The facets of the Birkhoff polytope of $W(F_4)$

Mathieu Dutour Sikiric

January 11, 2011

The group $W(\mathsf{F}_4)$ has 1152 elements. It is generated by 5 elements.

1. Generator 1 is:
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Generator 2 is:
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Generator 3 is:
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

4. Generator 4 is:
$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

5. Generator 5 is:
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We consider the facets of the convex hull of the group. There are 55872 facets. There are 2 orbits of facets

1. Orbit 1 has incidence 288 stabilizer of size 2304. One representative inequality is $Tr(XA) \leq 1$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $rank(A) = 1$.

2. Orbit 2 has incidence 36 stabilizer of size 24. One representative inequality is $Tr(XA) \leq 1$ with

$$A = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $rank(A) = 3$.