Geometry of numbers, equivariant L-type and coverings

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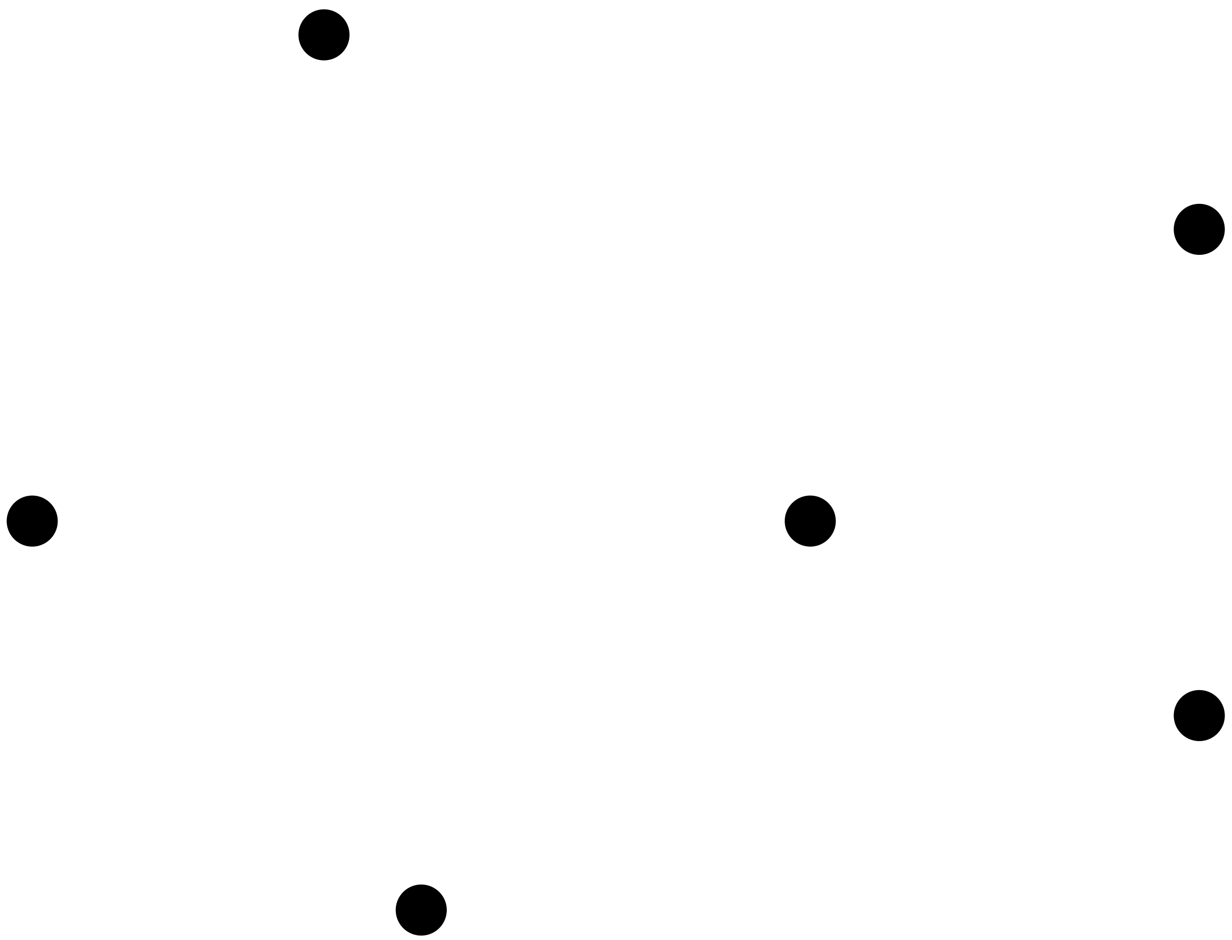
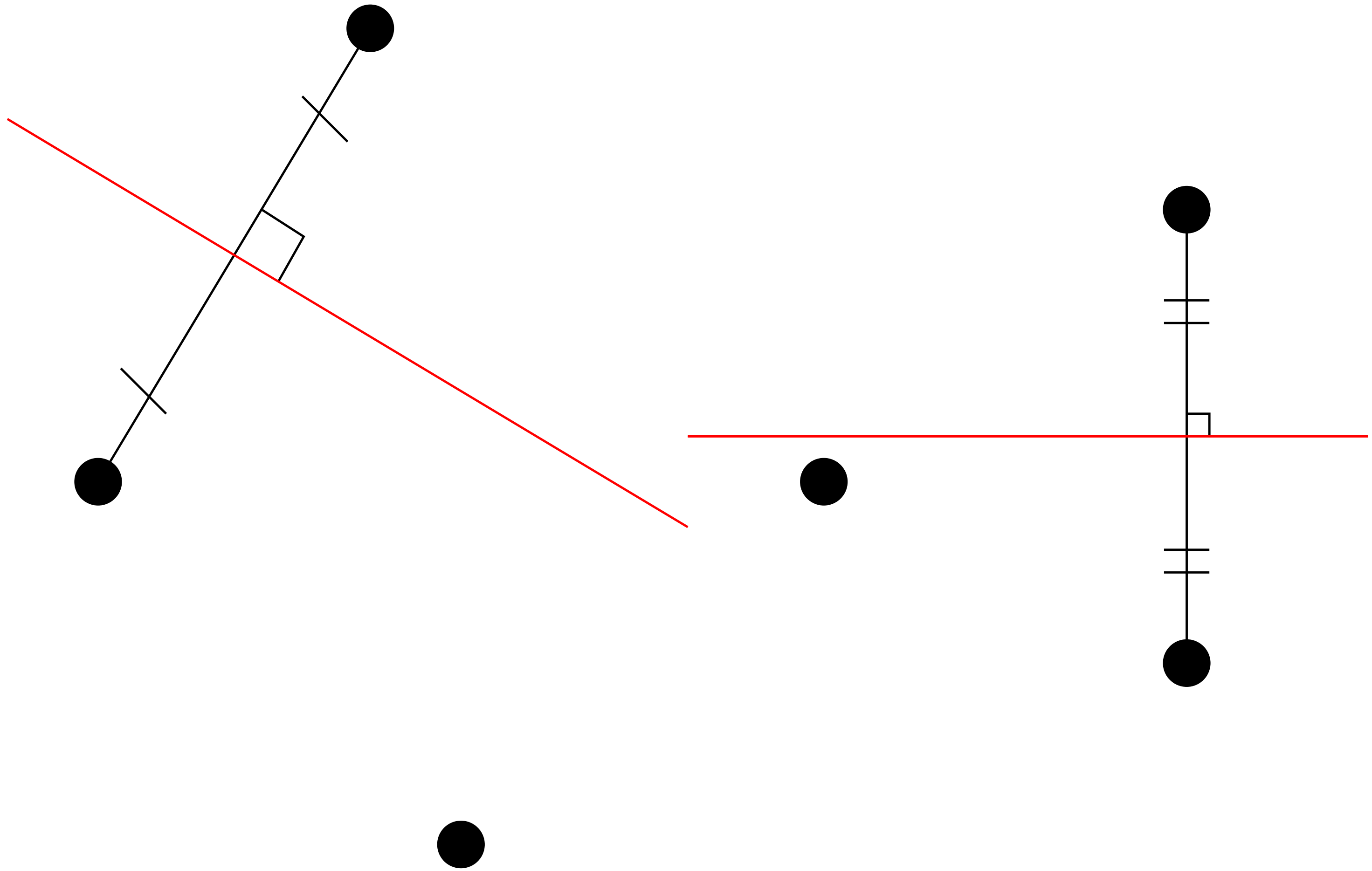
Voronoi and Delaunay polytopes

A finite set of points



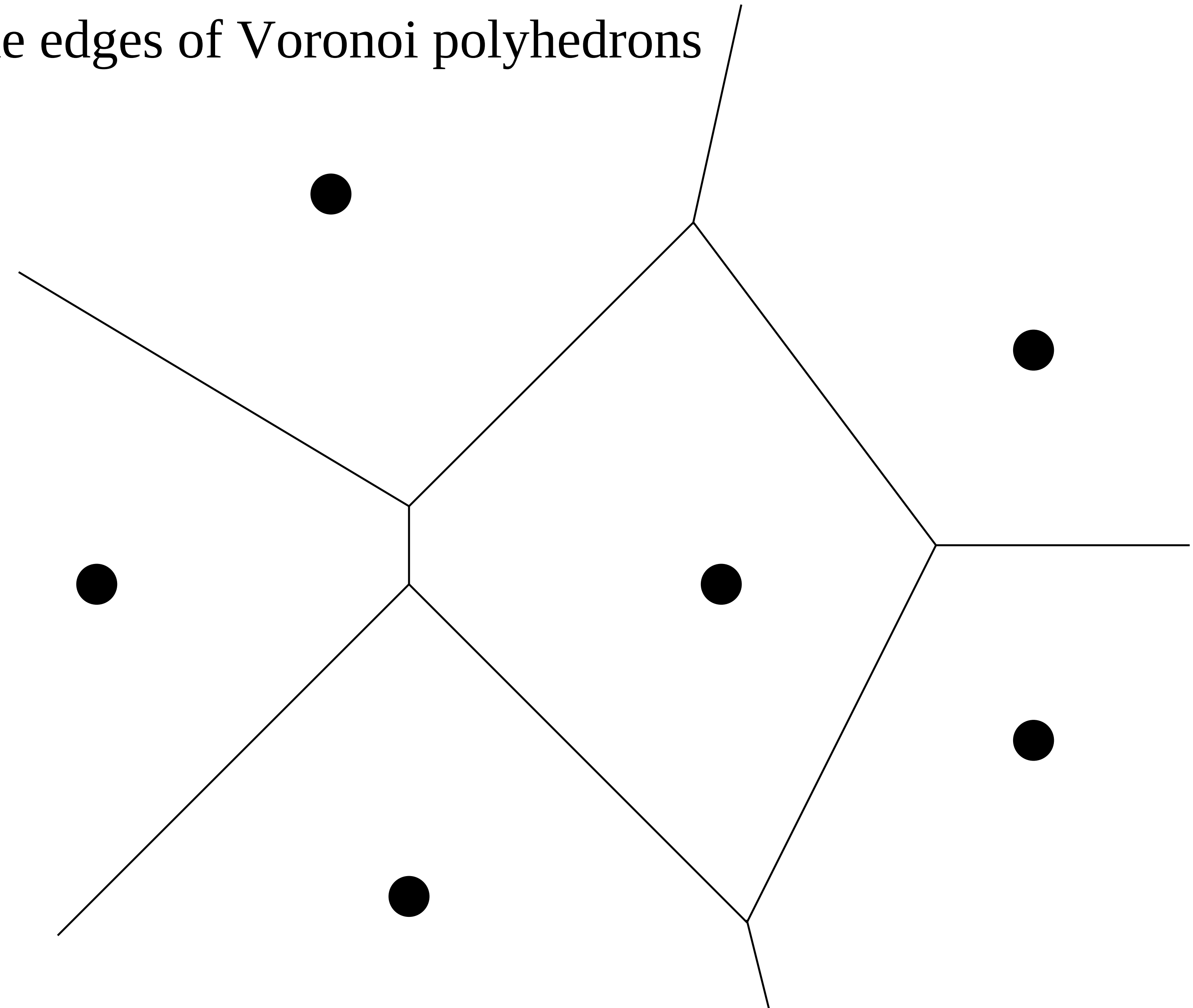
Voronoi and Delaunay polytopes

Some relevant perpendicular bisectors



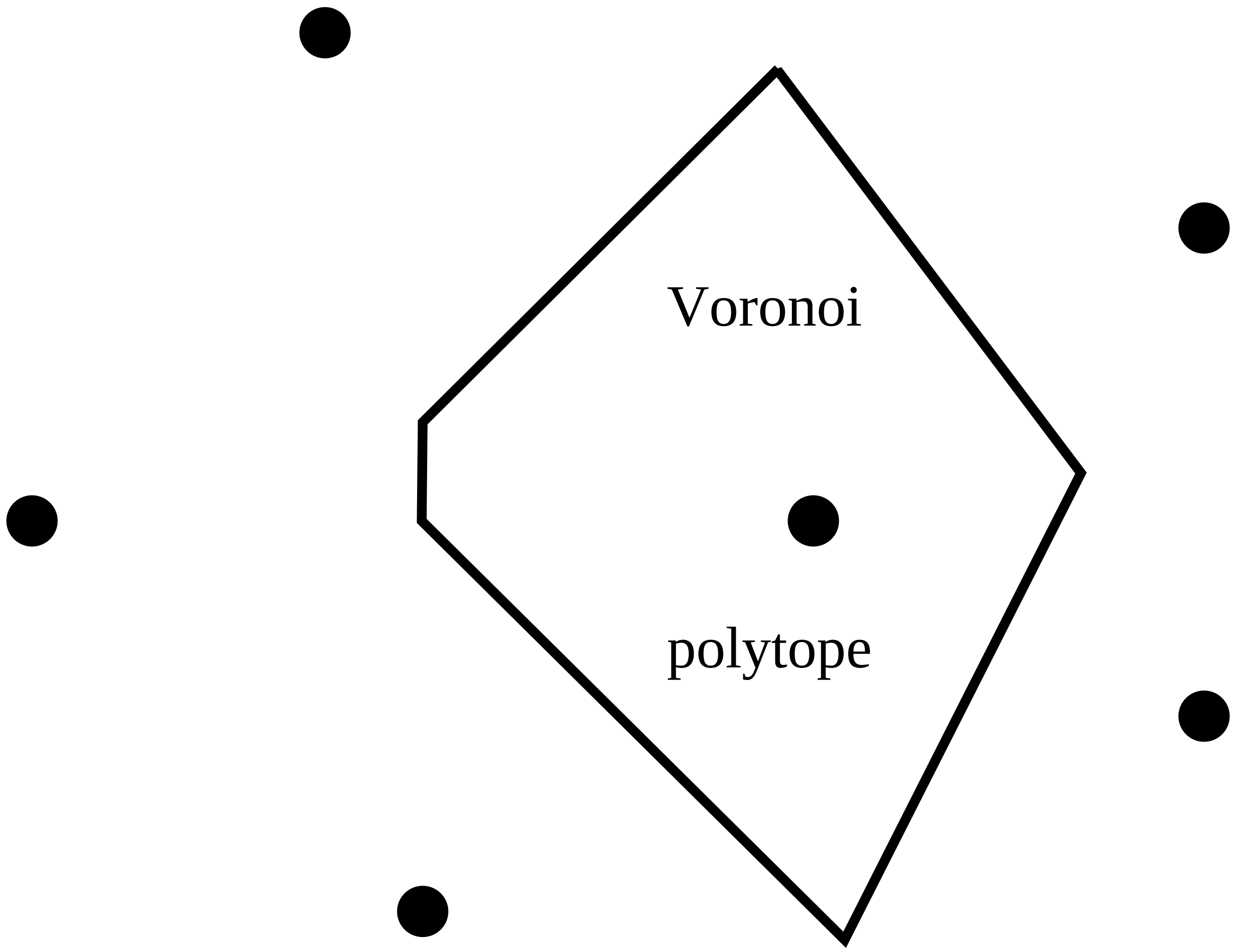
Voronoi and Delaunay polytopes

The edges of Voronoi polyhedrons



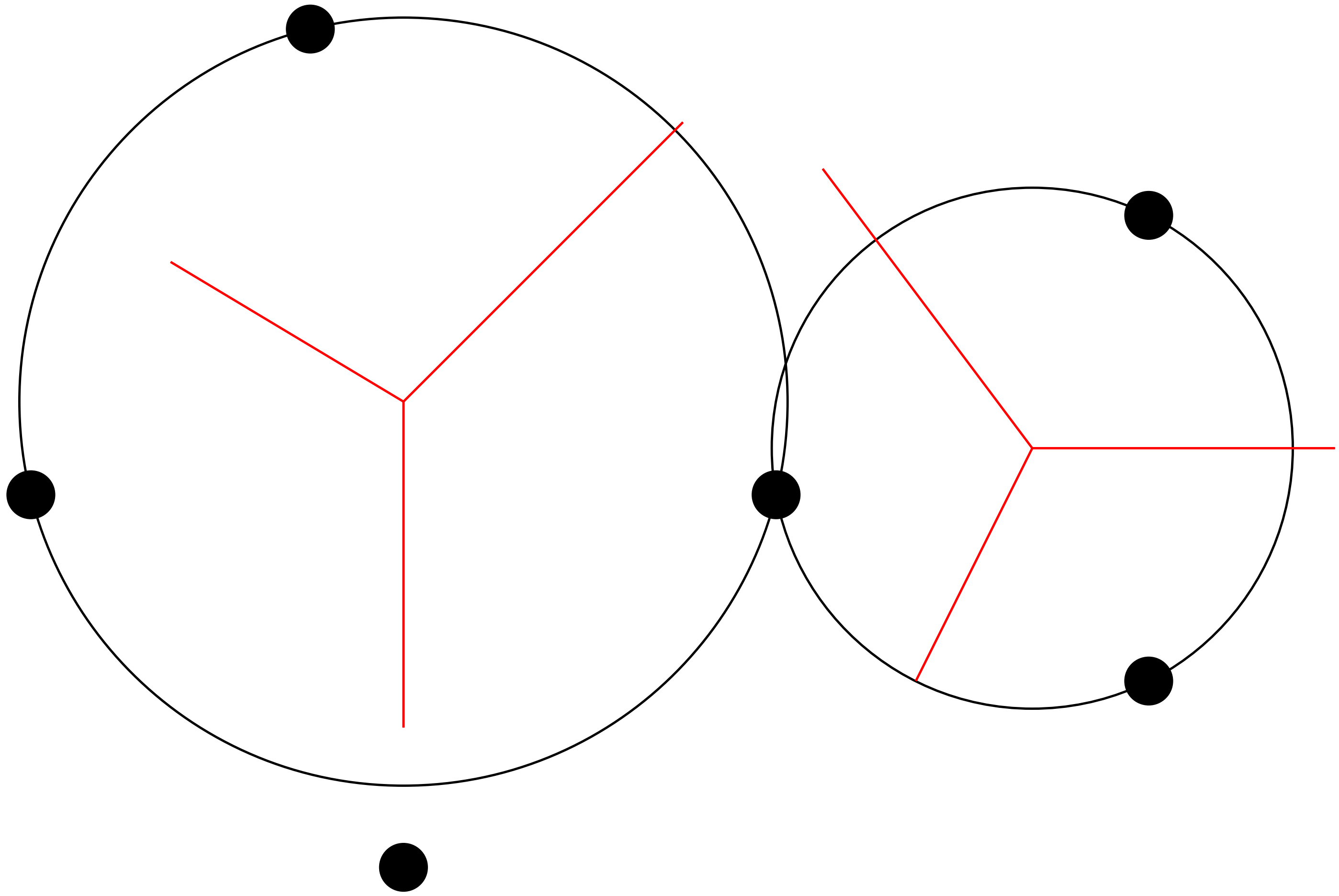
Voronoi and Delaunay polytopes

Voronoi polytope



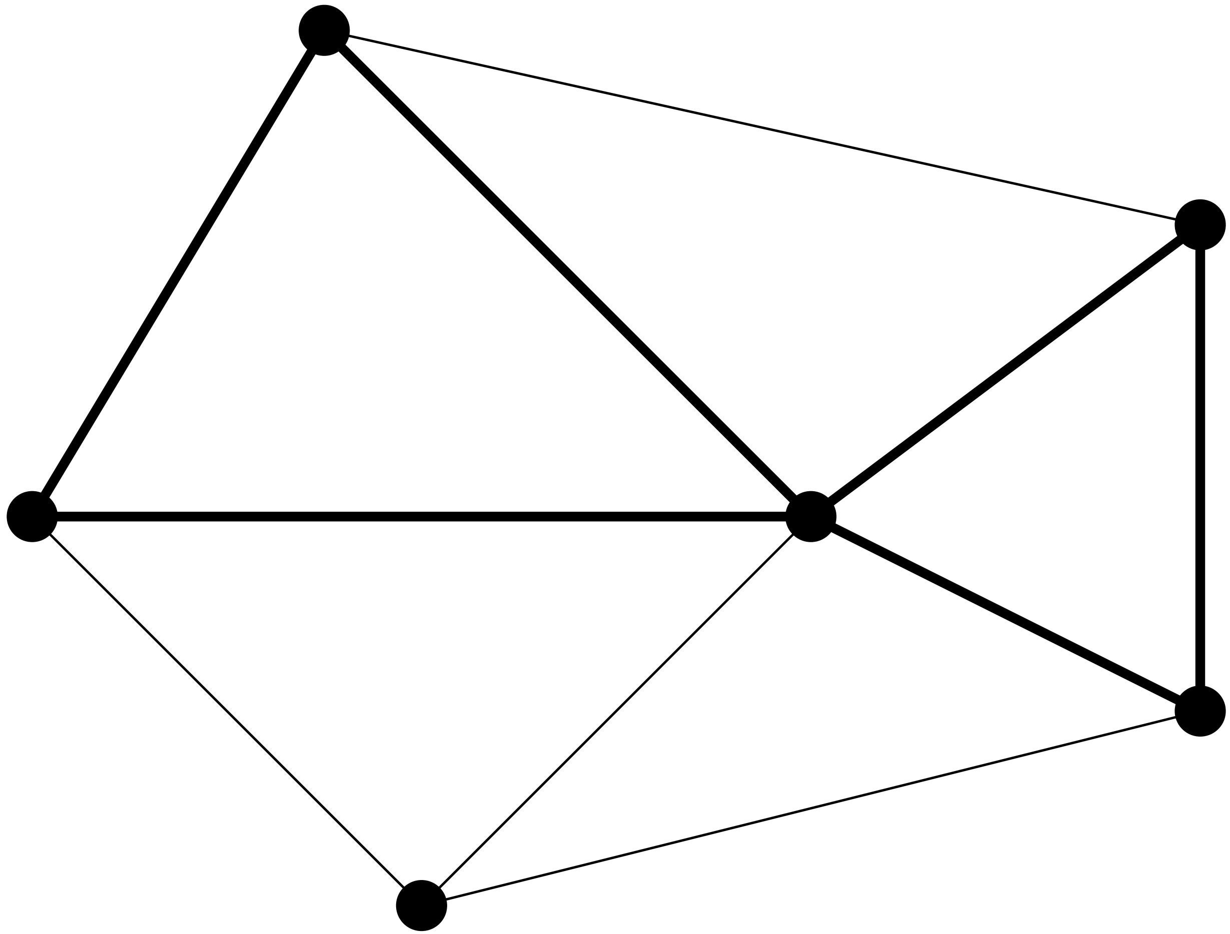
Voronoi and Delaunay polytopes

Empty spheres



Voronoi and Delaunay polytopes

Delaunay polytopes



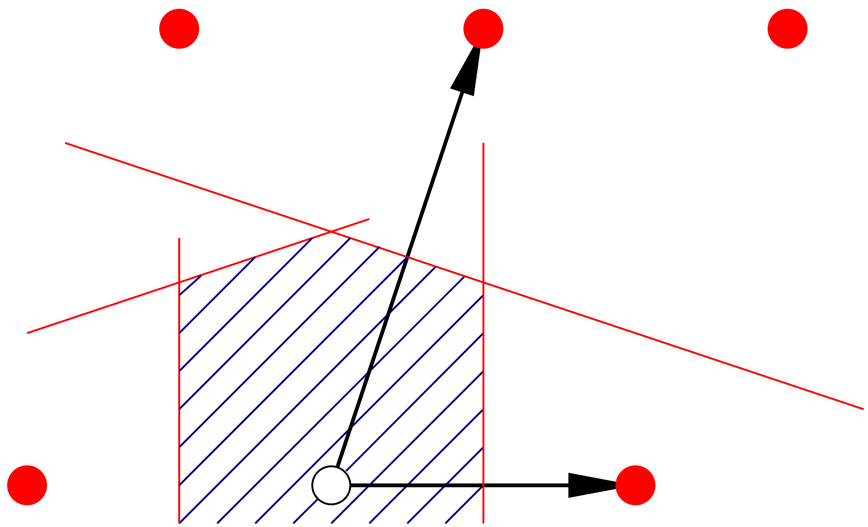
I. Delaunay polytopes

in

lattices

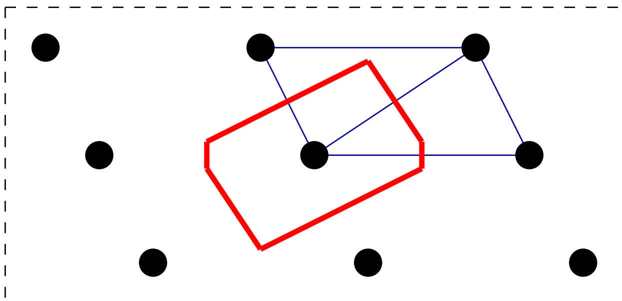
The Voronoi polytope of a lattice

- A Lattice L is a rank n subgroups of $\mathbb{R}^n$, i.e. of the form
$$L = v_1\mathbb{Z} + \cdots + v_n\mathbb{Z}$$
- The Voronoi cell $\mathcal{V}$ of L is defined by $\langle x, v\rangle \leq \frac{1}{2}||v||^2$ for $v \in L$.
- $\mathcal{V}$ is a polytope, i.e. it has a finite number of vertices (of dimension 0), faces and facets (of dimension $n-1$)



Voronoi and Delaunay in lattices

- Vertices of Voronoi polytope are center of empty spheres which defines Delaunay polytopes.
- Voronoi and Delaunay polytopes define dual tesselations of the space $\mathbb{R}^n$ by polytopes.
- Every k-dimensional face of a Delaunay polytope is orthogonal to a $(n-k)$-dimensional face of a Voronoi polytope.



- Given a lattice L, it has a finite number of orbits of Delaunay polytopes under translation.

Lattices with two Delaunay polytopes

- Take $L = \mathbb{Z}^n$; Delaunay:

Name	Center	Nr. vertices	Radius
Cube	$(\frac{1}{2})^n$	2^n	$\frac{1}{2}\sqrt{n}$

- Take $D_n = \{x \in \mathbb{Z}^n \,|\, \sum_{i=1}^n x_i$ is even$\}$; Delaunay:

Name	Center	Nr. vertices	Radius
Half-Cube	$(\frac{1}{2})^n$	$\frac{1}{2}2^n$	$\frac{1}{2}\sqrt{n}$
Cross-polytope	$(1, 0^{n-1})$	$2n$	1

- Take $E_8 = D_8 \cup (\frac{1}{2}^8) + D_8$; Delaunay:

Name	Center	Nr. vertices	Radius
Simplex	$(\frac{5}{6}, \frac{1}{6}^7)$	9	$\sqrt{\frac{8}{9}}$
Cross-polytope	$(1, 0^7)$	16	1

Gram matrix and lattices

- What really matters for lattice is their isometry class, i.e., if u is an isometry of $\mathbb{R}^n$ then the lattices L and uL have the same geometry.
- Denote $S^n_{>0}$ the cone of real symmetric positive definite $n \times n$ matrices and $S^n_{\geq 0}$ the positive semidefinite ones.
- Lattice L spanned by $v_1, \dots, v_n$ corresponds to

$$G_v = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0} \ .$$

  G_v depends only on the isometry class of L.
- Given $M \in S^n_{>0}$, one can find vectors $v_1, \dots ; v_n$ such that $M = G_v$.

Gram matrix and lattices

- Two matrices M, M' are arithmetically equivalent if there exist $P \in GL_n(\mathbb{Z})$ such that

$$M' = PM^tP\ .$$

- For any two basis of a lattice L, G_v and $G_{v'}$ are arithmetically equivalent.
- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- Algebraically $S^n_{>0} = O_n(\mathbb{R}) \backslash GL_n(\mathbb{R})$ and lattices are identified with $S^n_{>0}/GL_n(\mathbb{Z})$
- In practice it is preferable to think and draw in terms of lattices, but to compute in terms of matrices $A \in S^n_{>0}$.

II. Computational techniques

Search of Delaunay polytopes

- Given a matrix $M \in S^n_{>0}$, we want to compute its Delaunay Decomposition.
- We want to use the symmetries of M.
- In all cases considered M will be integral, but the algorithms do not depend significantly on this.

Closest Vector Problem

- Given a lattice L, a vector c, find all vectors $v \in L$ such that

$$||v - c|| \text{ is minimal}$$

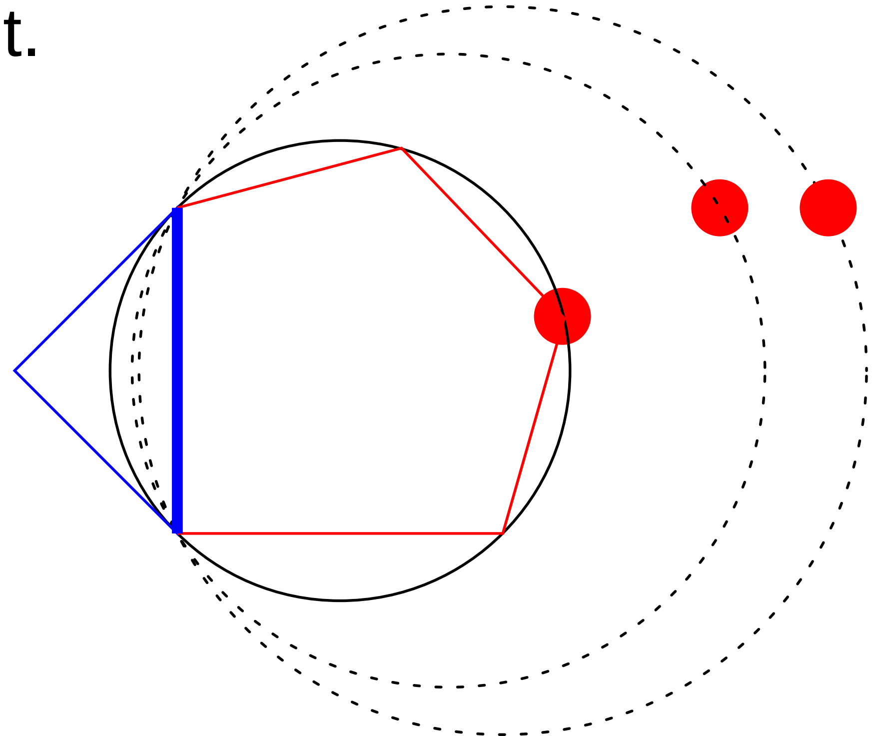
or in other term, if $M \in S^n_{>0}$ and $c \in \mathbb{R}^n$, find all $v \in \mathbb{Z}^n$ such that

$${}^t(v - c)M(v - c) \text{ is minimal}$$

- CVP is conjecturally a NP problem.
- Only way is to do an exhaustive search in a set of possible solutions, two programs:
  - Lattice-CVP (Dutour) use a hypercube, performing well up to dimension 10.
  - Voro (Vallentin) use an ellipsoid, performing well up to dimension, say 40.

Finding Delaunays

- Given a Delaunay polytope and a facet of it, there exist a unique adjacent Delaunay polytope.
- We use an iterative procedure:
  - Select a point outside the facet.
  - Create the sphere around it.
  - If there is no interior point finish, otherwise rerun with this point.



Finding Delaunay decomposition

- Find the isometry group of the lattice (program autom by Plesken & Souvignier).
- Find an initial Delaunay polytope (program finddel) by Vallentin and insert into list of orbits as undone.
- Iterate
  - Find the orbit of facets of undone Delaunay polytopes (GAP + lrs by Avis + Recursive Adjacency Decomposition method by Dutour).
  - For every facet, find the adjacent Delaunay polytope.
  - For every Delaunay test if they are isomorphic to existing ones. If not insert them to the list as undone.
  - Finish when every orbit is done.

Automorphism groups

- A Delaunay polytope P has two automorphism groups
  - The group $Isom(P)$ of isometries preserving the Delaunay.
  - The group $Aut(P)$ of lattice automorphism preserving the Delaunay.
- $Isom(P)$ is useful for computing the facet description.
- We need $Aut(P)$ for the computation of the Delaunay decomposition.
- We consider centers $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ with $c_i \in [0, 1[$.

$$Aut(P) = \mathsf{Stabilizer}(\mathsf{GRP}, \mathsf{c}, \mathsf{ActionMod1})$$

The trouble is that matrix actions are not easy: the above operation generates the full orbit.

Automorphism groups

- $Isom(P)$ contains $Aut(P)$. If $(v_i)_{1\leq i\leq N}$ are the vertices of P, then $Isom(P)$ is the group of permutations σ of N elements such that

$$||v_i - v_j|| = ||v_{\sigma(i)} - v_{\sigma(j)}||$$

  This group is "in general" easy to compute.
- If the index of P in L is 1, then $Isom(P) = Aut(P)$.
- The isomorphism question is treated similarly.

Some strategies

The computation of $Isom(P)$ is almost always relatively easy, then we can:

- Compute the intersection

  $$Aut(P) = Isom(P) \cap Aut(L)$$

  Well suited if $Aut(P)$ is a group of small index in $Isom(P)$.
- Iterate over all elements of $Isom(P)$, and select the ones which correspond to a matrix with integral coefficients. This yields $Aut(P)$.
  This strategy is well suited if $Isom(P)$ is small (say, 10000 elements)

Iterated stabilizer method

- Compute the center $c = (c_1, \ldots, c_n)$ and denote D the smallest integer such that $Dc \in \mathbb{Z}^n$.
- For every divisor D' of D, we can reduce the center modulo $\frac{1}{D'}$, the action is now modulo $\frac{1}{D'}$.
  Denote the stabilizer $Stab_{D'}$ of this action. Then $Stab(c)$ is a subgroup of $Stab_{D'}(c)$.
- The strategy is now to consider a series of divisors

$$D = D_1 > D_2 > \cdots > D_p = 1$$

and an associated series of stabilizers

$$Stab_{D_1}(c) \supseteq Stab_{D_2}(c) \supseteq \cdots \supseteq Stab_{D_p}(c)$$

$Stab_{D_i}$ is computed from $Stab_{D_{i-1}}$

Iterated stabilizer method

- The size of the orbit generated internally is

$$|Stab_{D_{i-1}}|/|Stab_{D_i}|$$

So, we might avoid the memory explosion, since the orbit generated are smaller.

- But there are several sequence $(D_1, \dots, D_p)$ possible. We do not know how to choose the one with the smallest orbits.

Autom method

- The program autom by Plesken & Souvignier can compute the group of matrices $P \in GL_n(\mathbb{Z})$ satisfying

$$PM_i{}^tP = M_i$$

with M_i some symmetric positive definite matrices.

- Given a Gram matrix $M \in S^n_{>0}$ of the lattice and a center c of a Delaunay, form the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & M^{-1} \end{pmatrix} \in M_{n+1,n+1}$$

- For the center c, form the matrix

$$B = ({}^tc')(c') \text{ with } c' = (1, c_1, \dots, c_n)$$

Autom method

- The group $Aut(P)$ is the automorphism group of the family $(A, B + A)$.
- All automorphism techniques have their isomorphism equivalent.
- Many problems happens:
  - We do not know a priori, which technique is the best. ➠ Use of heuristics in the computation.
  - In general this is the most difficult part of the Delaunay Decomposition computation.
  - For some lattices (like Coxeter lattices) it is best to use ad hoc methods.

Computing dual description

- cdd and lrs are general purpose programs for finding dual descriptions, which does not work for some polytopes.
- For symmetric convex cones, it suffices to compute orbits of facets
- The key idea of the Adjacency Decomposition Method is:
  - compute some initial facet (by linear programming) and insert the orbit into the list of orbits.
  - compute the adjacent facets to this facet (this is a dual description computation) and insert them into the list of orbits if they are new.
  - the algorithm finish when all orbits are finished.

Computing dual description

- The algorithm provides an improvement over a straightforward application of cdd and lrs
- Technically, we represent the group as permutation group on the extreme rays. Then, we use two following functions

Stabilizer(GroupExt, ListInc, OnSets);

RepresentativeAction(GroupExt, ListInc1, ListInc2, OnSets);

The important thing is to use the action OnSets, which is extremely efficient and uses backtrack search, i.e. in practice we never build the full orbit.

Further strategies

- Using the Adjacency Decomposition method we can usually find a conjecturally complete list of facets. However in many cases, there remain a few orbits of facets that are particularly difficult to compute.
- If the number of untreated orbits is lower than $n-1$, then we can use following theorem and conclude. Balinski theorem The skeleton of a n-dimensional polytope is n-connected, i.e. the removal of any set of $n-1$ vertices leaves it connected.
- Otherwise, we can apply the Adjacency decomposition method to the remaining orbits of facets. This strategy is Recursive Adjacency Decomposition method

Banking methods

- When one applies the Adjacency decomposition method, recursively, we can met some identical facets several times.
- The idea is to store the dual description of facets in a bank and when a computation happen to make call to that bank to see if it already done.
- So, one wants to compute dual description of some faces of a polyhedral cone. The key point is that this computation is intrisic, i.e. independent over what polytope the face belong to.

Coxeter lattices

- The lattice A_n is defined as

$$A_n = \{x \in \mathbb{Z}^{n+1} \text{ such that } \sum x_i = 0\}$$

- If r divides $n+1$, then writes $q = \frac{n+1}{r}$ and define the lattice A_n^r by

$$A_n^r = A_n \cup v_{n,r} + A_n \cup \ldots \cup (r-1)v_{n,r} + A_n$$

with

$$v_{n,r} = \frac{1}{r}\sum_{i=1}^{n+1} e_i - \sum_{i=1}^{q} e_i$$

- The dual of A_n^r is A_n^q.

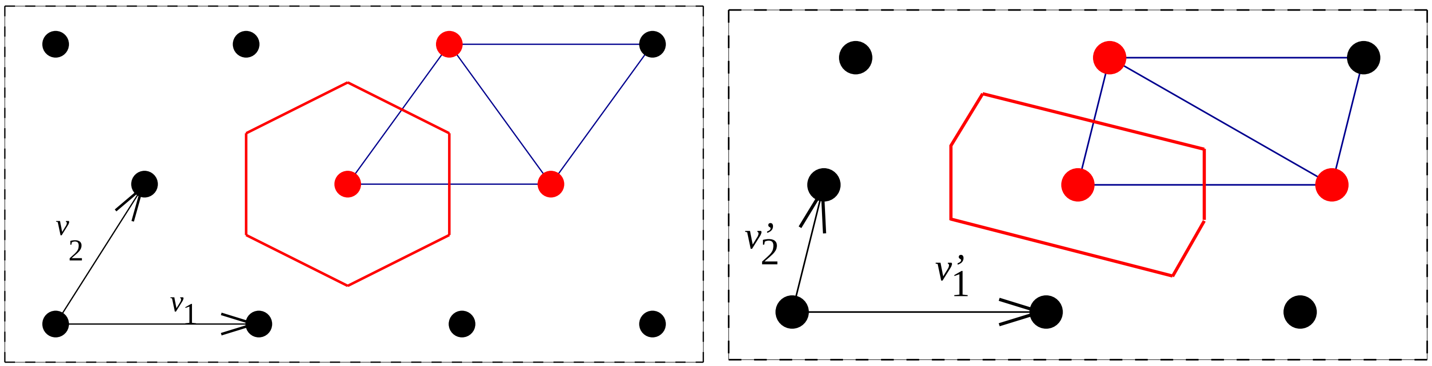
Coxeter lattices

- The Closest Vector problem for A_n^r is easy to solve.
- $A_8^3 = E_8$, $A_7^2 = E_7$.
- If $n \geq 9$, then the automorphism of A_n^r is $\pm I_n \times Sym(n+1)$.
- As a consequence the Isomorphism and Automorphism problem of Delaunay polytopes of those lattices is very easy to solve.
- For example A_{21}^2 has 21 orbits, one orbit is formed of Delaunay polytopes with 40698 vertices.
- Is there a general analytical description of the Delaunay decomposition of A_n^r?

III. L-type domain

L-type domains

- Take a lattice L and select a basis $v_1, \dots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



are part of the same L-type domain.

- A L-type domain is the assignment of Delaunay polytopes, so it is also the assignment of the Voronoi polytope of the lattice.

Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$||w_i - c|| = r \text{ i.e. } w_i^T M w_i - 2 w_i^T M c + c^T M c = r^2$$

- Substracting one obtains

$$\{w_i^T M w_i - w_j^T M w_j\} - 2\{w_i^T - w_j^T\} M c = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into linear inequalities on M.

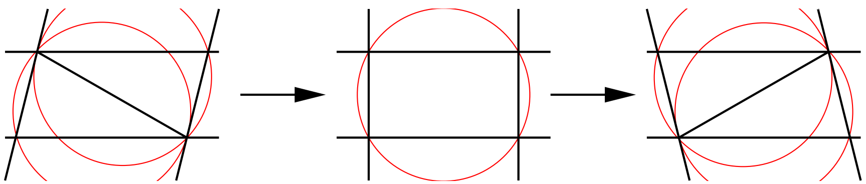
Defining inequalities

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding L-type is of dimension $\frac{n(n+1)}{2}$, they are called primitive
- A primitive L-type domain is essentially the data of all its defining simplices.
- Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. Denote $S(c, r)$ the sphere around V. If one writes $w = \sum_{i=1}^n \lambda_i v_i$ with $1 = \sum_{i=1}^n \lambda_i$, then one has

$$||w - c|| \geq r^2 \Leftrightarrow {}^t w M w - \sum_{i=1}^n {}^t v_i M v_i \geq 0$$

Equivalence and enumeration

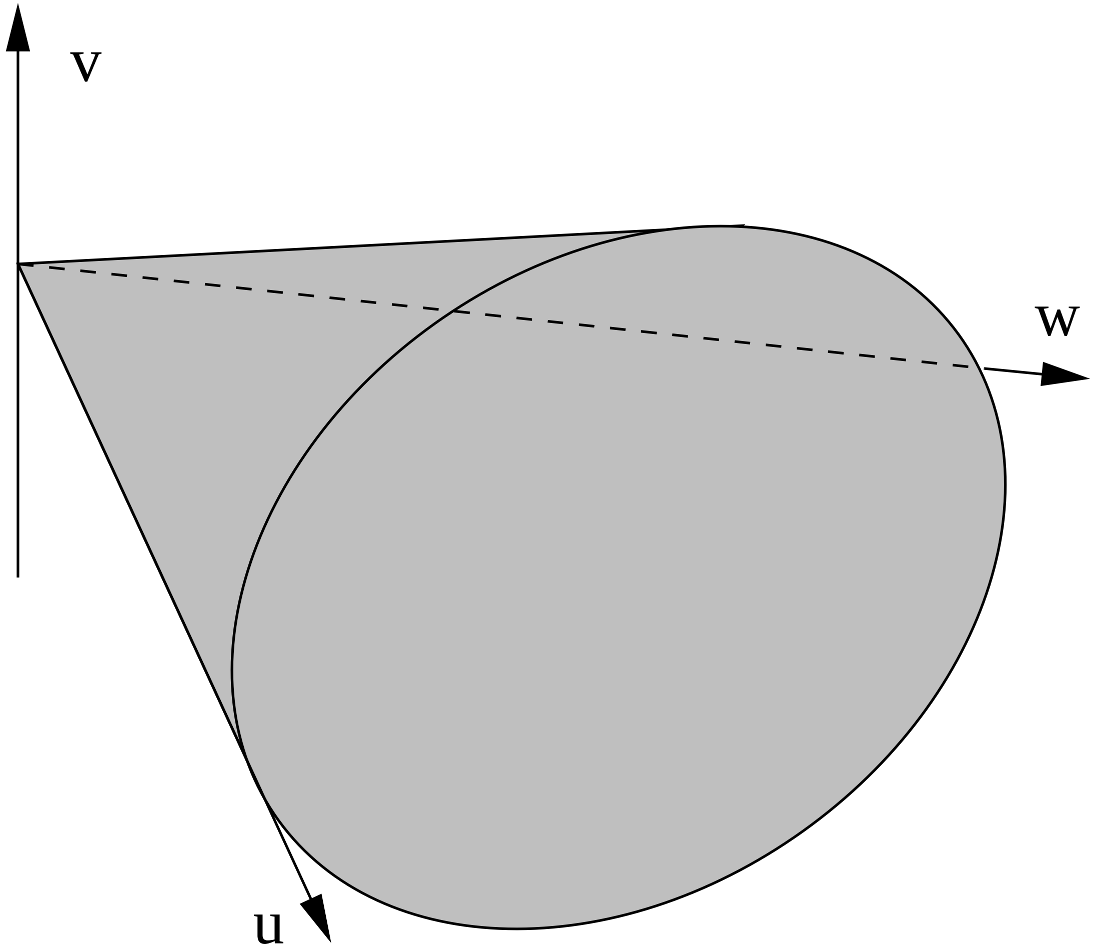
- Voronoi's theorem The inequalities obtained by taking adjacent simplices suffice to describe all inequalities.
- The group $GL_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive L-type domains.
- Voronoi proved that after this action, there is a finite number of primitive L-type domains.
- Bistellar flipping creates new triangulation. In dim. 2:



- Enumerating primitive L-types is done classicaly:
  - Find one primitive L-type domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.

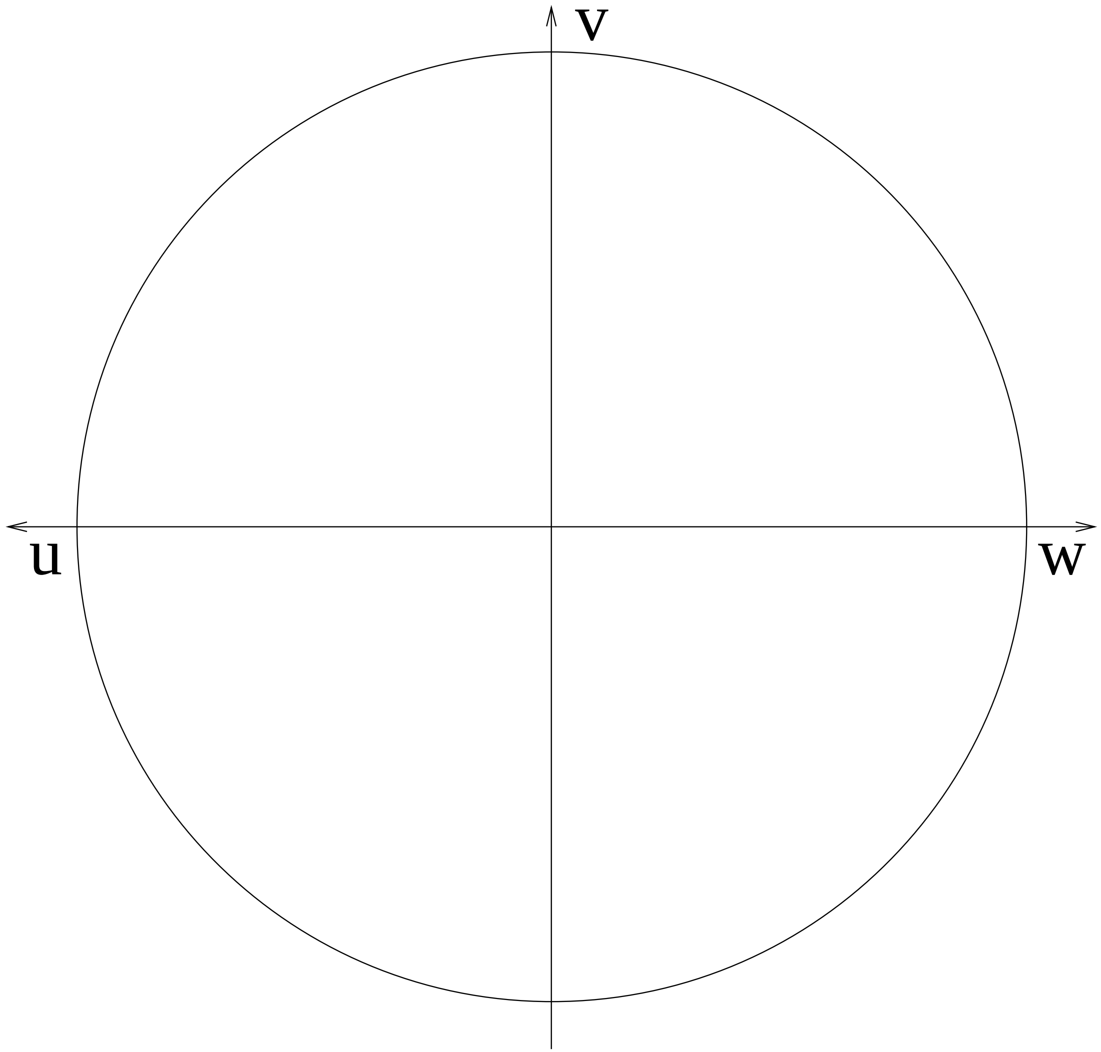
The partition of $S^2_{>0} \subset \mathbb{R}^3$

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



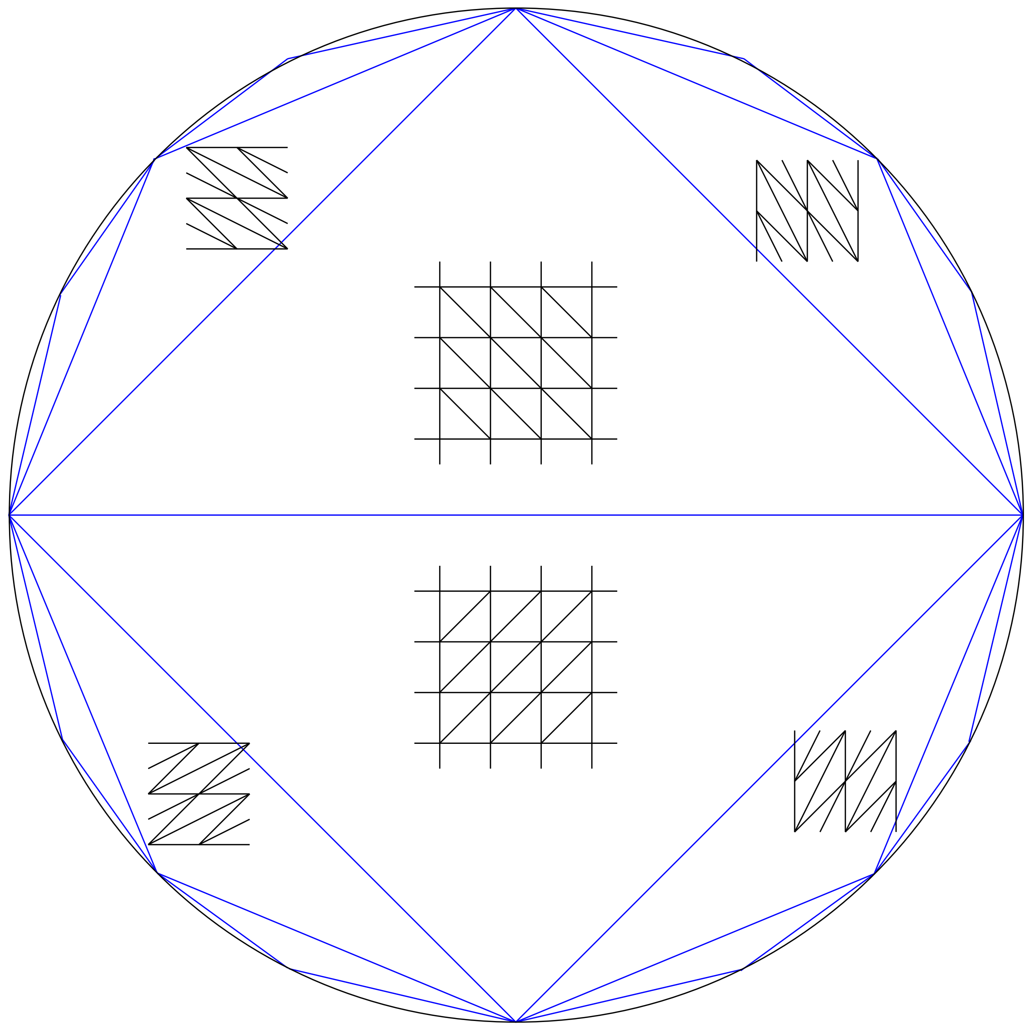
The partition of $S^2_{>0} \subset \mathbb{R}^3$

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{>0} \subset \mathbb{R}^3$

Primitive L-types in $S^2_{>0}$:



Rational boundary, tiling property

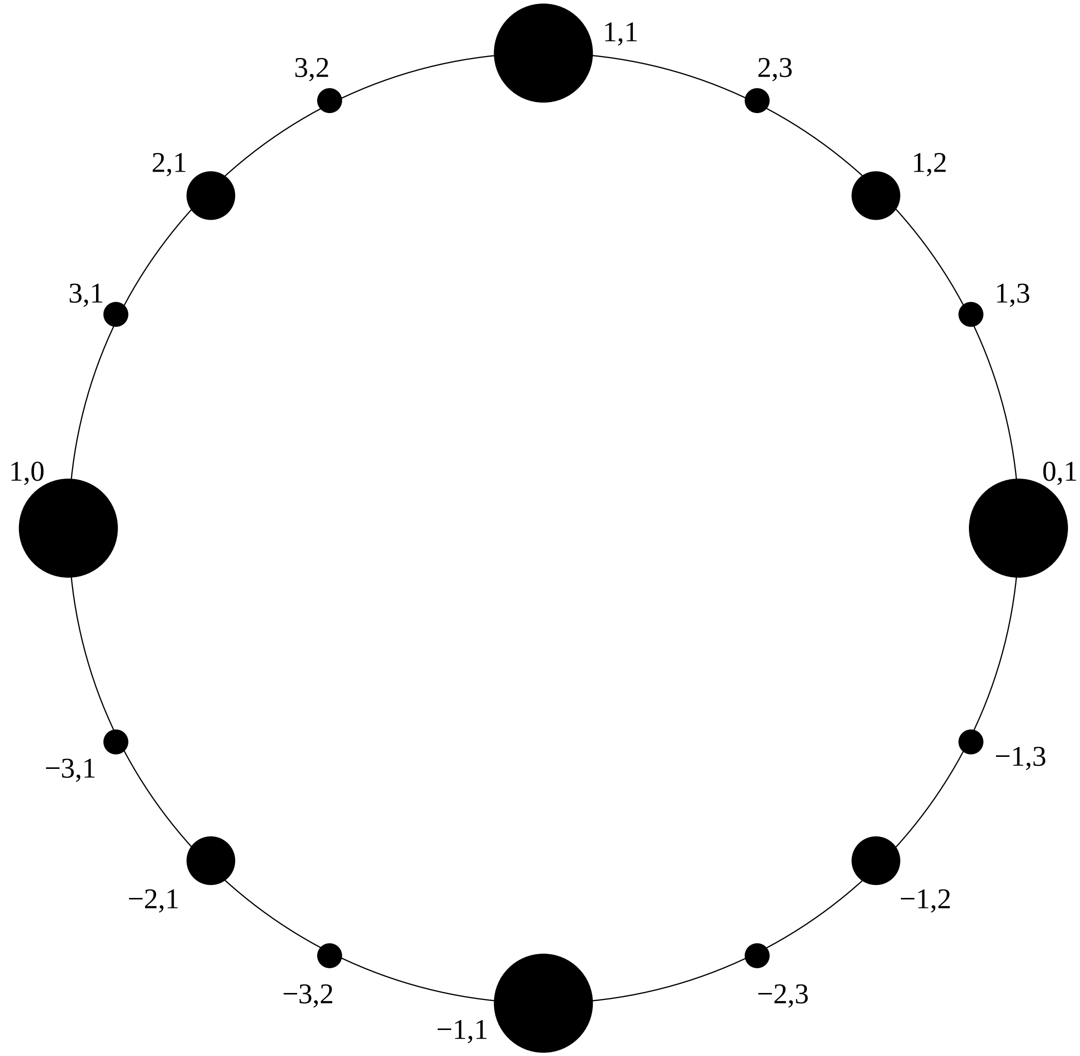
- A matrix A belongs to the rational boundary of $S^n_{>0}$ if there exist $d \leq n$, $B \in S^d_{>0}$ and $P \in GL_n(\mathbb{Z})$ such that

$$PA^tP = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

  This rational boundary is denoted $S^n_{\geq 0(rat)}$
- Voronoi's theorem: $S^n_{\geq 0(rat)} = Cone\{v^t v \mid v \in \mathbb{Z}^n\}$
- If a symmetric matrix A satisfies all linear inequalities defining a L-type domain, then in fact, $A \in S^n_{\geq 0(rat)}$.
- The L-type domains form a face-to-face tiling of $S^n_{\geq 0(rat)}$.

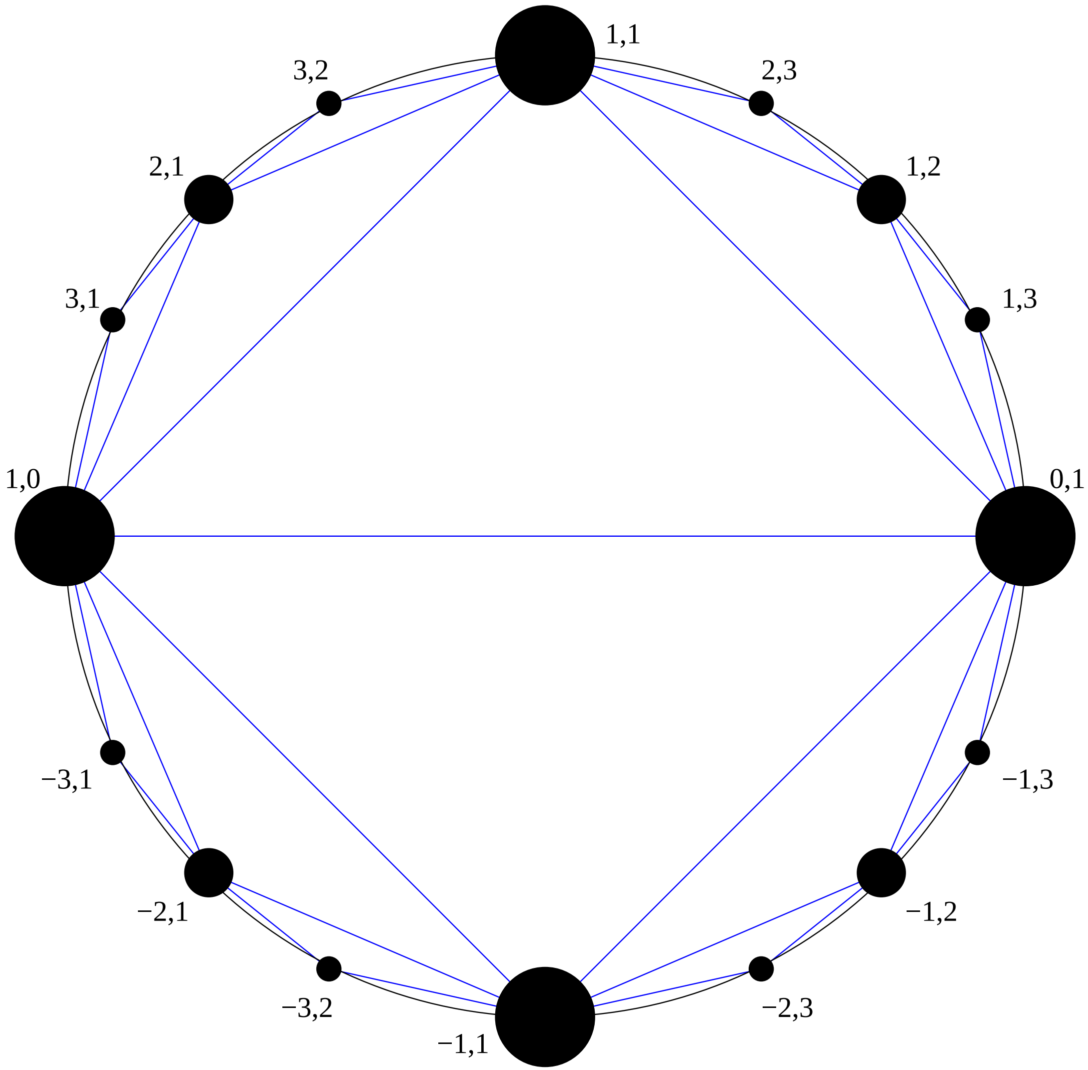
The rational boundary $S^2_{>0(rat)}$

All rank 1 matrices $v^t v$ with $v \in \mathbb{Z}^2$:

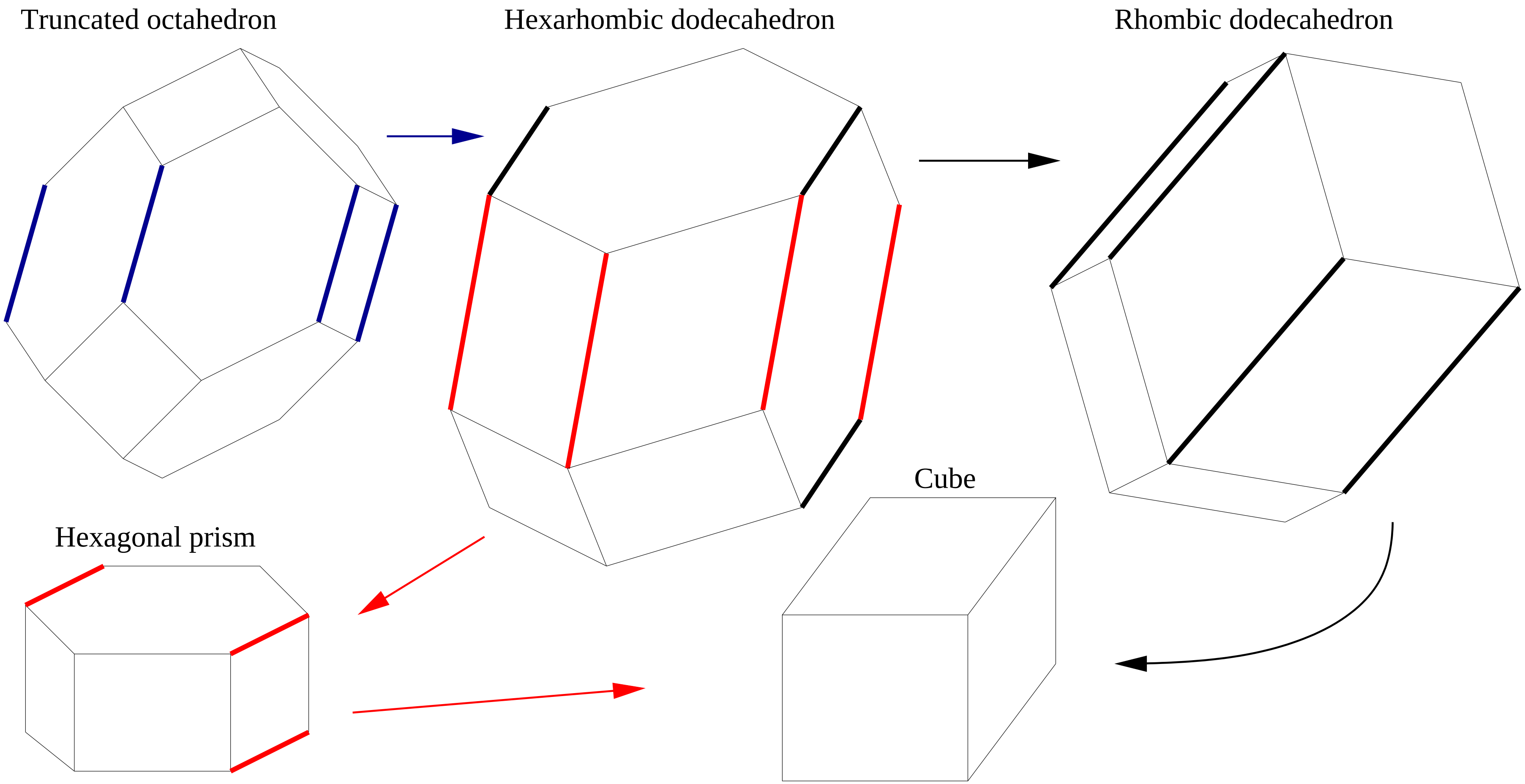


The rational boundary $S^2_{>0(rat)}$

Primitive L-type tiling of $S^2_{>0(rat)}$



3-dimensional Voronoi polytopes



Rigid lattices

A lattice is rigid (notion introduced by Baranovski & Grishukhin) if its L-type domain has dimension 1.

- One rigid in dimension 1: $\mathbb{Z}$.
- No rigid lattices in dimension 2 and 3.
- one rigid lattice in dimension 4: it is D_4.
- (Baranovski & Grishukhin, Engel) 7 rigid lattices in dimension 5.
- Dutour & Vallentin: In dimension 6, we obtained 25263 rigid lattices. Probably many more.

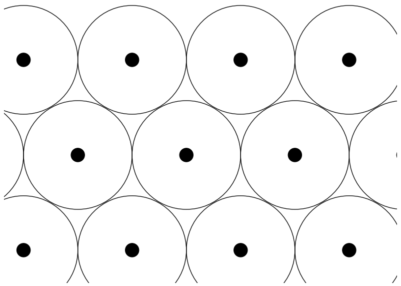
Enumeration of L-types

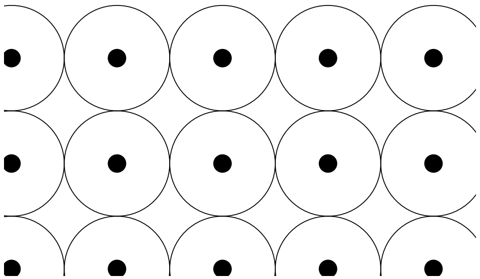
Dimension	Nr. L-type	Nr. primitive	Nr rigid lattices
1	1	1	1
2	2	1	0
3	5 Fedorov	1 Fedorov	0
4	52 DeSh	3 Voronoi	1
5	179377 Engel	222 BaRy, Engel & Gr	7 $\uparrow$ BaGr
6	?	$\geq 2.5.10^6$ Engel, Va	$\geq 2.10^4$ DuVa
7	?	?	?

IV. Covering and optimization

Lattice packings

- We consider packing by n-dimensional balls of the same radius, whose center belong to a lattice L.





- The packing density has the expression

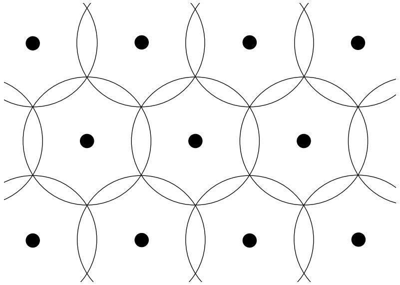
$$\alpha(L) = \frac{\lambda(L)^n \kappa_n}{det(L)} \leq 1$$

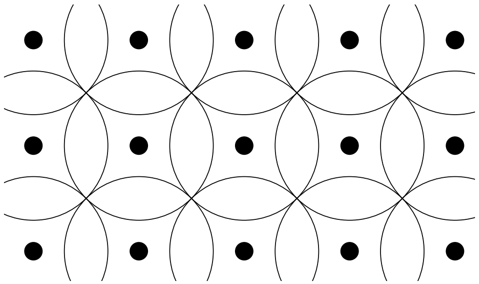
with κ_n being the volume of the unit ball B^n and

$$\lambda(L) = \frac{1}{2} \min_{v \in L - \{0\}} ||v||$$

Lattice covering

- We consider covering of $\mathbb{R}^n$ by n-dimensional balls of the same radius, whose center belong to a lattice L.





- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \kappa_n}{det(L)} \geq 1$$

with $\mu(L)$ being the largest radius of Delaunay polytopes and κ_n the volume of the unit ball B^n.

- Objective is to minimize $\Theta(L)$. Solution for $n \leq 5$: A_n^*.

Lattice packing-covering

L is a n-dimensional lattice.

- We want a lattice, such that the sphere packing (resp, covering) obtained by taking spheres centered in L with maximal (resp, minimal) radius are both good.
- The quantity of interest is

$$\frac{\Theta(L)}{\alpha(L)} = \{\frac{\mu(L)}{\lambda(L)}\}^n \geq 1$$

- Lattice packing-covering problem: minimize $\frac{\Theta(L)}{\alpha(L)}$.

Dimension	Solution		
2	A_2^*	4	H_4 (Horvath lattice)
3	A_3^*	5	H_5 (Horvath lattice)

Optimization problem

We want to find the best lattice packing, covering, packing-covering.

- The lattice packing problem is solved by the theory of perfect forms and perfect domain. See “premier mémoire” by Voronoi (1908) and book by Martinet for the search of optimal lattice packings.
- Thm. Given a L-type domain LT, there exist a unique lattice, which minimize the covering density over LT.
- Thm. Given a L-type domain LT, there exist a lattice (possibly several), which minimize the packing-covering density over LT.
- See “*Semidefinite programming approaches to lattice packing and covering problems*” by Schürmann & Vallentin

Radius of Delaunay polytope

- Fix a primitive L-type domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \dots, D_m$.
- Thm. For every $D_i = Conv(0, v_1, \dots, v_n)$, the radius of the Delaunay polytope is at most 1 if and only if

$$\begin{pmatrix} 4 & \langle v_1, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_n, v_n\rangle \\ \langle v_1, v_1\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\ \langle v_2, v_2\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_n\rangle & \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \dots & \langle v_n, v_n\rangle \end{pmatrix} \in S^{n+1}_{\geq 0}$$

  by Delone, Dolbilin, Ryshkov and Shtogrin.
- The condition is a semidefinite condition.

Covering problem

- Fix a primitive L-type domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \ldots, D_m$.
- Minkowski The function $-\log\det(M)$ is strictly convex on $S^n_{>0}$.
- Solve the problem
  - M in the L-type (linear condition),
  - the Delaunay D_i have radius at most 1 (semidefinite condition),
  - minimize $-\log\det(M)$ (strictly convex).
- The above problem is solved by the interior point methods implemented in MAXDET by Vandenberghe, Boyd & Wu. Unicity comes from the strict convexity of the objective function.

Packing covering problem

- We fix a primitive L-type domain.
- A shortest vector is an edge of a Delaunay. So, from the Delaunay decomposition, we know which vectors $v_1,\dots,$ v_p can be shortest.
- We consider the problem on $(M, m) \in S^n_{>0} \times \mathbb{R}$
  - M belong to the L-type domain (linear constraint)
  - all Delaunay have radius at most 1 (semidefinite condition)
  - $m \leq ||v_j||^2 = v_j^t M v_j$ for all i (linear constraint)
  - maximize m.
- The maximal value of m gives the maximal length of shortest vector and so the best packing-covering over a specific primitive L-type domain. A priori no unicity.

V. L-types of $S^n_{>0}$-spaces

$S^n_{>0}$-spaces

- A $S^n_{>0}$-space is a vector space $\mathcal{SP}$ of S^n, which intersect $S^n_{>0}$.
- We want to describe the Delaunay decomposition of matrices $M \in S^n_{>0} \cap \mathcal{SP}$.
- Motivations:
  - The enumeration of L-types is done up to dimension 5, perhaps possible for dimension 6 but certainly not for higher dimension.
  - We hope to find some good covering, and packing-covering by selecting judicious $\mathcal{SP}$. This is a search for best but unproven to be optimal coverings.
- A L-type in $\mathcal{SP}$ is an open convex polyhedral set included in $S^n_{>0} \cap \mathcal{SP}$, for which every element has the same Delaunay decomposition.

Rigidity and primitivity

- $(\mathcal{SP}, L)$-types form a polyhedral tessellation of the space $\mathcal{SP} \cap S^n_{>0}$.
- If $M \in \mathcal{SP} \cap S^n_{>0}$, then the rigidity degree of M is the dimension of the smallest L-type containing M, it is computed using the Delaunay decomposition of M.
- A $(\mathcal{SP}, L)$-type is primitive if it is full-dimensional in $\mathcal{SP}$.
- A $(\mathcal{SP}, L)$-type is rigid if it is one dimensional.
- Algorithm for finding a primitive $(\mathcal{SP}, L)$-type domain
  - Generate a random element in $S^n_{>0} \cap \mathcal{SP}$.
  - Compute its Delaunay decomposition.
  - Finish when the dimension of the $(\mathcal{SP}, L)$-type is maximal.

Testing equivalence of $(\mathcal{SP}, L)$-type

- Given a primitive $(\mathcal{SP}, L)$-type domain, find its extreme rays e_i and normalize the corresponding matrices by imposing that they have integer coefficients with $gcd = 1$.
- We associate to the $(\mathcal{SP}, L)$-types the matrix in $S^n_{>0} \cap \mathcal{SP}$: $M = \sum_i e_i$
- Two primitive $(\mathcal{SP}, L)$-type domains $\mathcal{LT}_1$ and $\mathcal{LT}_2$ are isomorphic if there a matrix P such that

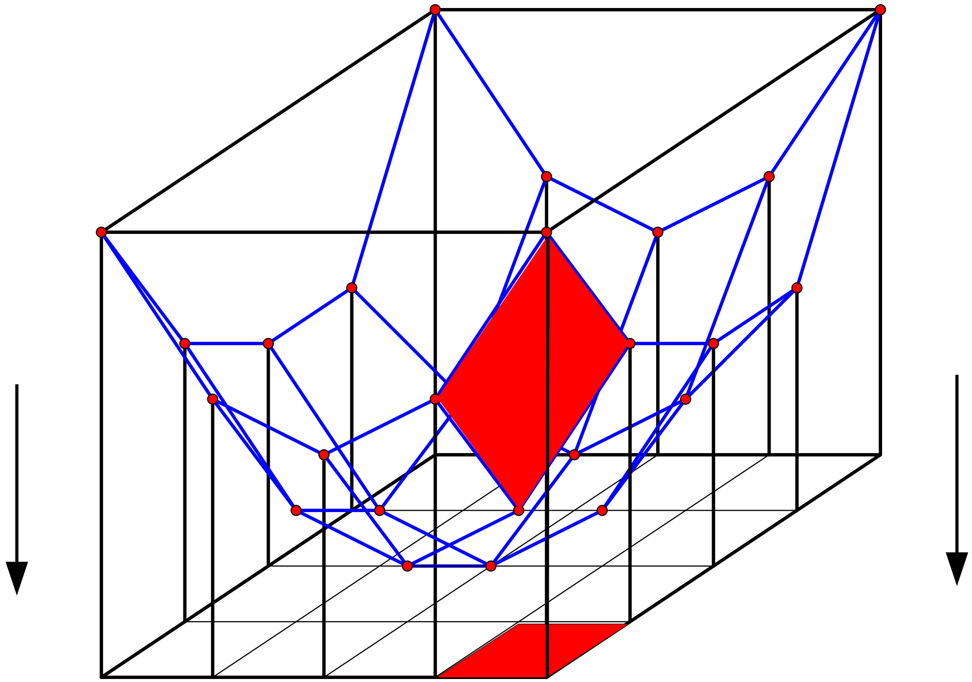
$$PM_1{}^tP = M_2 \text{ and } P\mathcal{SP}^tP = \mathcal{SP}$$

First equation is solved by program Isom and we iterate over the possible solutions for testing the second.

Lifted Delaunay decomposition

- The Delaunay polytopes of a lattice L correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertex-set:

$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{d+1} \ .$$

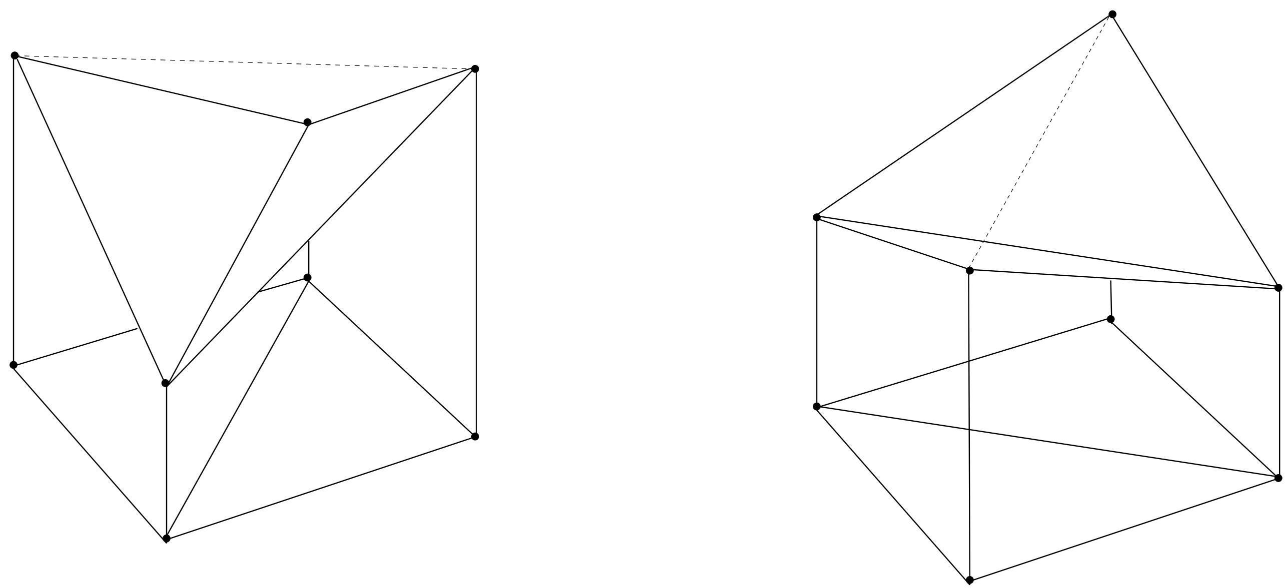


Flipping

- Take a primitive $(\mathcal{SP}, L)$-type domain with Delaunay polytopes $D_1, \dots, D_p$.
- If F is a facet of D_i and D' is the other Delaunay polytope, then it defines an inequality $f_{D',D_i}(M) \geq 0$. This form a finite set of defining inequalities of the $(\mathcal{SP}, L)$-type.
- We can extract relevant inequalities, which correspond to facets of the $(\mathcal{SP}, L)$-type. Select a relevant ineq. $f(M) \geq 0$.
- One has $f(M) = \alpha_1 f_{D_{j(1)},D'_1}(M) = \dots = \alpha_r f_{D_{j(r)},D'_r}(M)$ for some $\alpha_i > 0$ and some Delaunay D'_i adjacent to $D_{j(i)}$ on a facet F_i.
- If one moves to $f(M) = 0$, then all F_i disappear and the corresponding Delaunays merge.

Geometrical expression

- The “glued” Delaunay form a Delaunay decomposition for a matrix M in the $(\mathcal{SP}, L)$-type satisfying to $f(M) = 0$.
- The flipping break those Delaunays in a different way.
- Two triangulations of $\mathbb{Z}^2$ correpond in the lifting to:



- The polytope represented is called the repartitioning polytope.

Flipping of a repartitioning polytope

- Given a Delaunay decomposition D, the graph $G(D)$ is formed of all Delaunays with two Delaunay d_i, d_j adjacent if:
  - d_i and d_j share a facet
  - the inequality $f_{d_i,d_j}(M) = \alpha f(M)$ for $\alpha > 0$
- For every connected component C of this graph, the repartitioning polytope $R(P)$ is the polytope with vertex set

$$\{(v, {}^t v M v) \text{ with } v \text{ a vertex of a Delaunay of } C\}$$

- Combinatorially flipping correspond to switching from the lower facets to the higher facets of the lifted merging of Delaunay polytopes.

Enumeration technique

- Find a primitive $(\mathcal{SP}, L)$-type domain, insert it to the list as undone.
- Iterate
  - For every undone primitive $(\mathcal{SP}, L)$-type domain, compute the facets.
  - Eliminate redundant inequalities.
  - For every non-redundant inequality realize the flipping, i.e. compute the adjacent primitive $(\mathcal{SP}, L)$-type domain. If it is new, then add to the list as undone.

VI. Applications

Subgroups of $GL_n(\mathbb{Z})$

- A subgroup G of $GL_n(\mathbb{Z})$ is contained into a maximal finite subgroup.
- For every n, there is a finite number of maximal finite subgroup of $GL_n(\mathbb{Z})$ up to conjugacy.
- The actual enumeration of groups is done up to dimension 31 (Zassenhaus, Plesken, Pohst, Nebe).
- By finding conjugacy classes of subgroups of those maximal finite subgroups, we get a classification of all finite subgroups of $GL_n(\mathbb{Z})$.

Space of invariant forms

- Given a subgroup G of $GL_n(\mathbb{Z})$, define

$$\mathcal{SP}(G) = \left\{ X \in S^n \text{ such that } gX^tg = X \text{ for all } g \in G \right\}$$

- Given a $S^n_{>0}$-space $\mathcal{SP}$, define

$$Aut(\mathcal{SP}) = \left\{ \begin{array}{c} g \in GL_n(\mathbb{Z}) \text{ such that} \\ gX^tg = X \text{ for all } X \in \mathcal{SP} \end{array} \right\}$$

- For a maximal irreducible finite group, one has $dim\mathcal{SP}(G) = 1$.
- A Bravais group satisfies to $Aut(\mathcal{SP}(G)) = G$.

Equivariant L-type domains

- Equivariant L-type domains are L-types of a $S^n_{>0}$-space $\mathcal{SP}(G)$ for G Bravais.
- Thm. (Zassenhaus) One has the equality

$$\{g \in GL_n(\mathbb{Z}) \mid g\mathcal{SP}(G)^t g = \mathcal{SP}(G)\} = N_{GL_n(\mathbb{Z})}(G)$$

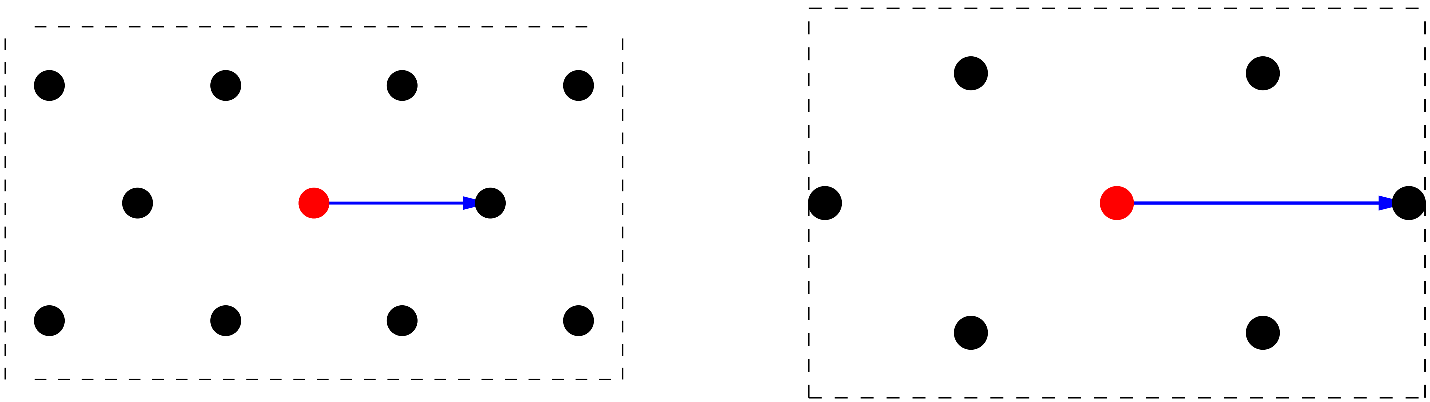
- Thm. For a given finite group $G \in GL_n(\mathbb{Z})$, there are a finite number of equivariant L-types under the action of $N_{GL_n(\mathbb{Z})}(G)$.
- $\mathcal{SP}(G)$ is defined by rational equations. If a T-space $\mathcal{SP}$ is defined by rational equations, does it has a finite number of L-types under $Aut(\mathcal{SP})$?

Small dimensions

- Dimension 6:
  - Vallentin found a better lattice covering than A_6^* in the vicinity of E_6^*.
  - No better in Bravais groups of rank 4.
- Dimension 7:
  - Vallentin & Schürmann found a better lattice covering than A_7^* in the vicinity of E_7^*.
  - No better in Bravais groups of rank 4.
- Dimension 8:
  - Vallentin & Schürmann proved that E_8 is not a local optimum of the covering density.
  - Conjecture (Zong) E_8 is the best lattice packing-covering in dimension 8.

Extension of Coxeter lattices

- Anzin & Baranovski computed the Delaunay decompositions of the lattices A_9^5, A_{11}^4, A_{13}^7, A_{14}^5, A_{15}^8 and found them to be better coverings than A_n^*.
- We do extension along short vectors

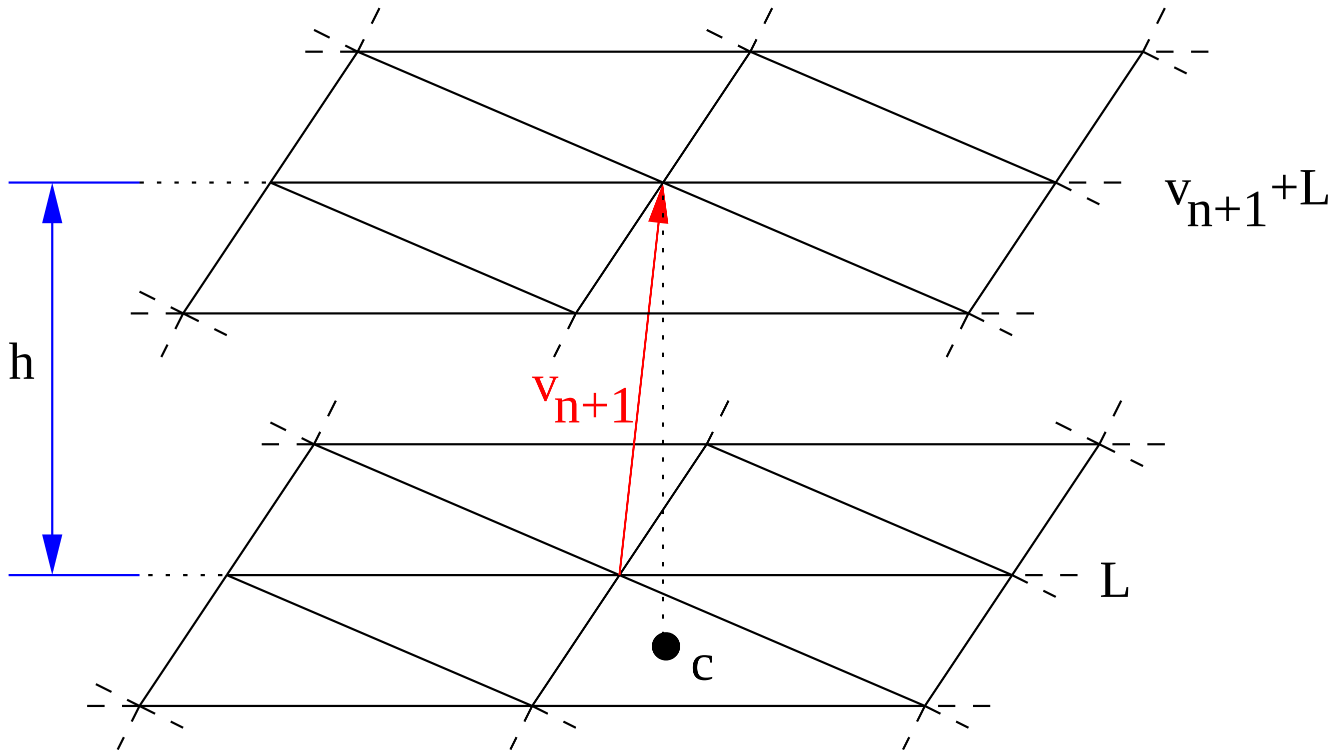


or compute in the Bravais space of short vectors.

- We manage to find record coverings in dimension 9, 11, 13, 14 and 15.

Lamination

- Given a n-dim. lattice L, create a $n+1$-dim. lattice L':



- The point c is fixed and we adjust the value of h.
- This defines a T-space.

Lamination

- In terms of Gram matrices

$$Gram(L) = A \text{ and } Gram(L') = \begin{pmatrix} A & A^t c \\ cA & \alpha \end{pmatrix}$$

  c is the projection of the vector $(0, \ldots, 0, 1)$ on the lattice L.

- The symmetries of L' are the symmetries of L preserving the center c and if $2c \in \mathbb{Z}^n$ the othogonal symmetry

$$\begin{pmatrix} I_n & 0 \\ 2c & -1 \end{pmatrix}$$

- c can be chosen as center of a Delaunay.

Lamination

- (Conway & Sloane) For the packing problem, one finds that the best lattice containing L as a section is defined by taking c to be a deep hole (i.e. Delaunay polytope of maximal radius). They obtain a family Λ_n of lattices.
- For the covering problem things are not so simple.
  - One cannot solve the general problem with c unspecified, since it has no symmetry and too much parameters
  - One restriction is to assume the value of c, this makes a rank 2 problem.
- Doing lamination over A_9^5 and A_{11}^4 one gets a record covering in dimension 10 and 12.

Best known lattice coverings

d	lattice	covering density Θ			
1	$\mathbb{Z}^1$	1	13	L_{13}^c	7.762108
2	A_2^*	1.209199	14	L_{14}^c	8.825210
3	A_3^*	1.463505	15	L_{15}^c	11.004951
4	A_4^*	1.765529	16	A_{16}^*	15.310927
5	A_5^*	2.124286	17	A_{17}^9	12.357468
6	L_6^c	2.464801	18	A_{18}^*	21.840949
7	L_7^c	2.900024	19	A_{19}^{10}	21.229200
8	L_8^c	3.142202	20	A_{20}^7	20.366828
9	L_9^c	4.268575	21	A_{21}^{11}	27.773140
10	L_{10}^c	5.154463	22	Λ_{22}^*	≤ 27.8839
11	L_{11}^c	5.505591	23	Λ_{23}^*	≤ 15.3218
12	L_{12}^c	7.465518	24	*Leech*	7.903536

VI. Single Delaunay

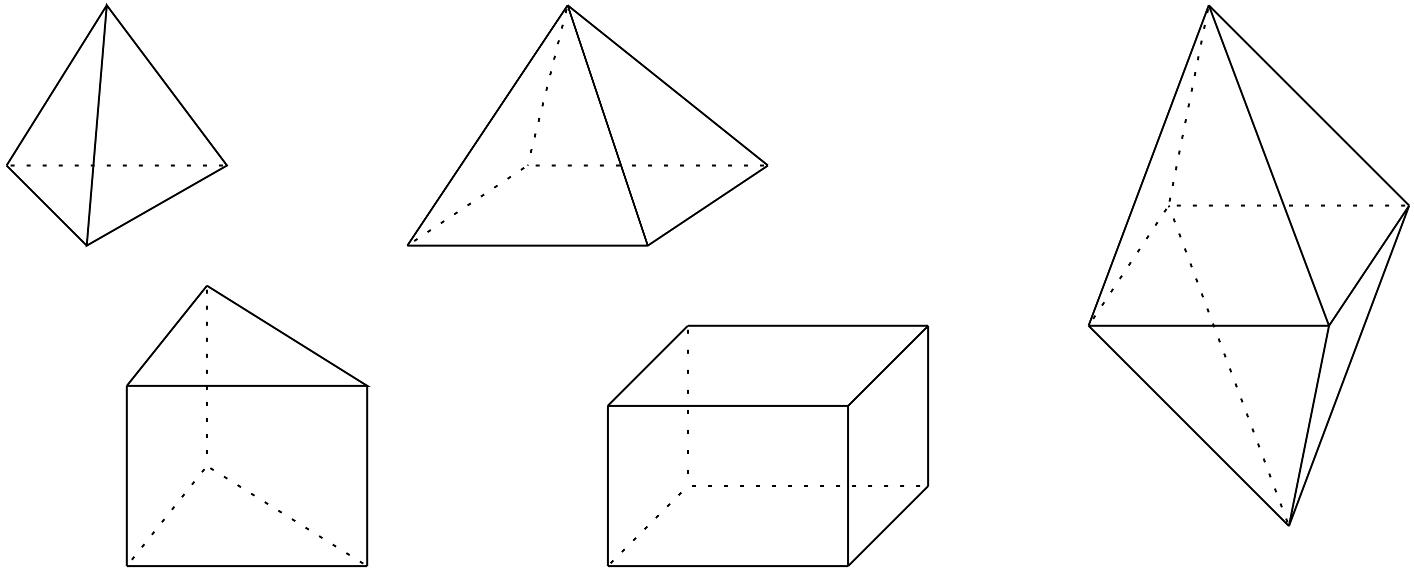
The problem

- We want to describe the possible Delaunays of a n-dimensional lattice.
- An affine basis of an n-dimensional polytope P is $\{v_0, \dots, v_n\}$ such that for every vertex v of P, there is

$$b_i \in \mathbb{Z}, \text{such that } b_0 + \dots + b_n = 1$$
$$\text{and } b_0 v_0 + b_1 v_1 + \dots + b_n v_n = v$$

- Not all Delaunay have an affine basis. All Delaunay polytopes of dimension ≤ 6 have an affine basis.
- We assume the existence of an affine basis.

Classification of Delaunays



dim	Nr of types		Computing time
2	2	Fedorov	
3	5	Fedorov	23s
4	19	Erdahl-Ryshkov	52s
5	138	Kononenko	5m
6	6241	Dutour	50h

Extreme Delaunay

- A Delaunay is extreme if it admits a unique lattice containing it, i.e. if the combinatorics determine the structure.
- Dutour: In dimension 6, the unique extreme Delaunay is Schlafli polytope, which is unique Delaunay of root lattice E_6.
- In dimension 7, conjecturally two Delaunay polytopes exist.
- Research in progress: In dimension 8, there is at least 27 such lattices.
- Research project: generalize the above theory to general Delaunay polytopes.

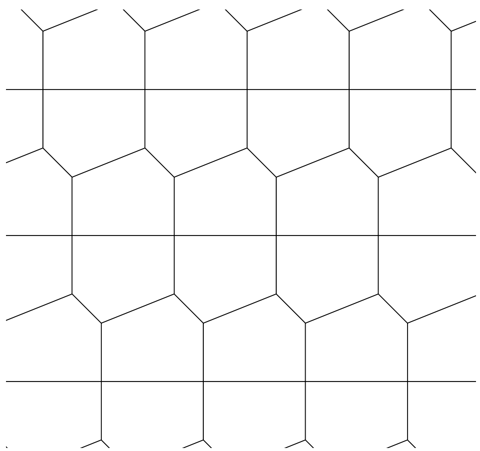
VII. Crystallographic structures

A generalization

- Given a lattice L, a crystallographic structure is a subset of $\mathbb{R}^n$ of the form

$$\mathcal{CS} = \Cup_{i=0}^{m} x_i + L$$

- For example if G is a Bravais group of $\mathbb{R}^n$ and x a point, then the orbit Gx is a crystallographic structure.
- A stereohedron is a polytope whose orbit under a space group form a face to face tiling of the space.



L-type theory, linear

- If one add a quadratic form to $\mathbb{R}^n$ then one can consider the Delaunay decomposition of $\mathcal{CS}$.
- The Delaunay decomposition can be computed with almost the same algorithms.
- Every Delaunay is circumscribed by an empty sphere. The condition that external point are outside the sphere translate into linear inequalities.
- One has a L-type theory for those sets.
- Search for periodic but non-lattice packings and coverings.

L-type theory, nonlinear

- Fix a quadratic form on $\mathbb{R}^n$ and make the x_i vary.
- The notion of L-type exist naturally in that context. They partition the parameter space.
- The theory obtained is no longer linear
- The key for a combinatorial attack is to have the solution of the problem: among polynomial inequalities

$$f_i(x) \geq 0 \text{ for } 1 \leq i \leq p$$

  find the non-redundant ones.
- Project: Use real algebraic geometry software for solving above problem and realize enumeration of L-type of stereohedron.

Thank You