope defined by a set $\mathcal{LF}$ of inequalities for which we want its vertex set $\mathcal{LV}$.

- lrs: it iterates over all admissible basis in the simplex algorithm of linear programming
  - It is a tree search, no memory limitation.
  - Ideal if the polytope has a lot of vertices.
- cdd/ppl: it adds inequalities one after the other and maintain the double description throughout the computation
  - All vertices and facets are stored memory limitation.
  - Good performance if the polytope has degenerate vertices.
- pd: We have a partial list of vertices, we compute the facets with lrs. If it does not coincide with $\mathcal{LF}$ then we can generate a missed vertex by linear programming.
  - It is a recommended method only if there is less vertices than facets.
- Other technique of beneath/beyond exist
- So, in general, choosing the right method is really difficult.

The adjacency decomposition method

Input: The vertex-set of a polytope P and a group G acting on P.
Output: $\mathcal{O}$, the orbits of facets of P.

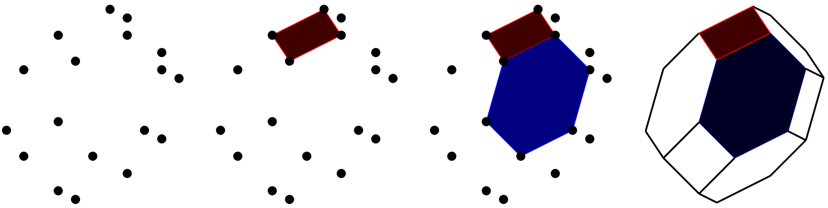
- Compute some initial facet F (by linear programming) and insert the corresponding orbit into $\mathcal{O}$ as undone.
- For every undone orbit O of facet:
  - Take a representative F of O.
  - Find the ridges contained in F, i.e. the facets of the facet F (this is a dual description computation).
  - For every ridge R, find the corresponding adjacent facet F' such that $R = F \cap F'$.
  - For every adjacent facet found test if the corresponding orbit is already present in $\mathcal{O}$. If no insert it as undone.
  - Mark the orbit O as done.
- Terminate when all orbits are done.

Reinvented many times (D. Jaquet 1993, T. Christof and G. Reinelt 1996).

General feature of the algorithm

It is a graph traversal algorithm:

- The algorithm starts by computing the orbits of lowest incidence, which are the one for which the dual description is easiest to be done.
- Sometimes it seems that no end is in sight, we get a lower bound on the number of orbits.
- At the end, only the orbits of highest incidence remains.
- The method can be applied recursively.

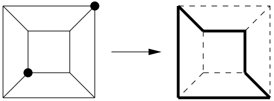


Permutation groups

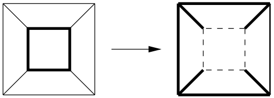
- Polytopes of interest have usually less than 1000 vertices $v_1, \ldots, v_N$, their symmetry group can be represented as a permutation of their vertex-set.
- The first benefit is that permutation group algorithms have been well studied for a long time and have good implementation in `GAP`.
  - A. Seress, *Permutation group algorithms*, Cambridge University Press, 2003.
  - D.F. Holt, B. Eick and E.A. O'Brien, *Handbook of computational group theory*, Chapman & Hall/CRC, 2005.
- The second benefit is that a facet of a polytope thus corresponds to a subset of $\{1, \ldots, N\}$ and that permutation group acting on sets have a very good implementation in `GAP`.
- In some extreme cases ($\#$ vertices > 100000) permutation groups might not work as quietly and other methods have to be used.

Balinski theorem and linear programming

- Balinski theorem The skeleton of a n-dimensional polytope is n-connected, i.e. the removal of any set of $n-1$ vertices leaves it connected.
- So, if the number of facets in remaining orbits is at most $n-1$, then we know that no more orbits is to be discovered.



- Theorem: For a polytope P, if one removes all the edges of the skeleton contained in a proper face F then the graph on the remaining edges is still connected.
- So if all remaining facets contain a common vertex then we do not need to continue further.



III. Metric and cut polytopes of graphs

Cut polytope of a graph

Let us take a graph G on n vertices. For a subset $S \subset \{1, \ldots, n\}$ and an edge $e = (x, y)$ of G we define a cut metric to be

$$\delta_S^G(e) = \begin{cases} 1 & \text{if } |S \cap e| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- We have $\delta_S = \delta_{\{1,\ldots,n\}-S}$ and $\delta_\emptyset = 0$.
- We define $\mathrm{CUTP}(G)$ to be the convex hull of the δ_S. It has 2^{n-1} vertices if G is connected.
- The dimension of $\mathrm{CUTP}(G)$ is equal to $|E|$, i.e. the number of edges of G.
- The polytope $\mathrm{CUTP}(G)$ can be interpreted as the projection of $\mathrm{CUTP}(K_n)$ on $\mathbb{R}^E$.

Metric polytope of a graph

Let us take a graph G on n vertices and we want to define the metric polytope.

- One possibility is to define $\mathrm{METP}(G)$ as the projection of $\mathrm{METP}(K_n)$ on $\mathbb{R}^E$ but this is a little difficult to work with.
- It turns out that we can express in a nice way the facets of $\mathrm{METP}(G)$:
- For an edge e not contained in any triangle we add the inequalities $0 \leq d(e) \leq 1$.
- For any chordless cycle C and odd sized set $F \subset C$ the inequality is
$$d(F) - d(C - F) \leq |F| - 1$$
where $d(U) = \sum_{u \in U} d(u)$.

Seymour theorem

Theorem: We have $\text{MET}(G) = \text{CUT}(G)$ if and only if G has no K_5 minor.

- The result was extended to polytope case by Barahona.
- It is an especially beautiful theorem that allows to compute the facets of many cut polytopes.
- It remains an isolated result:
  - Seymour, P. D., *Matroids and multicommodity flows*, European Journal of Combinatorics 2 (1981) 257–290.
- The smallest case where $\text{MET}(G) \neq \text{CUT}(G)$ is $G = K_5$. The additional facet inequality of $\text{CUT}(K_5)$ that needs to be added is the pentagonal inequality introduced in
  - M. E. Tylkin (=M. Deza), *On Hamming geometry of unitary cubes*, Soviet Physics Dokl. 5 (1960) 940–943
- Maybe the theorem can be generalized.

Bell polytopes

- For a family of list of integers $(L_1, \dots, L_r)$ a notion of Bell polytope $B(L_1, \dots, L_r)$ can be defined and a question is how to compute the facet inequalities of those polytopes.
- The dimension of many of them are too large to be computed.
- We can restrict ourselves to the facets having some symmetries
- By using only the conjugacy class of elements defining polytope of dimension as most 20 we find many symmetric facets for the cases

$\{\{2,2\},\{2,2\},\{2,2,2\}\}$	$\{\{2,2,2\},\{2,2,2,2\}\}$
$\{\{2,2\},\{2,2\},\{2,2\},\{2,2\}\}$	$\{\{2,2\},\{2,2\},\{2,2,2,2\}\}$
$\{\{3,3,2\},\{3,3,3\}\}$	$\{\{3,3,3\},\{3,3,2\}\}$
$\{\{4,4\},\{4,4\}\}$	$\{\{5,2,2\},\{5,2,2\}\}$

- Some special Bell polytopes can be interpreted as $\mathrm{CUTP}(G)$ and this was done for $G = K_{1,4,4}$

IV. Hypermetrics

Hypermetrics

A function $d : \{1, \dots, n\}^2 \mapsto \mathbb{R}$ belongs to the hypermetric polytope $\mathrm{HYPP}(K_n)$ if and only if for all $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ with $\sum_i b_i = 2s + 1$ with $s \in \mathbb{Z}$ we have

$$\sum_{i<j} b_i b_j d(i,j) \leq s(s+1)$$

- If one limits oneself to the inequalities with $s = 0$ then one gets the hypermetric cone.
- The hypermetric cone can be interpreted in term of Delaunay polytope and the hypermetric polytope in term of centrally symmetric Delaunay polytope.
- Facets and vertices of the hypermetric cone and polytope were computed up to $n = 8$.
- No generalization to graphs were found.
- M. Deza, M. Dutour Sikirić, *The hypermetric cone on eight vertices and some generalizations*, Journal of Symbolic Computations (to appear).

V. Quasi-metrics
Cones and polytopes

Quasi-metrics on K_n

Given a set X of points a quasi-metrics on X is a function $d : X \times X \mapsto \mathbb{R}$ such that

- $d(x, x) = 0$ for all $x \in X$,
- $d(x, y) \geq 0$ for all $x, y \in X$,
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The quasi-metric cone $\mathrm{QMET}(K_n)$ is the cone of quasi-metrics on $\{1, \dots, n\}$. Its dimension is $n(n-1)$. We studied it in

- M. Deza, M. Dutour, E. Panteleeva, Small cones of oriented semi-metrics, American Journal of Mathematical and Management Sciences 22 (2002) 199–225.

If we add the following constraints

$$d(x, y) \leq 1 \text{ and } d(x, y) + d(y, z) + d(z, x) \leq 2 \text{ for all } x, y, z \in X$$

then we obtain the quasi-metric polytope $\mathrm{QMETP}(K_n)$.

Symmetries

- For a set $S \subset \{1, \ldots, n\}$ the oriented switching is defined as

$$F_S(d)(x,y) = \begin{cases} 1 - d(y,x) & \text{if } |S \cap \{x,y\}| = 1 \\ d(x,y) & \text{otherwise.} \end{cases}$$

- The reversal operation is defined as

$$R(d)(x,y) = d(y,x)$$

- The reversal and symmetric group $Sym(n)$ define a symmetry group of QMET(K_n) of order of $2n!$.
- The reversal, oriented switchings and $Sym(n)$ define a symmetry group of QMETP(K_n) of order of $2^n n!$.

Quasi-metrics on graphs

If we have an undirected graph G then we define $Dir(E(G))$ to be the set of directed edges of $E(G)$. That is each edge $e = (i, j)$ corresponds to an oriented edge (i, j) and (j, i). We define QMET(G) and QMETP(G) to be the projection of QMET(K_n) and QMETP(K_n) on $Dir(E(G))$.

Theorem: QMET(G) is described as the cone of functions $d \in \mathbb{R}^{Dir(E(G))}$ satisfying to

- $0 \leq d(i, j)$ for $(i, j) \in Dir(E(G))$
- For each cycle $c = (v_1, \dots, v_m)$ of G the inequality

$$d(v_1, v_m) \leq d(v_1, v_2) + d(v_2, v_3) + \dots + d(v_{m-1}, v_m)$$

There is a similar descriptions for QMETP(G).

Weighted quasi-metrics and cuts

- A quasi-metric is called weighted if there exist a function $w : X \mapsto \mathbb{R}$ such that

$$d(x,y) + w(x) = d(y,x) + w(y) \text{ for all } x, y \in X.$$

- The cone and polytopes of weighted quasi-metrics are called WQMET(G) and WQMETP(G). This defines an interesting subcase between METP(G) and QMETP(G).
- For a set $S \subset \{1, \dots, n\}$ we define the (weighted) oriented cut

$$\delta'^{G}_{S}(x,y) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

- We define OCUTP(G) to be the convex hull of the $F_S(\delta'^{G}_{T})$ and it has at most 2^{2n-2} vertices.
- We have OCUTP(K_n) = WQMETP(K_n) for $n \leq 4$ but we have OCUTP(G) $\neq$ WQMETP(G) for some graph G which do not have K_5 minor, e.g. $K_5 - K_2$ or $Prism_3$.

V. Hemimetric

m-Hemimetrics

Metrics are an abstraction of distance between 2 points. What about the notion of area, volume, etc? We define an m-hemidistance on $m+1$ points $x_1, \ldots, x_{m+1} \in X$ to be a function satisfying

- $d(x_1, \ldots, x_{m+1}) \geq 0$ for all $x_1, \ldots, x_{m+1} \in X$
- $d(x_{\sigma(1)}, \ldots x_{\sigma(m+1)}) = d(x_1, \ldots, x_{m+1})$ for all $x_1, \ldots, x_{m+1} \in X$ and $\sigma \in Sym(m+1)$

But what about the equivalent of the triangle inequality? The naive extension is to consider

$$d(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})$$

for all $x_1, \ldots, x_{m+2} \in X$. This definition was used in:

- M. Dutour, M. Deza, Cones of metrics, *hemi-metrics and super-metrics*, Annals of the European Academy of Sciences (2003) 141–162

The simplex inequalities

- If $m = 1$ then the cycle inequalities are implied by the triangle inequalities:

$$\begin{array}{rcl} d(v_1, v_m) & \leq & d(v_1, v_2) + d(v_2, v_m) \\ & \leq & d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_m) \\ & \leq & d(v_1, v_2) + d(v_2, v_3) + \dots + d(v_{m-1}, v_m) \end{array}$$

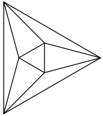
  But such a decomposition does not necessarily exist for $m > 1$.
- A closed manifold is a family M of $m+1$-subsets of $\{1, \dots, n\}$ such that for any m-set S the number of cells in M containing S is even.
- For a closed manifold $M = (S_1, \dots, S_M)$ the simplex inequality is

$$d(S_i) \leq \sum_{1 \leq j \leq M, i \neq j} d(S_j).$$

Hemimetrics on simplicial complexes

- We define $Set(n, m)$ to be the set of all $m+1$-subsets of $\{1, \ldots, n\}$.
- A m-dimensional simplicial complex is a subset of $Set(n, m)$ for some n.
- For a simplicial complex $\mathcal{K}$ we define the cone $\text{HMET}(\mathcal{K})$ to be the set of functions satisfying all simplicial inequalities induced by all closed submanifolds M of $\mathcal{K}$.
- Theorem: The cone $\text{HMET}(\mathcal{K})$ is the projection of $\text{HMET}(Set(n, m))$ on $\mathbb{R}^{\mathcal{K}}$.
- For the case $m = 2$ and 6 points, the octahedron gives an inequality in $\text{HMET}(Set(6, 2))$

$$d(S_1) \leq \sum_{i=2}^{8} d(S_i)$$



which cannot be decomposed into inequalities over the simplex.

THANK

YOU