# $C$-types, a generalization of $L$-types

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# I. Delaunay polytopes and $L$-type theory

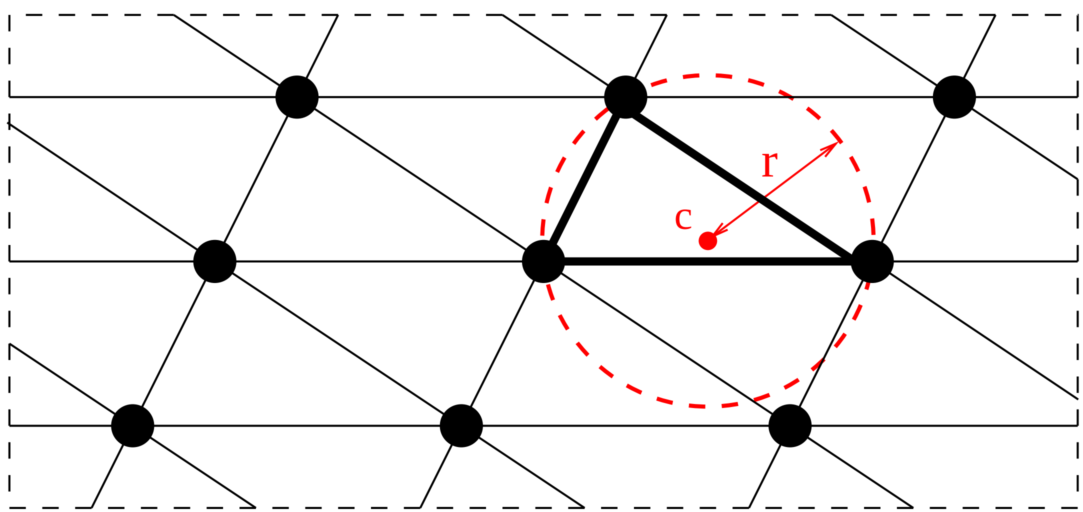
# Empty sphere and Delaunay polytopes

A sphere $S(c, r)$ of radius $r$ and center $c$ in an $n$-dimensional lattice $L$ is said to be an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope $P$ in a lattice $L$ is a polytope, whose vertex-set is $L \cap S(c, r)$.



# Gram matrix and lattices

- Take $u$ an isometry of $\mathbb{R}^n$. $D$ is a Delaunay polytope of a lattice $L$ if and only if $u(D)$ is a Delaunay polytope of $u(L)$. We want to study isometry classes of lattices.
- Denote by $S^n$ the vector space of real symmetric $n \times n$ matrices and by $S^n_{>0}$ the convex cone of positive definite ones.
- Lattice $L$ generated by $v_1, \dots, v_n$ corresponds to

$$G_v = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0} \ .$$

  $G_v$ depends only on the isometry class of $L$.
- Given $M \in S^n_{>0}$, one can find vectors $v_1, \dots, v_n$ such that $M = G_v$.

# Gram matrix and lattices

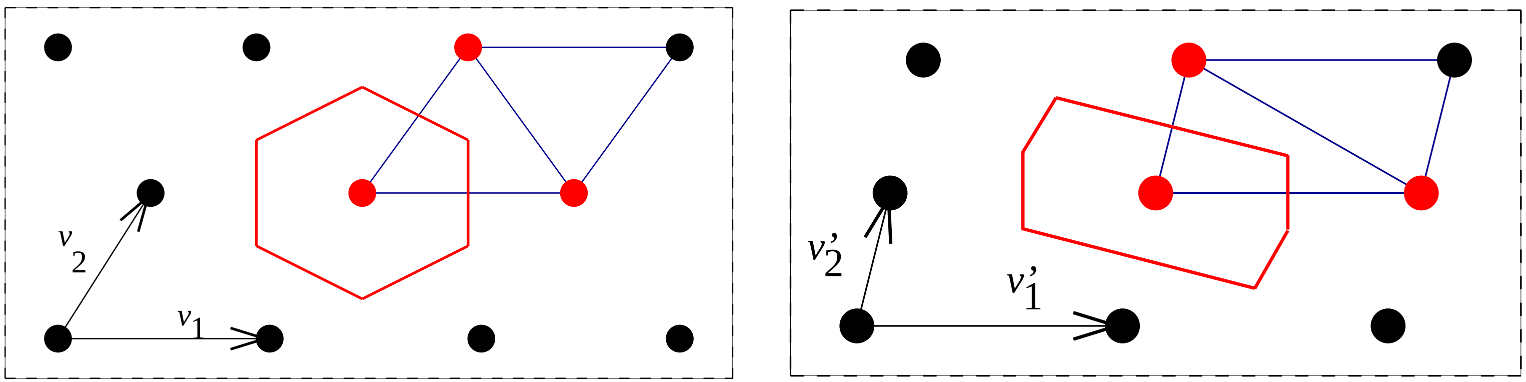
- Two matrices $M, M'$ are arithmetically equivalent if there exist $P \in GL_n(\mathbb{Z})$ such that

$$M' = P^T M P \; .$$

- For any two basis $\mathbf{v}$, $\mathbf{v}'$ of a lattice $L$, $G_{\mathbf{v}}$ and $G_{\mathbf{v}'}$ are arithmetically equivalent.
- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice it is preferable to think and draw in terms of lattices, but to compute in terms of matrices in $S^n_{>0}$.
- In the following, the Delaunay decomposition of a matrix $M \in S^n_{>0}$ is the Delaunay decomposition of $\mathbb{Z}^n$ with respect to the scalar product $x^T M y$.

# $L$-type domains

- A $L$-type domain is the set of matrices $M \in S^n_{>0}$ with the same Delaunay decomposition.
- Geometrically this means that the Gram matrices $G_{\mathbf{v}}$, $G_{\mathbf{v}'}$ of following lattices $L$ and $L'$

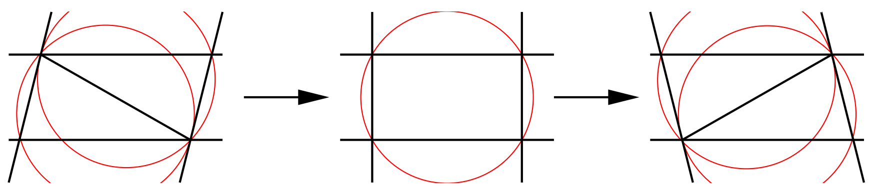



are part of the same $L$-type domain.

- Specifying Delaunay polytopes, means putting some linear equalities and inequalities on the Gram matrix $G_{\mathbf{v}}$.
- A priori, infinity of inequalities but a finite number suffices.

# Equivalence and enumeration

- If there is no equalities, i.e. if all Delaunays are simplices, then the $L$-type is called primitive.
- The group $GL_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive $L$-type domains.
- Voronoi proved that there is a finite number of primitive $L$-type domains up to arithmetic equivalence.
- Bistellar flipping creates new triangulations. In dim. $2$:



- Enumerating them is done classically:
  - Find one primitive $L$-type domain.
  - Find the adjacent ones by bistellar flipping and reduce by arithmetic equivalence.

# II. $C$-types

(by Ryshkov & Baranovs

# $C$-primitivity

- If $D$ is a Delaunay polytope, an edge $e = [v_1, v_2]$ of $D$ between two vertices $v_1$ and $v_2$ of $D$ is a face of $D$.
- The edge $e$ is encoded by its middle vector $m(e) = \frac{1}{2}(v_1 + v_2)$. Up to translation, one can assume that $m(e) \in \{0, \frac{1}{2}\}^n$.
- A parity class is a vector $c \in \{0, \frac{1}{2}\}^n - \{0\}$; we denote by $\mathcal{PC}$ the set of all parity classes.
- The matrix $M \in S^n_{>0}$ is said to be $C$-primitive if for every $c \in \mathcal{PC}$, there exist an edge $e = [v_1, v_2]$ of the Delaunay decomposition of $M$ such that $m(e) = c$.

# $C$-rigidity index

- If $c \in \mathcal{PC}$, denote by $N(c)$ the vectors, which are closest to $c$. $N(c)$ is a centrally symmetric face of a Delaunay polytope of $\mathbb{Z}^n$.
- $c$ is at equal distance from all points in $N(c)$ so there is $\lambda > 0$ such that

$$(v-c)^T M(v-c) = \lambda \text{ for all } v \in N(c) .$$

  This makes linear equalities on $M$.
- The $C$-rigidity index is defined as the dimension of the space defined by those equalities.

# $C$-type

- Denote $\mathcal{FS}(\mathbb{Z}^n)$ the family of all finite subsets of $\mathbb{Z}^n$.
- A $C$-type is
  - a function $N : \mathcal{PC} \to \mathcal{FS}(\mathbb{Z}^n)$ with
  - $N(c)$ being a collection of vertices in $\mathbb{Z}^n$, which is invariant by the action $x \mapsto 2c - x$.
- A $C$-type is called primitive if for every $c \in \mathcal{PC}$, one has $N(c) = \{v_1, v_2\}$.
- A primitive $C$-type can be encoded by the family

$$\{v_2 - v_1 \mid c \in \mathcal{PC}\}$$

  Primitive $C$-types can be reconstructed from this information.

# $C$-type domain

- A $C$-type is called realizable if there exists a matrix $M \in S^n_{>0}$ having centrally symmetric faces of Delaunay being in this $C$-type.
- We will consider only realizable $C$-types. Associated to a realizable $C$-type, there is its $C$-type domain, i.e. the set of matrices $M \ni S^n_{>0}$ whose centrally symmetric faces are this $C$-type.
- A $C$-type domain is primitive if and only if its centrally symmetric faces are simply edges.

# Matrix expression

- Take a $C$-type $\mathcal{CT}$ and $M$ in the $C$-type domain. For any $c \in \mathcal{PC}$, one should have
  - Take $v_0 \in N(c)$; for any $v \in N(c)$:

$$(v-c)^T M (v-c) = (v_0 - c)^T M (v_0 - c)$$

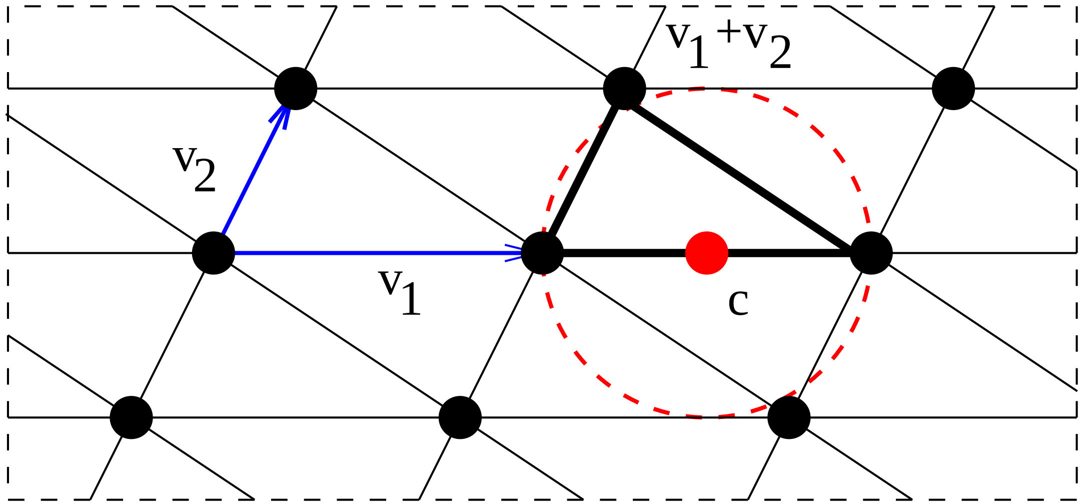
  - For any $v \in \mathbb{Z}^n - N(c)$:

$$(v-c)^T M (v-c) > (v_0 - c)^T M (v_0 - c)$$

- The first part makes linear equalities, the second part makes linear inequalities.
- Hence $C$-type domains are convex cone in $S^n_{>0}$.

# Dimension $2$ example

- Take a lattice $L = \mathbb{Z}v_1 + \mathbb{Z}v_2$:




- The condition that $v_1 + v_2$ is outside of the edge $[v_1, 2v_1]$ of center $c = \frac{3}{2}v_1$ yields

$$||v_1 + v_2 - \frac{3}{2}v_1|| > ||v_1 - \frac{3}{2}v_1||$$
$$\text{i.e. } ||v_2||^2 - \langle v_1, v_2\rangle \geq 0$$

- In fact in dimension $2$, $C$-types coincide with $L$-types.

# General theorem

- The group $GL_n(\mathbb{Z})$ acts on the set of $C$-type domains by arithmetic action

$$\begin{array}{rcl} GL_n(\mathbb{Z}) \times S^n_{>0} & \rightarrow & S^n_{>0} \\ (P, M) & \mapsto & P^T M P \end{array}$$

- Thm.(Ryshkov & Baranovskii)
  - $C$-type domains are polyhedral cones.
  - $C$-type domains realize a face-to-face tesselation of $S^n_{>0}$
  - The $L$-type domain tesselation of $S^n_{>0}$ is a finite refinement of the $C$-type domain tesselation.
  - In a fixed dimension, there are a finite number of $C$-type domains up to equivalence.

# Results

- All results on $C$-type were obtained by Ryskov & Baranovskii.
- They used it as technical tool for enumerating the $L$-types in dimension $5$.

| dim | Primitive $L$-types | Authors | Primitive $C$-types |
|---|---|---|---|
| 2 | 1 | Dirichlet (1860) | 1 |
| 3 | 1 | Fedorov (1885) | 1 |
| 4 | 3 | Voronoi (1908) | 3 |
| 5 | 222 | BaRy (1976), Engel & Gr (2002) | 76 |

# Proofs

- If $M \in S^n$ satisfy all the condition of a $C$-type then $M \in S^n_{\geq 0}$.
  proof: for $v \in \mathbb{Z}^n$ and $\lambda \in \mathbb{Z}$ one has
  $$(\lambda v - c)^T M(\lambda v - c) \geq (v_0 - c)^T M(v_0 - c) \text{ with } v_0 \in N(c)$$
  passing to the limit $\lambda \to \infty$, one obtains $v^T M v \geq 0$.
- Every $L$-type domain $\mathcal{LT}$ is contained in a unique $C$-type domain $\mathcal{CT}$ denoted by $\phi(\mathcal{LT})$.
  proof: The $L$-type domain $\mathcal{LT}$ defines all the Delaunay polytopes. Computing their centrally symmetric faces, one obtains a $C$-type domain.
- A $C$-type $\mathcal{CT}$ is the union of $L$-type domains.
  proof: if $M \in \mathcal{CT}$, then $M \in \mathcal{LT}$ with $\mathcal{LT}$ a $L$-type domain. By the above $\mathcal{LT} \subset \mathcal{CT}$.

# Proofs

A $C$-type $\mathcal{CT}$ contains a finite number of $L$-type domains.

- proof: There are a finite number of primitive $L$-type domains up to equivalence. Take $\mathcal{LT}_1, \dots, \mathcal{LT}_r$ some representatives.
- $\mathcal{LT}_i \subset \phi(\mathcal{LT}_i) = \mathcal{CT}_i$.
- Now it suffices to prove that for only a finite number of $P \in GL_n(\mathbb{Z})$ one has $P^T \mathcal{LT}_i P \subset \mathcal{CT}_i$.
- Take a Delaunay polytope $D$ of $\mathcal{LT}_i$ and find a basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ made of edges of $D$
- If $P^T \mathcal{LT}_i P \subset \mathcal{CT}_i$ then $P(e_1, \dots, e_n)$ is a family of edges of $\mathcal{CT}_i$. So, there is a finite number of possible $P$.

# Proofs

- $C$-type domains are polyhedral cone.
  proof: We know that $C$-type domains are finite union of $L$-type domains. Since $C$-type domains are convex, they are necessarily polyhedral.
- $C$-type domains realize a face-to-face tesselation of $S^n_{>0}$.
  proof: $S^n_{>0}$ is an union of $C$-type domains. They are defined by linear inequalities, so automatically, this makes a face-to-face tiling.

# III. Algorithms

# General algorithms

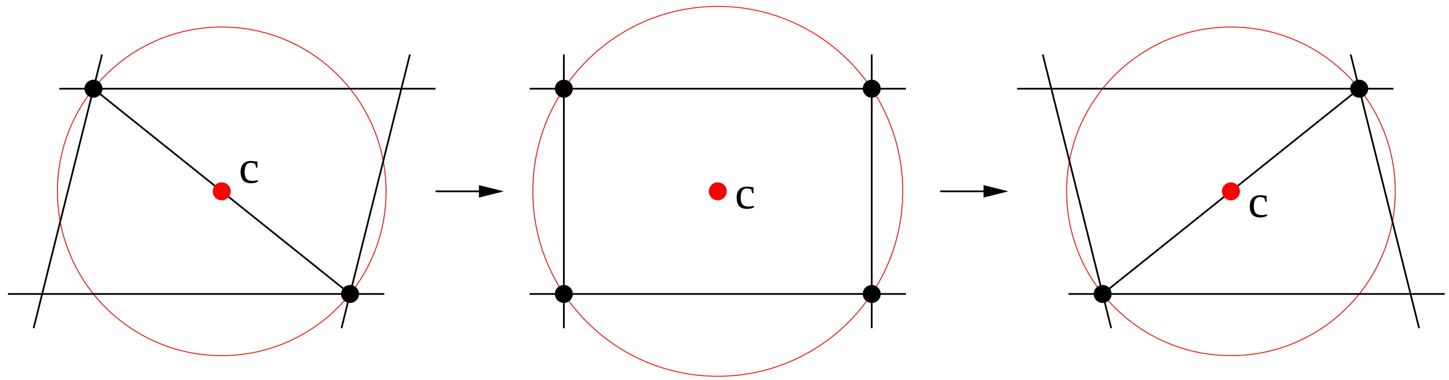
- We want to enumerate primitive $C$-type domains, the strategy used is
  - Find a primitive $C$-type domain and insert it into the list of primitive $C$-type domains.
  - For every undone primitive $C$-type domain,
    - Compute the non-redundant inequalities defining it.
    - For every facet, find the adjacent $C$-type domain.
    - For every adjacent $C$-type domain, do an isomorphism test with the elements in the existing list and insert them if they are new.

# Obtaining primitive $C$-type domain

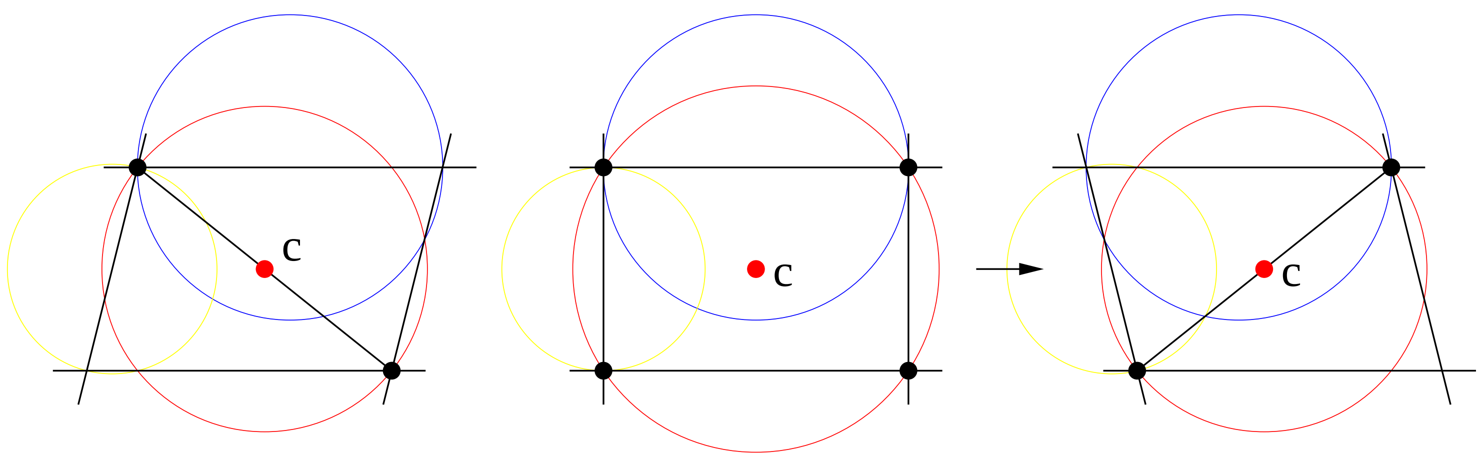
- The algorithm is similar to the one for $L$-types.
- Iterate the following
  - Find a random integral matrix, compute its Delaunay decomposition.
  - If one of the Delaunay has a centrally symmetric face, which is not an edge, then we know that the $C$-rigidity index is less than $\frac{n(n+1)}{2}$ and we restart the computation.
  - Otherwise, return the corresponding $C$-type.
- This algorithm is of Las Vegas type, i.e. it always return a correct answer but the running time is not known.

# The geometrical picture

- Geometrically the flipping consists in dim. $2$ of:

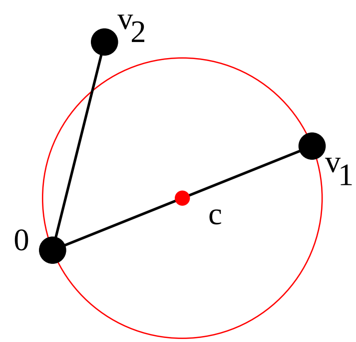



- If one puts the three parity classes in dim. $2$:



# Finding non-redundant inequalities

- Find all doubles $(v_1, v_2)$ such that $[0, v_1]$ and $[0, v_2]$ are edges of the Delaunay decomposition.




- For any double $(v_1, v_2)$, define the linear inequality

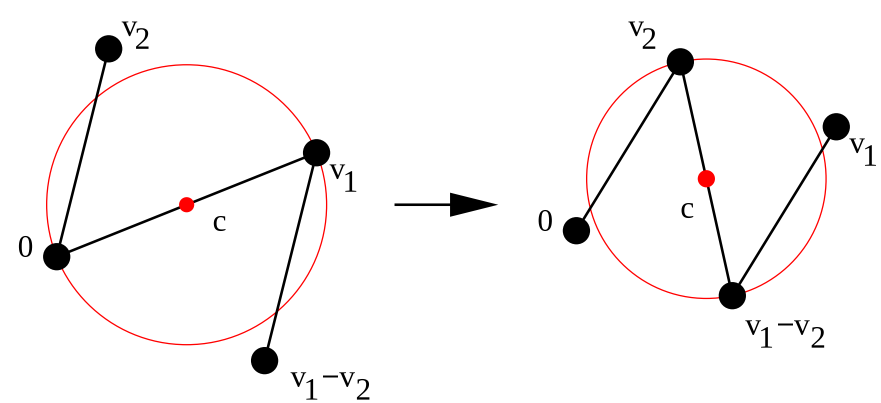
$$(v_2 - c)^T M (v_2 - c) \geq (v_1 - c)^T M (v_1 - c) \text{ with } c = \frac{1}{2}(v_1)$$

Denote by $f_{v_1,v_2}(M) \geq 0$ the corresponding inequality.

- This form a finite set $\mathcal{S}$ of inequalities. We extract a non-redundant set from $\mathcal{S}$ by linear programming.

# The $C$-flipping

- Take a primitive $C$-type domain $\mathcal{CT}$ and a non-redundant inequality $f(M) \geq 0$. We want to flip $\mathcal{CT}$ along the facet defining equality $f(M) = 0$.
  - Find all double $(v_1, v_2)$ such that there is $\alpha > 0$ with $f_{v_1,v_2}(M) = \alpha f(M)$.
  - For every such double replace the edge $[0, v_1]$ by the edge $[v_2, v_1 - v_2]$.




  - We then get the adjacent primitive $C$-type domain.

# Testing equivalence

- Associate to $\mathcal{CT}$ with edge vectors $(v_1, \ldots, v_{2^n-1})$ the vector family

$$V(\mathcal{CT}) = (v_1, -v_1, \ldots, v_{2^n-1}, -v_{2^n-1})$$

- Two $C$-type domains $\mathcal{CT}$ and $\mathcal{CT}'$ are equivalent if there exist a matrix $P \in GL_n(\mathbb{Z})$ such that $\mathcal{CT}' = P^T \mathcal{CT} P$.
- In other words $\mathcal{CT}$ and $\mathcal{CT}'$ are equivalent if and only if there exist a matrix $P \in GL_n(\mathbb{Z})$ such that $PV(\mathcal{CT}) = V(\mathcal{CT}')$.
- The automorphism group question is expressed similarly.

# Lemma

If $\mathcal{CT}$ is a $C$-type, then its edges $\mathbb{Z}$-generates $\mathbb{Z}^n$.

- proof: Take $\mathcal{LT}$ a $L$-types such that $\phi(\mathcal{LT}) = \mathcal{CT}$. If $v$ and $v'$ are two vertices of $\mathbb{Z}^n$, then we can find a sequence of vertices $v = v^0, \dots, v^N = v'$ such that $v^i$ and $v^{i+1}$ belong to the same Delaunay polytope.
- For any two vertices $w$, $w'$ of a Delaunay polytope $D$ one can find a sequence of vertices $w = w^0, \dots, w^M = w'$ such that $w^i$ and $w^{i+1}$ form an edge of $D$.
- Every edge of $D$ corresponds to an edge of the $C$-type. Hence,

$$v' - v = \sum_e \lambda_e e \text{ with } \lambda_e \in \mathbb{Z}$$

# Algorithm

- To the $C$-type with vector family $V(\mathcal{CT})$ one associates

$$M_{\mathcal{CT}} = \sum_{v \in V(\mathcal{CT})} v v^T$$

- Associates to $\mathcal{CT}$ the edge colored graph $G(\mathcal{CT})$ on $V(\mathcal{CT})$ with edge colors

$$p_{v,v'} = v^T M_{\mathcal{CT}}^{-1} v' \quad \text{for any} \quad v, v' \in V(\mathcal{CT})$$

- There exist a matrix $P \in GL_n(\mathbb{R})$ such that $PV(\mathcal{CT}) = V(\mathcal{CT}')$ if and only if the edge-colored graph $G(\mathcal{CT})$ and $G(\mathcal{CT}')$ are isomorphic.
- $V(\mathcal{CT})$ and $V(\mathcal{CT}')$ are $\mathbb{Z}$-generating, so $P \in GL_n(\mathbb{Z})$.

# IV. Generalization

# $S^n_{>0}$-spaces

- A $S^n_{>0}$-space $\mathcal{SP}$ is a vector space of $S^n$, which intersect $S^n_{>0}$.
- We want to study the centrally symmetric faces of matrices $M \in \mathcal{SP} \cap S^n_{>0}$.
- Example of possible spaces are

$$\mathcal{SP}(G) = \{X \in S^n \mid g^T X g = X \text{ for all } g \in G\}$$

with $G$ a finite subgroup of $GL_n(\mathbb{Z})$.

# $G$-invariant faces

- Centrally symmetric faces are faces, which are invariant by a transformation $x \mapsto w - x$ with $w \in \mathbb{Z}^n$.
- If $G$ is a finite subgroup of $GL_n(\mathbb{Z})$, why not consider the faces that are invariant under $G$?

# $k$-faces

- $L$-types are the specification of all Delaunay, i.e. of $n$-dimensional faces.
- $C$-types are the specification of centrally symmetric faces but in primitive case, it is the specification of $1$-dimensional faces.
- Would it be possible to extend the theory to the case of $k$-dimensional faces with $1 < k < n$?
- After that one would want a subspace version of it

# V. First generalization

# Settings

- Take $\mathcal{SP}$ a $S^n_{>0}$-space.
- We want to describe the centrally symmetric faces of Delaunay decomposition of matrices in $\mathcal{SP} \cap S^n_{>0}$.
- A $(\mathcal{SP}, C)$-type is defined as the assignation of centrally symmetric faces of the Delaunay tesselation. A $(\mathcal{SP}, C)$-type domain is the corresponding convex cone.
- A $(\mathcal{SP}, C)$-type domain is obtained as intersection of a $C$-type domain (in $S^n_{>0}$) with $\mathcal{SP}$. They are thus polyhedral domains.
- Two $(\mathcal{SP}, C)$-type domains $\mathcal{CT}_1$ and $\mathcal{CT}_2$ are called equivalent if there exist $P \in GL_n(\mathbb{Z})$ such that $P^T \mathcal{CT}_1 P = \mathcal{CT}_2$.

# Equivariance and finiteness

- If $G$ is a finite subgroup of $GL_n(\mathbb{Z})$, then

$$\mathcal{SP}(G) = \{M \in S^n \mid g^T M g = M \text{ for all } g \in G\}$$

- Thm.(Zassenhaus): One has the equality

$$\{g \in GL_n(\mathbb{Z}) \mid g\mathcal{SP}(G)^t g = \mathcal{SP}(G)\} = N_{GL_n(\mathbb{Z})}(G)$$

- Thm.(DSV): Take $\Delta$ a polyhedral face-to-face tiling of $S^n_{>0}$, which is invariant under $GL_n(\mathbb{Z})$ and has a finite number of classes. If $G$ is a finite subgroup of $GL_n(\mathbb{Z})$ then $\Delta \cap \mathcal{SP}(G)$ has a finite number of classes under action of $N_{GL_n(\mathbb{Z})}(G)$.
- Thm. For a given finite group $G \in GL_n(\mathbb{Z})$, there are a finite number of $C$-types under the action of $N_{GL_n(\mathbb{Z})}(G)$.

# Finiteness

- Suppose $\mathcal{SP}$ is an $S^n_{>0}$, define

$$Stab(\mathcal{SP}) = \left\{ \begin{array}{c} g \in GL_n(\mathbb{Z}) \text{ such that } \\ g\mathcal{SP}^t g = \mathcal{SP} \end{array} \right\}$$

- We know some examples where $\mathcal{SP}$ is irrational such that
  - $Stab(\mathcal{SP}) = \pm I_n$
  - $\mathcal{SP}$ contains an infinite number of $(\mathcal{SP}, C)$-type domains.

  And so contain an infinite number of $C$-types after action of $Stab(\mathcal{SP})$.

- But we know no example with $\mathcal{SP}$ rational and an infinite number of $(\mathcal{SP}, C)$-types after action of $Stab(\mathcal{SP})$.

# General algorithms

- A $(\mathcal{SP}, C)$-type domain is called primitive if it has maximal dimension in $\mathcal{SP}$.
- We fix a $S^n_{>0}$-space $\mathcal{SP}$ and we want to enumerate primitive $(\mathcal{SP}, C)$-type domains, the strategy used is
  - Find a primitive $(\mathcal{SP}, C)$-type domain and insert it into the list of primitive $(\mathcal{SP}, C)$-type domains
  - For every undone primitive $(\mathcal{SP}, C)$-type,
    - Compute the non-redundant inequalities defining it
    - For every facet, find the adjacent $C$-type domain.
    - For every adjacent $(\mathcal{SP}, C)$-type domain, Do an isomorphy test with elements in the existing list and insert them if they are new.
- Finding primitive $(\mathcal{SP}, C)$-type domain is easy: take element at random and finish when it is ok.

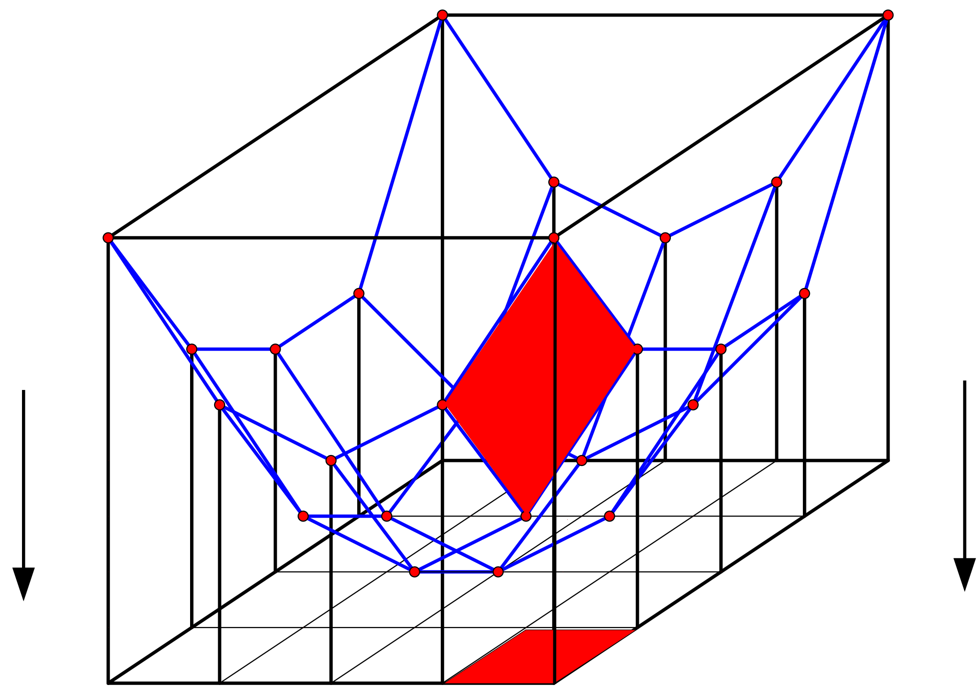
# Linear inequalities

- We have the family of $N(c)$ and we want to find the corresponding inequality.
- The first step consists in finding the facets of $N(c)$. For every such facet $F$, find all centers $c'$, such that $N(c')$ and $N(c)$ share $F$. Saying that vertices of $N(c')$ are outside the sphere around $N(c)$ makes one linear inequality. Denote this inequality by $f_{c,c'}(M) \geq 0$.
- There is a finite number of such inequalities.
- We extract the set of non-redundant inequalities from this finite set.

# Lifted Delaunay decomposition

- The Delaunay polytopes of a lattice $L$ correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertex-set:

$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{d+1} \ .$$



- Faces of Delaunay polytopes $\Leftrightarrow$ faces of $\mathcal{C}(L)$

# Oriented graph

- Take a $(\mathcal{SP}, C)$-type and suppose we know all the non-redundant inequalities of the $(\mathcal{SP}, C)$-type domain. Take $f(M) \geq 0$ one such inequality.
- Construct an oriented graph $G$ on $\mathcal{PC}$ by

  $c \to c'$ if and only if there is $\alpha > 0$ with $f_{c,c'}(M) = \alpha f(M)$

- Take an oriented graph $G$, the directed component $DC(v)$ of a vertex $v$ is the set of vertices $v'$ of $G$ such that there exist a path

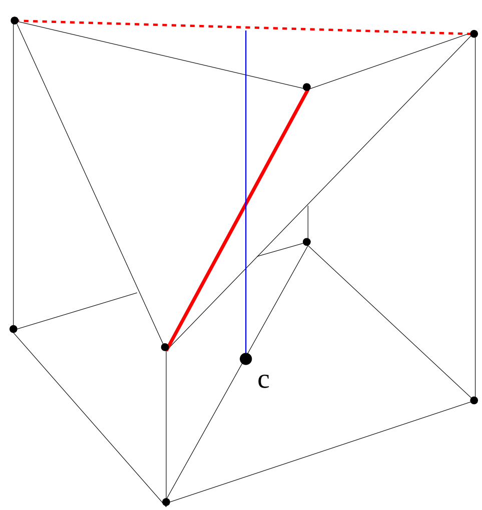
$$v = v^0 \to v^1 \to \cdots \to v^N = v'$$

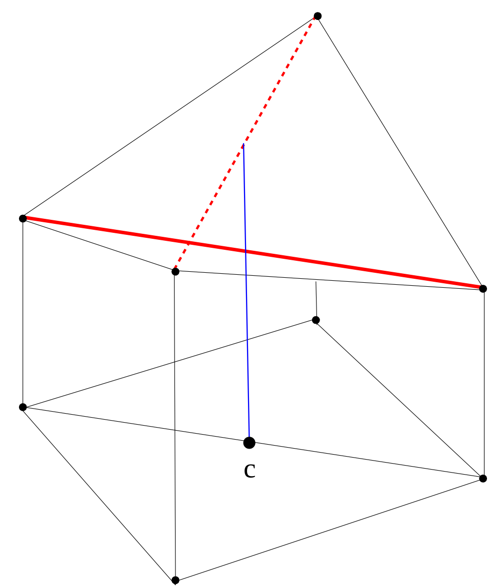
# $(\mathcal{SP}, C)$-repartitionning polytope

- For every directed component $DC(c)$ of this graph, the $(\mathcal{SP}, C)$-repartitioning polytope $RP(c)$ is the polytope with vertex-set

$$(v, {}^t v M v) \text{ with } v \text{ a vertex of a Delaunay of } DC(c)$$

- Every face of a Delaunay of the form $N(c')$ with $c' \in DC(c)$ correspond to a face of $RP(c)$.



# Faces of $RP(c)$

- If $c' \in DC(c)$, then define the affine line $c' + \mathbb{R}_+ e_{n+1}$ in $\mathbb{R}^{n+1}$ and create the intersection

$$RP(c) \cap c' + \mathbb{R}_+ e_{n+1} = [c' + \lambda_1 e_{n+1}, c' + \lambda_2 e_{n+1}]$$

- $N(c')$ is the smallest face containing $c' + \lambda_1 e_{n+1}$.
- $N'(c')$ is the smallest face containing $c' + \lambda_2 e_{n+1}$.
- Consider $\lambda$ such that $c' + \lambda e_{n+1} \in RP(c)$. Then there exist $x_v$, such that

$$\begin{cases} c' + \lambda e_{n+1} & = & \sum_{v \in V(RP(c))} x_v v \quad \text{with} \quad x_v \geq 0 \\ 1 & = & \sum_{v \in V(RP(c))} x_v \end{cases}$$

$\lambda_1$, $\lambda_2$ and $N'(c')$ are found by linear programming.

# The $(\mathcal{SP}, C)$-flipping

- Take a $(\mathcal{SP}, C)$-type domain and $f(M) \geq 0$ a relevant inequality of $\mathcal{CT}$.
- The $(\mathcal{SP}, C)$-flipping of $\mathcal{CT}$ along $f(M) = 0$ is realized in the following way:
  - Find all oriented directed component $DC(c)$
  - For every $c' \in DC(c)$, if $\lambda_1 \neq \lambda_2$, do linear programming and change $N(c')$ by $N'(c')$.
  - We then get the new $(\mathcal{SP}, C)$-type domain.

# VI. Second generalization

# $G$-parity classes

- We take $G$ a finite subgroup of $GL_n(\mathbb{Z})$ and consider the space $\mathcal{SP}(G)$. We do not assume that $-I_n \in G$.
- The $G$-parity classes are the vectors $c \in \mathbb{R}^n$ such that for all $g \in G$ one has $gc - c \in \mathbb{Z}^n$.
- We want a finite number of $G$-parity classes
- This means that we want the system of equation $gx = x$ with $g \in G$ implies $x = 0$.
- For all $x \in \mathbb{R}^n$ one has $\sum_{g \in G} gx = 0$.

# Nearest neighbors

- We assume that the solution of the equation $gx = x$ for all $g \in G$ is only $0$.
- The set $N(c)$ of nearest neighbors to a $G$-parity class is $G$-invariant.
- By above property one will have

$$\frac{1}{|N(c)|} \sum_{v \in N(c)} v = c$$

- All the preceding theory generalizes by replacing parity classes by $G$-parity classes. Also, one can take a linear subspace $\mathcal{SP}$ of $\mathcal{SP}(G)$.

THANK

YOU